Politecnico di Torino

Massimo Bini

Thesis Work

Optimal Targeting in Social Networks

Supervisors:
Fabrizio Dabbene
Paolo Frasca
Chiara Ravazzi

Abstract

Motivated by problems arising in social networks, this thesis work considers a competition between two strategic agents who try to maximally influence a population by targeting a finite number of non-strategic/regular agents. It is assumed that regular agents adjust their opinion through a distributed averaging process, whereas the strategic agents present a fixed belief, towards which they try to shift the average opinion of the overall network by identifying the optimal targets to connect to. More specifically, the competition is set from the perspective of one of the strategic agents, as the optimization problem of selecting at most k regular agents to connect to in order to shift the network average opinion. Such a problem is known to be computationally hard, and effective heuristics are needed to reduce its complexity.

The core of this thesis work consists in the technical results upon which these heuristics are built. In particular, the problem is tackled in two ways. On one hand, an alternative proof (with respect to the literature) of the objective function submodularity is provided, so as to reduce the problem to a greedy heuristic – in this way, rather than solving one computationally challenging optimal targeting problem, it is possible to find a suboptimal solution by solving k separate single targeting problems. On the other hand, I exploit the underlying graph structure to solve special instances of the problem, upon which I build more refined heuristics.

As a first instance, I exploit the electrical analogy to provide the analytical solution for the single targeting problem over line graphs (STP). Then, the peculiar behavior of the objective function over such graphs is proved to be extendable to generic tree graphs, by considering the tree branches as pseudo-line graphs. From these findings, I define an algorithm that finds the optimal solution in a much faster way with respect to a brute-force approach. Upon this, I extend the algorithm to singular targeting problems over tree-like/sparse graphs, and then to the more general optimal targeting problem (OTP), by combining the STP solutions in a greedy manner.

Subsequently, as a second instance, the analytical solution for the OTP over complete graphs is provided. The result here is important, since it provides an immediate and simple answer to the question whether it is convenient or not to block the opponent's influence by targeting the same nodes. Upon this, another heuristic is built, by extending the simple strategy to

block the opponent or not, depending on a similar criteria, to generic graphs. Moreover, two additional heuristics are built based on the intuition that high degree nodes are the most influential ones. The former restricts the greedy search to a subset of nodes, reducing the complexity, whereas the latter consists in a zero-cost heuristic that targets the k_+ nodes with highest degree.

Finally, a scheme summarizing all the possible heuristics is provided in order to combine them effectively. Indeed, different depending on the cost vs accuracy trade-off and the underlying graph, some heuristics are better than others.

Contents

Nomenclature

Acronyms

OTP Optimal Targeting Problem

STP Singular Targeting Problem

Symbols

B Incidence matrix

C Conductance matrix

c_i Cardinality of the subtree rooted at node $i \in \{\text{path from } v^- \text{ to } v^+\}$ made up by the nodes $j \in \mathcal{I}^{<iv^-} \cap \mathcal{I}^{<iv^+}$ of the tree $\mathcal{T}$

d Degree vector

D Degree matrix

D_C Diagonal matrix of edge conductances, $(D_C)_{ee} = C_e$

$D_{C\mathbb{1}}$ Diagonal matrix of node total conductances, $D_{C\mathbb{1}} = diag(C\mathbb{1})$

$\mathcal{E}$ Edge set

$F_+(\mathcal{A})$ Sum of the asymptotic opinions of the regular individuals when $\mathcal{A}$ is the set of nodes linked to the strategic agent $+$

$\mathcal{F}$ Computational complexity of the objective function F_+

$\mathcal{G}$ Graph

$\mathcal{G}|_{\mathcal{R}}$ Graph restricted to the regular set $\mathcal{R}$

$G_+(\mathcal{A})$ Sum of the asymptotic opinions' sign of the regular individuals when $\mathcal{A}$ is the set of nodes linked to the strategic agent +

$\mathcal{G}_C$ Electrical network associated to graph $\mathcal{G}$

H Fundamental matrix, $(I - Q^{11})^{-1}$

$\mathcal{I}$ Node set of the tree graph $\mathcal{T}$

$\mathcal{I}^{<ij}$ Subset of $\mathcal{I}$'s nodes that form the subtree rooted at node i that does not contain node j, along with the path from i to j, apart from i

k_+ Number of edges that agent + can place

k_- Number of edges that agent − can place

L Laplacian matrix

$\mathcal{N}_i$ Neighborhood of agent i

N Number of regular nodes

Q Normalized weight matrix

Q^{11} $\mathcal{R} \times \mathcal{R}$ block of the normalized weight matrix

Q^{12} $\mathcal{R} \times \mathcal{S}$ block of the normalized weight matrix

$\mathcal{R}$ Regular agents node set

R^{eff} Effective resistence

$\mathcal{S}$ Strategic agents node set

$\mathcal{T}$ Tree graph

$\hat{\mathcal{U}}$ $:= \mathcal{U} \cup v$

$\mathcal{V}$ Node set of $\mathcal{G}$

V Voltage vector

v^+ Node linked to strategic node +

v^- Node linked to strategic node −

W	Weight matrix of $\mathcal{G}$
W^{11}	$\mathcal{R} \times \mathcal{R}$ block of the weight matrix
W^{12}	$\mathcal{R} \times \mathcal{S}$ block of the weight matrix
$x(t)$	Opinion vector at time t
$x^{\mathcal{R}}(t)$	Opinion state vector restricted to the regular agents set at time t
$x^{\mathcal{S}}(t)$	Opinion state vector restricted to the strategic agents set at time t
$\bar{x}$	Asymptotic opinion state vector
$\bar{x}^{\mathcal{R}}$	Asymptotic opinion state vector restricted to the regular set
$\bar{x}^{(\mathcal{A})}$	Asymptotic opinion state vector when $\mathcal{A}$ is the set of nodes directly connected to the strategic agent $+$
$+$	Agent $+$
$-$	Agent $-$
$\mathbb{1}$	All-ones vector
η	Input current vector
ϕ	Current flow vector

Chapter 1

Introduction

During the last few years, the diffusion of opinions has seen an explosion in terms of relevance and reach. Thanks to the new connections made possible by social platforms, individuals nowadays interact with thousands of others, exchanging opinions, ideas, memes, and much more, in an unprecedented way. At the same time, alongside these new technologies, the propagation of systematic biases and misinformation arose significantly, becoming eventually a real threat to western democracies. As examples, let us think to the misleading ads proposed during the Brexit campaign in 2016 [1], or the Pizzagate conspiracy theory that went viral during the 2016 US Presidential Elections [2]. In this context, understanding how opinions spread among population, and how political agents can intervene to control such dynamics becomes crucial.

Opinion Dynamics Theory

The main approach to model opinion dynamics represents society as a social network of N interconnected agents, each communicating with her contacts. I consider that at an initial stage every agent has a belief around a topic, represented by a distribution $x(0)$ over the agents. Then, by interacting with her contacts, each agent updates her opinion by averaging her own with the ones of her neighbors in a distributed manner. This distributed averaging process is better known as the French-DeGroot model [3, 4, 5]. In this model, when the underlying graph representing the network of agents is particularly connected, the dynamics taking place leads in the long-run to a consensus among the population. However, in reality, it is difficult to see such a consensus to a societal extent. In order to overcome this issue, a generalization to the French-DeGroot model is considered [6]. According to this, some individuals are not updating their opinion, spreading over and over the same myopic idea. I denote these agents as *strategic*, and I interpret them as either individuals or media outlets who wish to influence others with their opinion. However, more in general, these agents can also be interpreted as stubborn individuals, who do not easily change their opinion, or opinion leaders, who have a strong influence over some communities and are less likely to change their opinion as time goes on. By introducing this differentiation in the agents' behavior, we can find in the long-run the disagreement in society that we usually see, while maintaining the simplicity of the French-DeGroot model [7, 8, 9].

The framework in which this thesis work takes place differentiates agents in these two categories and assumes that non-strategic/regular agents influence each other reciprocally, and experience equal influence from their neighbors. So, while regular individuals interact and mutually influence each other, strategic agents have the only effect of shifting population's opinion towards their own. Clearly, this is an ideal representation of reality. However, let us notice how the assumption of uninfluenceable individuals, as well as the one of network staticity, becomes more robust as the diffusion duration decreases. This means that such model of a static society of regular and strategic individuals would be much closer to reality as the opinion dynamics time span is short.

Nevertheless, a huge variety of different approaches are present in the literature: the updating function relating the opinion of some agents to the other opinions can be either linear or nonlinear; the time domain in which the diffusion takes place can be either discrete or continuous; the update timing among agents can be either synchronous or asynchronous; the agents' opinion/state can be either categorical or quantitative; the interactions between individuals can be pairwise, among nearest neighbors or among anyone else; etc. In this diversity of possibilities, different approaches are more peculiar to some domains rather than others, whereas some generalizations can even be useless, depending on the application. For example, linear update functions allow for a simple mathematical treatment by means of well-known linear algebra techniques, such as matrix theory, making them a powerful instrument for studying opinion formation. Conversely, nonlinear update functions are more customizable and can be effectively used for a broad variety of fields: from the diffusion of innovations to epidemics.

At the same time, some generalizations can even be indifferent to the analysis: e.g. in this work it is assumed that all agents update their opinion synchronously within a discrete-time domain. Both these assumptions have no impact on asymptotic opinions, making our work rather general.

Maximal Influence Competition

Building upon these notions, the main question that arises is the following: How is it possible to control these dynamics? To answer this question quantitatively, the problem is stated as a competition between two strategic agents – of opposite opinion – who try to lead population's opinion towards their own by targeting a certain number of regular agents. This competition is set from the perspective of one of the strategic agents, so as to be described by the optimization problem of selecting at most k regular agents to connect to in order to maximally shift the average asymptotic opinion of the social network. Notice that this representation in not restrictive in terms of connection with reality, since it is equivalent to a more general one: where there is an arbitrary number of strategic agents of opposite opinion. Indeed, by remarking the equivalence between an asymptotic opinions network and an electrical network, all the strategic agents of identical opinion can be considered as voltage sources and merged together, without affecting the long-run results. In this context, rather that thinking of two competitors as single individuals, we can think of them as two groups of people – made up of opinion leaders, media, influencers, stubborn individuals, etc. – holding opposite opinions and trying to make their idea prevail by optimally targeting new individuals.

In the last few years, a major interest has been arising around these influence maximization problems over networks, attracting researches with different kinds of background.Some used an approach similar to ours to the problem. [10] analyzed the same problem as the one presented in this work, complementing part of the work studied in this thesis. Similarly, [11, 12, 13, 14, 15] tackled an analogous problem where influence maximization is achieved by means of regular agents' replacements with strategic agents rather than targeting. Others, instead, tackled the same problem as ours but with a different approach. [16] studied the problem applied to optimal investment strategies for competing camps, and also extended it to a multiphase dynamic game [17, 18], within the Friedkin-Johnson framework [6]. Conversely, [19] tackled the problem using a game theoretical approach.

While [20] extended the model to three competitors: two extremists and one centrist. In addition, [21] treated the problem as a dynamic targeting game. On the other hand, also different optimal targeting problems have been studied: [22] presented a work where two firms seek to maximize the diffusion of a product in a society, and another one [23] where a planner wants to do the same in terms of an action.

Optimal Targeting Problem Results and Heuristics

The optimization problem tackled in this thesis is known to be computationally hard [10]. This means that a brute-force approach is unfeasible, and effective heuristics are needed to reduce its complexity.

The core of this thesis work consists in the theoretical results that have been discovered, and successively deployed to build these heuristics. Whereas, the main contributions to the literature are the resulting heuristics, which led to a significant improvement in the emerging solution of the optimization problem, in terms of both accuracy and expensiveness. The simplest way to tackle the problem is to reduce its complexity by making use of the submodularity of the objective function. In particular, both its monotonicity and submodularity are proved independently and without being aware of the results in [10], which appeared later – in addition, alternative version of the results are provided in Appendix A.1 and A.2, the latter being built upon the results in [11]. Thanks to these findings, one can bypass the computation of the optimal targeting problem (OTP), of complexity $O(N^{3+k})$, by simply computing the 1-best solutions of successive k singular targeting problems (STP) in a greedy manner. Such 1-best solutions are simply computed with a brute-force approach, by comparing all the N possible combinations. This allows to reduce the complexity to $O(kN^4)$, and leads eventually to a bounded approximation of the optimal solution [24].

A more complicated way to tackle the problem consists in exploiting the results for some special graph in order to build more refined heuristics. Our contribution here is entirely original to the literature, and leads to remarkable results. As a first kind of special graphs, the analytical solution for the STP over line graphs is provided. In particular, the computation of the objective function in terms of the target node position exhibits a peculiar concave behavior. This allows to find a closed-form solution for the STP over line graphs. Additionally, such peculiar behavior of the objective function is proved to be extendable to the branches of generic tree graphs, when treated as pseudo-line graphs. Here, if the tree is known, it is possible to find again a closed-form solution. But, in addition, these findings also allow to define an algorithm which finds the optimal solution for the STP over generic tree graphs in an efficient way. Subsequently, as a second kind of special graphs, the analytical solution for the OTP over complete graphs is provided, implying that, whatever the budget available to both players, it is possible to solve the OTP in a closed-form. This result is powerful, since it tells us in an immediate way whether it is convenient or not to block the opponent's influence by targeting the same nodes: if the overall number of edges available and placed by one strategic agent is strictly greater than the number of edges placed by the opponent, the optimal strategy consists in blocking their influence over the graph.

Finally, upon these results, a variety of heuristics are built, which will be used interchangeably depending on the previous knowledge about the underlying graph and the accuracy vs cost trade-off characterizing different heuristics. In order to present useful results, these heuristics are compared with zero-cost strategies, consisting in both the targeting of highest degree nodes, and a mix between targeting the highest degree nodes and a blocking approach – coming from the complete graph result. In this way we can have a proper comparison between the best (empirically determined) zero-cost solutions, with respect to expensive results. For the simple STP, upon the tree graph algorithm, a *tree-like heuristic* over generic graphs is built, allowing to cut the cost significantly in the maximum search, with more remarkable results as the graph is more tree-like/sparse. Next, for the OTP, the *blocking heuristic* is built by copying the mechanics of the complete graph result – by blocking the opponent influence over nodes whose degree is sufficiently high – and by placing the remaining nodes in a greedy manner. This is usually more accurate and less expensive than a purely greedy approach. Notice however, that this is not always true, especially when the graph is sparser. Moreover, it is possible to additionally reduce the heuristics complexity by implementing the greedy search in a smarter way: either simulating k successive tree-like heuristics, or by doing the k brute-force searches restricted to a subset of high-degree nodes. Concluding, these heuristics are summarized in a scheme of work telling which heuristic to choose, based on the cost vs accuracy trade-off, and the underlying graph.

Chapter 2

Problem Statement

2.1 Preliminaries and Graph Terminology

When studying networks of interconnected agents, the fundamental mathematical model is that of *graphs*. In such representation, the unitary elements making up the network are denoted as *nodes*, while the relationships among such entities are represented as links among them – denoted as *edges* or *links* – and can eventually stress a directionality, if *directed*, or not, if *undirected*. Moreover, such edges can be associated to values quantifying their strength. In this case we talk about *weighted* edges and the graph associated will be denoted as a *weighted graph*.

More formally, a *weighted graph* is defined as the triple

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$$

where

- $\mathcal{V}$ is the countable set of nodes associated
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, indicated as the ordered pair (i, j) if directed from node i to j
- $W \in \mathbb{R}_+^{\mathcal{V} \times \mathcal{V}}$ is the weight matrix associated, where $W_{ij} > 0$ indicates the strength of edge (i, j) and $W_{ij} = 0$ if and only if $(i, j) \notin \mathcal{E}$, i.e. if there is no edge (i, j).

Then, I will denote a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ as

- *undirected* if its weight matrix W is symmetric, i.e. $W_{ij} = W_{ji}$ for each $i, j \in \mathcal{V}$. In such case, the *undirected* edges will be denoted as unordered pairs $\{i, j\}$, corresponding to both the directed links (i, j) and (j, i).

- *unweighted* if its weight matrix $W \in \{0, 1\}^{\mathcal{V}\times\mathcal{V}}$, i.e. $W_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$. Note that unweighted graphs can be equivalently indicated by the couple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Let us now introduce some definitions associated to the nodes of the graph.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph, then

- the *out-neighborhood* and *in-neighborhood* of a node $i \in \mathcal{V}$ are defined as the sets of nodes to which i is linked when pointing outwards and inwards respectively, i.e.

$$\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\} \ , \qquad \mathcal{N}_i^- = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$$

  while the nodes in $\mathcal{N}_i$ and $\mathcal{N}_i^-$ are called the *out-neighbors* and *in-neighbors* of node i respectively. Note that when the two sets are identical – as for undirected graphs – I will simply refer to a sole *neighborhood*, univocally denoted as $\mathcal{N}_i$, whose elements are called *neighbors*.

- when a node i has no out-neighbors, i.e. it has no links pointing outwards other than possibly itself, it is called a *sink*.

- the *out-degree* and *in-degree* of a node are defined as

$$w_i = \sum_{j\in\mathcal{V}} W_{ij} \ , \qquad w_i^- = \sum_{j\in\mathcal{V}} W_{ji}$$

  Note that for unweighted graphs, as I will analyze, such degrees correspond to the cardinality of the out-neighborhood and in-neighborhood of a node i respectively plus possibly itself. Whereas for undirected graphs such degrees are identical and I will simply refer to a sole *degree* as w_i.

Walks, Paths and Reachability

Let us now introduce a series of notions that exploit the graph representation and allow for intuitive interpretations of its key properties.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph, then

- a finite sequence of nodes $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ such that $\gamma_0 = i$, $\gamma_l = j$ and $(\gamma_{h-1}, \gamma_h) \in \mathcal{E}$ for each $h = 1, 2, \dots,$ is defined as the *walk* from node i to node j, while l is called the *length* of the walk.

- a walk $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ such that $\gamma_h \neq \gamma_k$ for each h, k such that $0 \leq h < k \leq l$ except for possibly $\gamma_0 = \gamma_l$, i.e. a walk that does not pass in the same node twice except for possibly its starting point, is called a *path*.

- a path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ such that $l \geq 3$ and $\gamma_0 = \gamma_l$, i.e. a non-trivial path starting and ending in the same point, is called a *cycle*.

- a node j is said to be *reachable* from node i if it exists a walk γ from i to j.

- the graph $\mathcal{G}$ is called *strongly connected* if for each $i \neq j$ in $\mathcal{V}$, there exists a walk from i to j, i.e. j is reachable from i.

- a subset of nodes $\mathcal{U} \subset \mathcal{V}$ is said to be *trapping* – or *absorbing* – in $\mathcal{G}$ if for every node $i \in \mathcal{U}$ and $j \in \mathcal{V} \setminus \mathcal{U}$ there exists no path from i to j.

- a subset of nodes $\mathcal{U} \subset \mathcal{V}$ is said to be *globally reachable* in $\mathcal{G}$ if for every node $j \in \mathcal{V} \setminus \mathcal{U}$ there exists a path from j to some node $i \in \mathcal{U}$.

Algebraic Graph Theory

Let us now introduce the basic linear algebraic tools used in this work to exploit the graph representation. Notice that for the whole document the vectors v are represented as column vectors while their transpose is denoted as $v^\top$. In addition, I will denote as $\mathbb{1}$ the all-ones column vector, whose size is determined according to its usage.

The first natural matrix of interest is obviously the weight matrix W, whose elements are the weights associated to each edge of the graph.

From W it is possible to obtain

- the *out-degree* and *in-degree* vectors

$$w = W\mathbb{1}\ , \qquad w^{-} = W\mathbb{1}$$

  whose elements are the out-degrees and in-degrees associated to each node respectively.

- the *out-degree matrix*, or simply the *degree matrix*

$$D = diag(w)$$

  which is the rewriting of the w vector into the main diagonal of the diagonal matrix D.

In addition to W, two other matrices turn out to be extremely relevant:

- the *normalized weight matrix*

$$Q = D^{-1}W$$

  which is non-negative and for which $Q\mathbb{1} = \mathbb{1}$, i.e. its rows sum to one. Matrices with such properties are called *stochastic*, since the elements of such rows can be associated to probabilities.

- the *Laplacian matrix*

$$L = D - W$$

  for which $L\mathbb{1} = 0$, i.e. its rows have zero sum. Matrices L with this property and such that $-L$ is Meltzer are indeed called *Laplacian.*

Additionally, it is also useful to restrict the *normalized weight matrix* Q to a subset $\mathcal{U}$ of nodes, denoting such new matrix as $\underline{Q}|_{\mathcal{U}}$. Then, it is easy to see that

- if $\mathcal{U}$ is trapping, $\underline{Q}|_{\mathcal{U}}$ is stochastic

- if $\mathcal{U}$ is not trapping, $\underline{Q}|_{\mathcal{U}}$ is *sub-stochastic*, i.e.

$$\underline{Q}|_{\mathcal{U}}\mathbb{1} \leq \mathbb{1} \qquad \underline{Q}|_{\mathcal{U}}\mathbb{1} \neq \mathbb{1}$$

Spectral Properties

Let us now exhibit one of the main results of non-negative matrices theory: the Perron-Frobenius theorem [25, 26]. Such result is related to the spectral properties of non-negative square matrices and its consequences on the graph theoretical framework are various. However, for brevity, I will simply present the results functional to the development of this work, also omitting the proof of this result, which can be found – in alternative ways – on various other works [25, 26, 27, 28, 29, 30, 31].

Theorem 2.1.1 (Perron-Frobenius)**.** *Let $A \in \mathbb{R}_+^{n\times n}$ be a non-negative square matrix. Then, there exist a unique non-negative eigenvalue $\lambda_A \in \mathbb{R}_+$ and non-negative vectors $x, y \in \mathbb{R}_+^n$ such that*

- $Ax = \lambda_A x$
- $A^\top y = \lambda_A y$
- $\forall\lambda$ *eigenvalue of A,* $|\lambda| \leq \lambda_A$

Such eigenvalue λ_A – being the eigenvalue with greatest module – is called the *dominant eigenvalue* of A, while the corresponding vectors x and y are called the *right* and *left dominant eigenvalues*, respectively.

This result can be written in terms of the normalized weight matrix, since it is a non-negative square matrix. Indeed, it possible to write down the following corollary as a simple rewriting of the Perron-Frobenius theorem. The proof is omitted for triviality.

Corollary 2.1.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E},, W)$ be a graph where $W \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$. Let $Q = D^{-1}W$ be its normalized weight matrix. Then, there exist a unique non-negative eigenvalue $\lambda_Q \in \mathbb{R}_+$ and non-negative eigenvectors $x, y \in \mathbb{R}_+^n$ such that*

- $Qx = \lambda_Q x$
- $Q^\top y = \lambda_Q y$
- $\forall\lambda$ *eigenvalue of Q,* $|\lambda| \leq \lambda_Q$

2.2 Competitive Opinion Dynamics

The focus of this work is related to the study of opinion dynamics controllability over networks. In this regard, let us represent a society as an influence network described by a graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$$

whose nodes $i \in \mathcal{V}$ represent its *individuals*, also denoted as *agents*, and the edges $(i, j) \in \mathcal{E}$ express the interactions among them. The intensity of such interactions are reported in the weight matrix W, where the entries W_{ij} represent how much agent i is positively influenced by agent j, while the W_{ii} terms determine how much agent i weights her own current opinion with respect to others' ones, i.e. it indicates her resistance to change.

Let us now assume that each individual has a state $x_i(t) \in [-1, +1]$, representing her opinion at time t, that at each time step $t = 1, 2, 3, \dots$ is updated as a response to the interaction with her neighbors, according to the update rule

$$x_i(t+1) = \sum_{j \in \mathcal{V}} Q_{ij} x_j(t) \qquad i \in \mathcal{V}, \quad t = 1, 2, \dots \tag{2.1}$$

indicating that the new state $x_i(t+1)$ of agent i corresponds to the weighted average of her neighbors' opinion $x_j(t)$ at time t and her own previous one $x_i(t)$. More compactly, equation (2.1) can be rewritten in the matrix form

$$x(t+1) = Qx(t) \qquad t = 1, 2, \dots \tag{2.2}$$

where $x(t)$ corresponds to the vector of all agents' opinion. Such equation in the social science context is known as the *DeGroot* opinion dynamics model [3, 4, 5] or, more generally, as the *linear averaging dynamics* on $\mathcal{G}$.

This behavior could also be taken to the limit: when an agent only considers her own opinion, i.e. $x(t+1) = x(t)$ for each $t = 1, 2, \dots$. Such individuals are defined as *strategic* and have a huge impact on the dynamics.

Indeed, it is convenient to handle this different behavior by splitting the population into:

- $\mathcal{R} \subseteq \mathcal{V}$: set of *regular* agents, who weight their opinion among their neighbors

- $\mathcal{S} = \mathcal{V} \setminus \mathcal{R}$: set of *strategic* agents, who maintain the same opinion

By rearranging the nodes in such a way that the regular nodes come first, it is possible to partition the weight matrix W in its $\mathcal{R} \times \mathcal{R}$, $\mathcal{R} \times \mathcal{S}$, $\mathcal{S} \times \mathcal{R}$ and $\mathcal{S} \times \mathcal{S}$ blocks, as denoted below

$$W = \begin{array}{c} \quad \mathcal{R} \quad\;\; \mathcal{S} \\ \begin{pmatrix} W^{11} & W^{12} \\ 0 & I \end{pmatrix} \end{array} \begin{array}{c} \\ \mathcal{R} \\ \mathcal{S} \end{array}$$

where W^{21} is substituted with the *null matrix* and W^{22} with the *identity matrix* of corresponding dimensions.

Moreover, let us make two strong assumptions on the regular agents behavior in order to keep the mathematics simple:

Assumption 1. $W^{11} = (W^{11})^\top$*, i.e.* $\mathcal{G}|_\mathcal{R}$ *undirected*

Assumption 2. $W^{11} \in \{0,1\}^{\mathcal{R}\times\mathcal{R}}$*, i.e.* $\mathcal{G}|_\mathcal{R}$ *unweighted*

where $\mathcal{G}|_\mathcal{R}$ is the graph restricted to regular agents.

The two assumptions tell us that each person reciprocally influences each other and that each individual is equally influenced by the individuals she interacts with, respectively.

On the other hand, let us also assume for the strategic agents that

Assumption 3. $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$

Assumption 4. $W^{12} \in \{0,1\}^{\mathcal{R}\times\mathcal{S}}$

which tell that each strategic agent has at least one link to a regular agent, but no more than one to the same target. Thanks to the latter assumption, the problem will be easier to state – in terms of node targeting – and it will present a quite reduced solution space, i.e. all the possible configurations of W^{12} in $\{0,1\}^{\mathcal{R}\times\mathcal{S}}$ such that $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$. Such assumption is both intuitive and motivated by experiments over different random graphs, which will suggest that multiple targeting onto the same node is not optimal.

The work presented here focuses on optimal targeting on a social network, expressed as a competition between two agents of opposite opinion. Such agents are represented as two strategic agents, denoted as $-$ and $+$, of opinion -1 and $+1$ respectively, while the other individuals of the social network are considered as regular.

Let us now assume that the population size is finite and equal to $N+2$, i.e. $|\mathcal{V}| = N+2$, and let us rearrange the nodes in such a way that the regular nodes come first, i.e.

- $\mathcal{R} = \{1, 2, 3, \dots, N\}$
- $\mathcal{S} = \{N+1, N+2\} := \{-, +\}$

where the two strategic nodes are denoted as $-$ and $+$ form now on.

Now, by partitioning the normalized weight matrix in its $\mathcal{R}\times\mathcal{R}$, $\mathcal{R}\times\mathcal{S}$, $\mathcal{S}\times\mathcal{R}$ and $\mathcal{S}\times\mathcal{S}$ blocks so that

$$Q = \begin{array}{c} \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} \\ \begin{pmatrix} Q^{11} & Q^{12} \\ 0 & I \end{pmatrix} \end{array} \begin{array}{c} \\ \mathcal{R} \\ \mathcal{S} \end{array}$$

the discrete-time update rule in vector representation becomes

$$\begin{aligned} x^{\mathcal{R}}(t+1) &= Q^{11}x^{\mathcal{R}}(t) + Q^{12}x^{\mathcal{S}}(t) \\ x^{\mathcal{S}}(t+1) &= x^{\mathcal{S}}(t) \end{aligned} \qquad t = 1, 2, \dots \tag{2.3}$$

where $x^{\mathcal{R}}(t)$ and $x^{\mathcal{S}}(t)$ are the partitions of the state vectors as

$$x(t) = \begin{pmatrix} x^{\mathcal{R}}(t) \\ x^{\mathcal{S}}(t) \end{pmatrix} \begin{array}{c} \mathcal{R} \\ \mathcal{S} \end{array}$$

Let us now assume that the graph restricted to regular nodes $\mathcal{G}|_{\mathcal{R}}$ is strongly connected, i.e. for each couple of node i and j there always exists a path in $\mathcal{G}|_{\mathcal{R}}$ going from i to j:

Assumption 5. $\mathcal{G}|_{\mathcal{R}}$ *strongly connected*

Such condition is required since we want $\mathcal{S}$ to be globally reachable for each possible configuration of strategic nodes connections. Indeed, the following proposition holds.

Proposition 2.2.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph, let $\mathcal{S} \subset V$ be a nonempty subset of nodes, let $\mathcal{R} = \mathcal{V} \setminus \mathcal{S}$, and let W^{12} be the $\mathcal{R} \times \mathcal{S}$ block of the weight matrix W. Then, for all the possible combinations of $W^{12} \in \{0,1\}^{\mathcal{R} \times \mathcal{S}}$ such that $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$, $\mathcal{S}$ is globally reachable iff $\mathcal{G}|_\mathcal{R}$ is strongly connected*

Proof. Assumption 3, i.e. $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$, tells that each node in $\mathcal{S}$ has at least one link to a node in $\mathcal{R}$. Without loss of generality, let us consider the case where all the nodes belonging to $\mathcal{S}$ are linked to the only one, same, node i in $\mathcal{R}$ – the worst-case scenario, in terms of connectivity. Thus, being S globally reachable, it means that for each $j \in \mathcal{R}$, i is reachable from j, since it is the only node through which S is reachable. However, the choice of such node i is arbitrary, considering all the combinations of W^{12}, leading to the definition of connectivity for $\mathcal{G}|_\mathcal{R}$. □

Such assumption – under which each regular node can reach at least one strategic node – leads to a fundamental result, as expressed in the following proposition: each regular agent, by following the DeGroot dynamics as expressed in (2.3), will reach a stationary opinion as time goes on, and such opinion will be a combination of the opinions of all the strategic agents she can reach.

Proposition 2.2.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph, let $\mathcal{S} \subset \mathcal{V}$ be a nonempty globally reachable subset of strategic nodes, let $\mathcal{R} = \mathcal{V} \setminus \mathcal{S}$ be the subset of regular nodes, and let Q^{11} be the $\mathcal{R} \times \mathcal{R}$ block of the normalized weight matrix Q. Then,*

(i) *Q^{11} is substochastic and asymptotically stable*

(ii) *$I - Q^{11}$ is invertible with non-negative inverse matrix*

$$H = (I - Q^{11})^{-1} = \sum_{k \geq 0} (Q^{11})^k$$

where H is called the fundamental matrix

(iii) *for every intial state vector $x(0) \in \mathbb{R}^\mathcal{V}$, the DeGroot dynamics with strategic nodes described in (2.3) satisfy*

$$\lim_{t \to +\infty} x^\mathcal{R}(t) = HQ^{12} x^\mathcal{S}$$

Proof. Starting from (i), the sub-stochasticity of Q^{11} follows from the facts that $Q^{11} = Q|_{\mathcal{R}}$ and $\mathcal{R}$ is not trapping, since $\mathcal{S}$ is globally reachable and nonempty. The second result of (i) makes use of *Perron-Frobenius* Corollary 2.1.2, which says that under this conditions it always exists a $y \geq 0$ such that $(Q^{11})^\top y = \lambda_{Q^{11}} y$, where $\lambda_{Q^{11}}$ is the dominant eigenvalue of Q^{11}. Then, let $\mathcal{U} \subseteq \mathcal{R}$ be the support of y. Since S is globally reachable, also $\mathcal{U}$ is not trapping, and thus sub-stochastic. Consequently, by summing over $i \in \mathcal{U}$, it is possible to obtain that

$$\begin{aligned}
\lambda_{Q^{11}} \sum_{i\in\mathcal{U}} y_i &= \sum_{i\in\mathcal{U}} \lambda_{Q^{11}} y_i \\
&= \sum_{i\in\mathcal{U}} \sum_{j\in\mathcal{V}} (Q^{11})_{ji} y_j \\
&= \sum_{i\in\mathcal{U}} \sum_{j\in\mathcal{U}} (Q^{11})_{ji} y_j \\
&< \sum_{j\in\mathcal{U}} y_j
\end{aligned}$$

where the last inequality holds since $\min_{i\in\mathcal{U}} \sum_{j\in\mathcal{S}} (Q^{11})_{ij} y_j < 1$, being Q^{11} sub-stochastic. This leads to $\lambda_{Q^{11}} < 1$, sufficient and necessary condition for Q^{11} to be asymptotically stable.

Point (ii) is a consequence of this result. Indeed, since $(Q^{11})^t \to 0$ as $t \to +\infty$, it is possible to write

$$\begin{aligned}
(I - \lim_{t\to+\infty} (Q^{11})^t) &= (I - Q^{11})(I + Q^{11} + (Q^{11})^2 + \dots) \\
I &= (I - Q^{11})(I + Q^{11} + (Q^{11})^2 + \dots)
\end{aligned}$$

implying that $(I - Q^{11})$ is invertible and its inverse is exactly the second term of the right member of the equation, i.e.

$$(I - Q^{11})^{-1} = \sum_{k\geq 0} (Q^{11})^k$$

which is non-negative, since it is a sum of positive terms.

Point (iii) instead is obtained by writing (2.3) explicitly. Indeed,

$$
\begin{aligned}
x^{\mathcal{R}}(t) &= Q^{11}x^{\mathcal{R}}(t-1) + Q^{12}x^{\mathcal{S}} \\
&= Q^{11}[Q^{11}x^{\mathcal{R}}(t-2) + Q^{12}x^{\mathcal{S}}] + Q^{12}x^{\mathcal{S}} \\
&= (Q^{11})^2x^{\mathcal{R}}(t-2) + [(Q^{11})^1 + (Q^{11})^0]Q^{12}x^{\mathcal{S}} \\
&\vdots \\
&= (Q^{11})^t x^{\mathcal{R}}(0) + [(Q^{11})^{t-1} + (Q^{11})^{t-2} + \cdots + (Q^{11})^0]Q^{12}x^{\mathcal{S}} \\
&= (Q^{11})^t x^{\mathcal{R}}(0) + \left[\sum_{k=0}^{t-1}(Q^{11})^k\right]Q^{12}x^{\mathcal{S}}
\end{aligned}
$$

that by taking the limit $t \to +\infty$ leads to

$$\lim_{t\to+\infty} x^{\mathcal{R}}(t) = HQ^{12}x^{\mathcal{S}}$$

□

This leads to the notion of *asymptotic opinion* state vector, defined as

$$\bar{x} := \lim_{t\to+\infty} x(t) = \begin{pmatrix} HQ^{12}x^{\mathcal{S}} \\ x^{\mathcal{S}} \end{pmatrix} \in \mathbb{R}^{\mathcal{V}}$$

where its regular agents partition can be expressed in the following equivalent ways:

$$\bar{x}^{\mathcal{R}} = HQ^{12}x^{\mathcal{S}} = (I - Q^{11})^{-1}Q^{12}x^{\mathcal{S}} = \left[\sum_{k\geq 0}(Q^{11})^k\right]Q^{12}x^{\mathcal{S}}$$

Nonetheless, in Chapter 3.2, by introducing the notions of random walks over graphs, I will allow for a probabilistic interpretation of such vector, that originates by noticing that

$$HQ^{12} = \mathbb{1}$$

Indeed, since Q is stochastic,

$$\begin{aligned} Q^{11}\mathbb{1} + Q^{12}\mathbb{1} &= \mathbb{1} \\ Q^{12}\mathbb{1} &= (I - Q^{11})\mathbb{1} \\ (I - Q^{11})^{-1}Q^{12}\mathbb{1} &= \mathbb{1} \end{aligned}$$

In such way, we can interpret the asymptotic opinion $\bar{x}_i$ of a regular agent $i \in \mathcal{R}$ as a convex hull of opinions x^S of the strategic nodes $s \in \mathcal{S}$, i.e.

$$x_i^{\mathcal{R}} = \sum_{s \in \mathcal{S}} (HQ^{12})_{is} x_s^{\mathcal{S}} \tag{2.4}$$

where

$$(HQ^{12})_{is} = \sum_{k \geq 0} \sum_{i_0 = i, i_1 \ldots i_k - 1 \in \mathcal{R}, i_k = s} \prod_{1 \leq h \leq k} Q_{i_{h-1}, i_h} \tag{2.5}$$

i.e. the coefficient is the sum of the probabilities of all the possible k-length paths from node $i \in \mathcal{R}$ to node $s \in \mathcal{S}$ where the first $k-1$ nodes of such path are regular ones, for all k.

This means that the asymptotic opinion of a regular agent i can be written as the sum over $s \in \mathcal{S}$ of the probabilities of going from node i to node s – multiplied by the state of such strategic node – through the random walk defined by the normalized weight matrix Q.

Lastly, the asymptotic opinion state vector can be equivalently expressed as the only vector satisfying the system

$$\begin{cases} \sum_{j \in \mathcal{V}} L_{ij} x_j = 0 \quad \forall i \in \mathcal{R} \\ x_i = x_i^{\mathcal{S}} \quad \forall i \in \mathcal{S} \end{cases} \tag{2.6}$$

where L_{ij} is the ij-th component of the *Laplacian* $L = D - W$. Indeed

$$\begin{aligned} L \begin{pmatrix} HQ^{12}x^{\mathcal{S}} \\ x^{\mathcal{S}} \end{pmatrix} &= D \begin{pmatrix} HQ^{12}x^{\mathcal{S}} \\ x^{\mathcal{S}} \end{pmatrix} - D \begin{pmatrix} Q^{11}HQ^{12}x^{\mathcal{S}} + Q^{12}x^{\mathcal{S}} \\ Q^{21}HQ^{12}x^{\mathcal{S}} + Q^{22}x^{\mathcal{S}} \end{pmatrix} \\ &= D \begin{pmatrix} (I - Q^{11})HQ^{12}x^{\mathcal{S}} - Q^{12}x^{\mathcal{S}} \\ x^{\mathcal{S}} - Q^{21}HQ^{12}x^{\mathcal{S}} - Q^{22}x^{\mathcal{S}} \end{pmatrix} \\ &= D \begin{pmatrix} 0 \\ x^{\mathcal{S}} - Q^{21}HQ^{12}x^{\mathcal{S}} - Q^{22}x^{\mathcal{S}} \end{pmatrix} \end{aligned}$$

2.3 Optimal Targeting

Since my focus is to investigate optimal targeting on a social network, let us set the problem as an optimization problem from the perspective of one of the strategic agents: $+$. In such context, I assume that the other strategic agent $-$, the adversary, is already linked to the population while $+$ could or could not be linked yet.

First of all, let us assume that the unitary edges of agent $-$ are already placed and let us define the number of such edges as

- $k_- := \sum_{j\in\mathcal{R}} W_{j-}$

On the other hand, we need to define both the number of edges k_+^0 already linked to $+$, and the available amount of edges k_+ that she has to add, such that the initial weight matrix $W^{(0)}$, that is the weight matrix before the targeting problem, is such that

- $k_+^0 := \sum_{j\in\mathcal{R}} W_{j+}^{(0)}$

while the final weight matrix W, that is the weight matrix after the targeting problem, is such that

- $k_+ + k_+^0 := \sum_{j\in\mathcal{R}} W_{j+}$

Indeed, I assume that at the beginning the k_- edges of agent $-$ and the k_+^0 edges of agent $+$ are placed, while agent $+$ faces the optimization problem of maximizing the sum of asymptotic opinions by placing these k_+ edges. Notice how the condition $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$ is equivalent to ask for k_- and $k_+^0 + k_+$ to be greater than one.

In order to formalize this problem from the perspective of the $+$ agent, let us denote by $\bar{x}_i^{(\mathcal{A})}$ the asymptotic opinion of agent i in the particular configuration of new edges linked to the strategic agent $+$, univocally identified by the weight matrix $W^{\mathcal{A}}$, where the new nodes targeted by $+$ are all the nodes belonging to the set $\mathcal{A}$. Then, the simplest choice for the objective function, since the goal of the $+$ agent is to move the other nodes' opinion towards her own, is the sum of all the regular agents' opinion, in terms of the set $\mathcal{A}$ of nodes targeted by $+$.

Optimal Targeting Problem (OTP): Let us define the objective function that we want to maximize as the *set function*

$$F_+ : 2^{\mathcal{R}} \to \mathbb{R}$$

whose domain is the family of nonempty sets of the possible combinations of regular agents. Such function takes as input the set of regular nodes that are targeted by the strategic agent + and it computes the average asymptotic opinion of the agents in such particular configuration, i.e.

$$F_+(\mathcal{A}) = \frac{1}{N} \sum_{i \in \mathcal{R}} \bar{x}_i^{(\mathcal{A})} \quad , \quad \mathcal{A} \subseteq \mathcal{R} \tag{2.7}$$

Then, the optimal targeting problem OTP can be defined as

$$\max_{\mathcal{A} \subseteq \mathcal{R}} F_+(\mathcal{A}) \tag{2.8}$$

which, in practice, is an influence maximization problem in terms of optimal targeting. In particular, if $k_+, k_- \geq 1$, I will refer to the general problem (2.8) as the *optimal targeting problem* (OTP) or *multiple targeting problem*, whereas, if $k_+, k_- = 1$, I will talk of *single targeting problem* (STP), and I will slightly change the notation.

Singular Targeting Problem (STP): : Let us denote by v^+ and v^- the unique nodes linked to + and − respectively, and by $F_+(v^+)$ the objective function. Then the single targeting problem STP is defined as

$$\max_{v^+ \in \mathcal{R}} F_+(v^+) \tag{2.9}$$

Chapter 3

Graphs, Random Walks and Electrical Analogy

3.1 Special Graphs

In this section let us define the main graphs that will be studied in the rest of the manuscript.

Tree Graph

A *tree graph* is a connected graph that is unweighted, undirected, and that has no cycles.

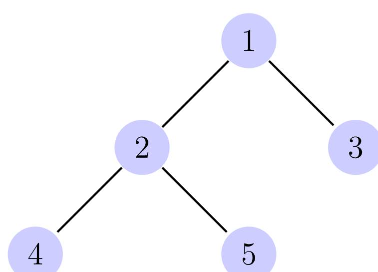


Figure 3.1: Tree graph with $N = 5$

Line Graph

A *line graph* is a tree graph where $N - 2$ nodes have degree two, and two nodes have degree one.

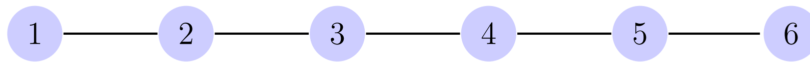


Figure 3.2: Line graph with $N = 6$

Complete Graph

A *complete graph* is a connected unweighted undirected graph where each node is linked to every other node.

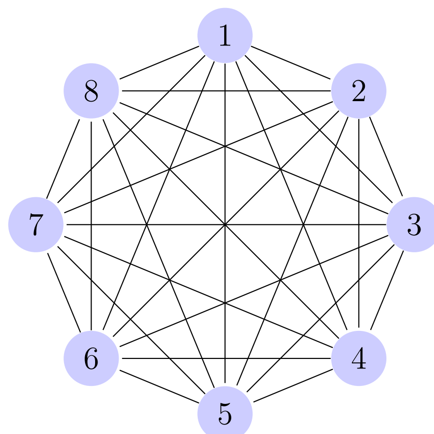


Figure 3.3: Complete graph with $N = 8$

3.2 Random Walks on Graphs

In this section I present the basic notions of Markov chains and their random walks representation on graphs. Indeed, when talking of a finite state space Markov chain, it is possible to associate this notion to the one of a random walk over a graph, where a state corresponds to a node and the overall finite state space to the node set. In particular, depending on the considered dynamics, it is possible to represent both discrete-time and continuous-time Markov chains. However, I will only present the former of the two, since in this work – studying asymptotic states – both representations are equivalent, i.e. asymptotic opinion states do not depend on the time domain of the update rule or – from the alternative point of view here presented – the time domain of the associated Markov chain.

Markov Chains and Random Walks

A Markov chain is a stochastic process where the probability distribution of future states is completely determined by the present state distribution, regardless of the past history of the stochastic process. This property is known as the *Markov property*.
In the graph theoretical framework, when the state space of the Markov chain is discrete and finite, it is possible to think of a Markov chain as a particle moving over the nodes of a graph in a random walk manner. Indeed, it is possible to imagine such particle as jumping from one node to another: if at time t the particle is located in node i, at time $t+1$ it could jump to one of its out-neighbors j with a probability proportional to the weight of the edge (i, j). In such way, it is possible to visualize a discrete-time Markov chain as a moving particle. Nevertheless, also the opposite is true: some dynamics over graphs can be equivalently expressed as a Markov chain and can be tackled by the tools of stochastic processes.

More formally, let $X(t)$ be a discrete-time stochastic process with discrete state space $\mathcal{X}$. Then, $X(t)$ is a *discrete-time Markov chain* if for any $i, j, i_0, i_1, \dots, i_{t-1} \in \mathcal{X}$ the Markov property

$$\mathbb{P}(X(t+1) = j|X(t) = i, X(t-1) = i_{t-1}, \dots, X(0) = i_0) = \mathbb{P}(X(t+1) = j|X(t) = i)$$

holds. This means that the probability of going from a state i to j – also called *transition probability* – only depends on the current state, while it does not depend on the history of past states.

Since I want to describe such concept from a graph theoretical point of view, let us assume that the state space $\mathcal{X}$ of the discrete-time Markov chain $X(t)$ is discrete and finite. Then, it is possible to associate such Markov chain to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where the finite state space $\mathcal{X}$ corresponds to the node set $\mathcal{V}$ and the transition probability $\mathbb{P}(X(t+1) = j|X(t) = i)$ corresponds to the entry Q_{ij} of the normalized weight matrix $Q = D^{-1}W$ – denoted as the *transition probability matrix*.
However, since the states of a Markov chain are probability distributions, in order to have a perfect equivalence, I need to introduce an initial probability distribution $\pi(0)$ over the node set V – representing the probability distribution of the particle's starting position. In such way, there is a perfect equivalence between discrete-time Markov chains with finite discrete state space and random walks over a weighted directed graph, by associating:

- the finite discrete state space $\mathcal{X}$ to the node set $\mathcal{V}$, i.e. $\mathcal{V} = \mathcal{X}$

- the transition probability matrix to the normalized weight matrix Q, whose entries satisfy

$$Q_{ij} = \mathbb{P}(X(t+1) = j | X(t) = i) \ , \qquad i, j \in \mathcal{V}$$

- the initial probability distribution of the Markov chain $X(0)$ to the initial probability distribution $\pi(0)$, whose entries satisfy

$$\pi_i(0) = \mathbb{P}(X(0) = i) \ , \qquad i \in \mathcal{V}$$

Consequently, by defining Q and $\pi(0)$, it is possible to determine the probability distribution of a trajectory $(X(t) = i_t, X(t-1) = i_{t-1} \dots, X(0) = i_0)$, by simply applying the Markov property. Indeed,

$$\begin{aligned}
&\mathbb{P}(X(t) = i_t, X(t-1) = i_{t-1} \dots, X(0) = i_0) \\
&\quad = \mathbb{P}(X(t) = i_t | X(t-1) = i_{t-1} \dots, X(0) = i_0) \cdot \mathbb{P}(X(t-1) = i_{t-1} \dots, X(0) = i_0) \\
&\quad = Q_{t-1,t} \cdot \mathbb{P}(X(t-1) = i_{t-1} | X(t-2) = i_{t-2} \dots, X(0) = i_0) \cdot \mathbb{P}(X(t-2) = i_{t-2} \dots, X(0) = i_0 \\
&\quad = Q_{t-1,t} \cdot Q_{t-2,t-1} \cdot \mathbb{P}(X(t-2) = i_{t-2}, X(t-3) = i_{t-3} \dots, X(0) = i_0) \\
&\quad \vdots \\
&\quad = \prod_{1 \le s \le t} Q_{i_{s-1}, i_s} \cdot \pi_{i_0}(0) \qquad (3.1)
\end{aligned}$$

where $t = 0, 1, 2, \dots$.

This justifies the description of the coefficient (2.5) weighting the strategic nodes' states for the calculation of the asymptotic opinions. Indeed, the probabilities of all the possible k-length walks towards strategic nodes can be described as random walk trajectories.

Hitting Time and Absorbing Probability

Another interesting notion is the one of *hitting time*. This value is a random variable that refers to the time spent by a Markov chain $X(t)$ to move from a state i to another state j and it is defined in the following way.

Let $X(t)$ be a Markov chain with finite discrete state space $\mathcal{X}$. Then, the *hitting time* on a node $i \in \mathcal{X}$ and the *hitting time* on a subset of states $\mathcal{S} \subseteq \mathcal{X}$ are the random variables defined by

$$T_j := \inf\{t \in \mathbb{N} : X(t) = j\} \ , \qquad T_{\mathcal{S}} := \inf\{t \in \mathbb{N} : X(t) \in \mathcal{S}\}$$

where the convention that the infimum of an empty set is equal to $+\infty$ is used. In practice, such hitting times T_j and $T_{\mathcal{S}}$ consist in the first times at which the markov chain $X(t)$ jumps to j or one of the states of $\mathcal{S}$, respectively.

On the other hand, this idea is strictly related to another notion: the one of *absorbing probability*. Indeed, when studying random walks on graphs – especially in presence of absorbing sets – a natural question that comes into mind is not only when such random walk will end up in a subset of nodes – if ever – but also where such subset is first hit. Indeed, for many applications, it is important to know which state of an absorbing set – also called *absorbing state* – is more probable to be hit first.

Formally, let $X(t)$ be a Markov chain with finite discrete state space $\mathcal{X}$, let $\mathcal{S} \subseteq \mathcal{X}$ be a subset of states, and let $i \in \mathcal{X}$ be the initial state of $X(t)$. Then, the *absorbing ptobability* in $s \in \mathcal{S}$ is defined by

$$\Gamma_{is} := \mathbb{P}(X(T_{\mathcal{S}}) = s|X(0) = i)$$

Which means that Γ_{is} represents the probability that the Markov chain $X(t)$, initiated at node i, hits the node $s \in \mathcal{S}$ before hitting any other node in $\mathcal{S} \setminus \{s\}$, i.e.

$$\Gamma_{is} = \mathbb{P}(T_{\mathcal{S}} = T_s|X(0) = i)$$

3.3 Electrical Analogy

In order to study analytically the targeting problem from the perspective of the + agent, it is convenient to use the electrical analogy, as I present below. Let us briefly recall the basic notions of such analogy.

First of all, let us consider a strongly connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{E}$ is the set of unordered couples $\{i, j\}$ rather than ordered ones as we previously used. Such graph can be seen as an electrical network $\mathcal{G}_{\mathcal{C}} = (\mathcal{V}, \mathcal{E}, C)$ where the weight matrix W is substituted by the conductance matrix $C \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$, where $C_{ij} = C_{ji}$ is now the conductance between the nodes i and j (notice how the reciprocity assumption must hold). Then, let us define

- $B \in \{0, +1, -1\}^{\mathcal{E}\times\mathcal{V}}$ the *incidence matrix*, such that

$$\begin{cases} B\mathbb{1} = 0 \\ B_{ei} \neq 0 \iff i \in e \ , \qquad e \in \mathcal{E} \end{cases}$$

- $D_C \in \mathbb{R}^{\mathcal{E}\times\mathcal{E}}$ diagonal matrix such that

$$(D_C)_{ee} = C_{ij} = C_{ji} \ , \qquad e = \{i,j\} \in \mathcal{E}$$

  leading to

$$B^\top D_C B = D_{C\mathbb{1}} - C$$

  where $D_{C\mathbb{1}} = diag(C\mathbb{1})$. Indeed $D_C B$ associates at each row of B the weight of the corresponding edge multiplying 1 or -1, while $B^\top D_C B$ generates the matrix that on each diagonal entry has the sum of all the conductances on such node, while on the ij-th entry it has the conductance value of edge $\{i,j\}$ of negative sign, if present.

- $\eta \in \mathbb{R}^{\mathcal{V}}$ the *input current vector* (positive if ingoing, negative if outgoing), such that $\eta^\top \mathbb{1} = 0$
- $V \in \mathbb{R}^{\mathcal{V}}$ the *voltage vector*
- $\Phi \in \mathbb{R}^{\mathcal{E}}$ the *current flow vector* (positive if going from i to j on (i,j))

In such way, the usual Kirchoff and Ohm's law can be expressed as

$$\begin{cases} B^\top \Phi = \eta \\ D_C B V = \Phi \end{cases}$$

leading to

$$\begin{aligned} B^\top(D_C B V) &= \eta \\ (D_{C\mathbb{1}} - C)V &= \eta \\ L(C)V &= \eta \end{aligned} \tag{3.2}$$

where $L(C) := D_{C\mathbb{1}} - C$ is the *Laplacian* of C. Since the graph is strongly connected, $L(C)$ has rank $|\mathcal{V}| - 1$ and $L(C)\mathbb{1} = 0$, making V, up to translations, the unique solution of the system. Also notice that $(L(C)V)_i = 0 \ \ \forall i \in \mathcal{V}$ such that $\eta_i = 0$. Consequently, the solution of Equation (3.2) in terms of voltages correspond to the one of Equation (2.6) in terms of asymptotic opinions, where the regular nodes are the ones with 0 input current, while the strategic ones are the nodes with input current different from 0. Indeed it is possible to interpret the asymptotic opinion of regular agents in a strongly connected undirected graph as the voltage in the corresponding regular node of the electrical network.

From now on, I will thus talk about this electrical equivalence where the agents are nodes in the electrical network and their asymptotic opinions are the associated voltages. In such equivalence, the strategic nodes $-$ and $+$ are considered voltage sources of value -1 and $+1$. Thus, the objective function of the optimal targeting problem (2.8) becomes

$$F_+(\mathcal{A}) = \frac{1}{N} \sum_{i \in \mathcal{R}} V^{(\mathcal{A})}(i)$$

where $V^{(\mathcal{A})}(i)$ is the voltage of node i when the set of nodes linked to $+$ is $\mathcal{A}$.

Remark. *Let us notice that in the electrical network it is possible to glue together nodes having the same voltage, i.e. substituting the graph with a simplified version. This allows for more interesting interpretations of this thesis work since, by considering two agents competing with each other, it is possible to equivalently represent the problem where two group of people of opposite opinion – represented as strategic agents – compete with each other, independently from the number of people involved.*

Chapter 4

Properties of the Linear Optimization Problem

In this chapter I will present two of the main properties of the objective function of the optimization problem (2.8): monotonicity and submodularity. Such properties will allow for some heuristics to find sub-optimal results, as I will show in Chapter 6.2 and Chapter **??**.

Notice how both proofs have been made with the same approach and following analogous steps, in order to be easier to understand. Nevertheless, different approaches could have been taken, in particular for monotonicity, for which I will put an alternative proof in Appendix A.1. For the proof of submodularity instead, a proof based on the work from Yildiz et al. [11] has also been made in Appendix A.2. Such proof it has not been used in order to develop a self-consistent work. However a part of the proof has been inspired by theirs, as it will be reminded hereafter. However, notice that the submodularity result is not an original contribution to the literature, since Yi et al. recently published in preprint a work with the same result [10].

4.1 Monotonicity of $F_+(\,\cdot\,)$

Proposition 4.1.1. *Let $F_+ : 2^{\mathcal{R}} \to \mathbb{R}$ be the objective function of the optimization problem (2.8), which is a set function. Let $\mathcal{A}, \mathcal{B}$ be node sets such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{R}$. Then,*

$$F_+(\mathcal{B}) \geq F_+(\mathcal{A})$$

i.e. F_+ is monotonic.

Proof. In Chapter 3.2 I showed that the asymptotic opinion of a regular agent following the DeGroot dynamics in (2.3) – representable as a convex hull of the strategic agents' opinion – can be formulated in a probabilistic framework. This means that – asymptotically – a regular agent i weights the opinion x_s of each strategic node s with a coefficient proportional to the probability that the random walk $X(t)$ initiated at node i hits the absorbing state s, with transition probability matrix Q. More formally,

$$\begin{aligned}
\bar{x}_i &= \sum_{s\in\mathcal{S}} (HQ^{12})_{is} x_s \\
&= \sum_{s\in\mathcal{S}} \left(\sum_{k\geq 0} \sum_{i_0=i,i_1\ldots i_k-1\in\mathcal{R}, i_k=s} \prod_{1\leq h\leq k} Q_{i_{h-1},i_h} \right)_{is} x_s \\
&= \sum_{s\in\mathcal{S}} \left(\lim_{t\to+\infty} \mathbb{P}(X(t)=s|X(0)=i) \right) x_s \\
&= \lim_{t\to+\infty} \mathbb{P}(X(t)=+|X(0)=i) - \lim_{t\to+\infty} \mathbb{P}(X(t)=-|X(0)=i) \\
&= p_i - (1-p_i) \\
&\quad \text{where } p_i := \lim_{t\to+\infty} \mathbb{P}(X(t)=+|X(0)=i) \\
&= 2p_i - 1
\end{aligned} \tag{4.1}$$

where I denoted by p_i the probability that the Markov chain $X(t)$ initiated in i hits the strategic node $+$ before hitting $-$.

Now, let us remember from Chapter 3.3 that – when talking of asymptotic opinions – it is possible to reason in terms of the equivalent electrical network. In such framework, the two strategic nodes are considered as voltage sources of opposite voltage and we know that two nodes of same voltage can be glued together. In the same way, a voltage source can also be split into several identical voltage sources without affecting the circuit. This is what I am going to do with the strategic node $+$ in order to complete the proof. Indeed, from the graph $\mathcal{G}$ with set $\mathcal{A}$ of nodes linked to $+$, I will consider the equivalent graph where each node p in $\mathcal{A}$ is linked to a different strategic node $+^{(p)}$ generated from the splitting of $+$, so as to have each strategic node $+^{(p)}$ of in-degree 1.

More formally, for each case $\mathcal{U} = \mathcal{A}, \mathcal{B}$ of node sets where $\mathcal{A} \subseteq \mathcal{B}$ – which are the set of nodes linked to the strategic $+$ – let us build a corresponding graph $\mathcal{G}(\mathcal{U}) = (\mathcal{V}(\mathcal{U}), \mathcal{E}(\mathcal{U}), W(U))$ where the strategic node $+$ is no more unique in general, but there are as many type-one strategic nodes $+^{(p)}$ as the cardinality of $\mathcal{U}$, each one of in-degree one. For clarity, let us rearrange the nodes of the graph putting the nodes of $\mathcal{A}$ first, followed by the remaining $|\mathcal{B} \setminus \mathcal{A}|$ nodes of $\mathcal{B}$ and, then, the remaining ones. Thus, by writing the node sets partitions of regular and strategic nodes as functions of the node set considered we can summarize the two cases in

$$\begin{cases} \mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{B}) = \mathcal{R} \\ \mathcal{S}(\mathcal{A}) = \{+^{(1)}, \dots, +^{(|\mathcal{A}|)}\} \cup \{-\} \\ \mathcal{S}(\mathcal{B}) = \mathcal{S}(\mathcal{A}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} \cup \{-\} \end{cases}$$

where $\mathcal{R}$ is the set of regular nodes of $\mathcal{G}$, while $\mathcal{R}(\mathcal{U})$ and $\mathcal{S}(\mathcal{U})$ are the sets of regular and strategic nodes of $\mathcal{G}(\mathcal{U})$ respectively such that $\mathcal{V}(\mathcal{U}) = \mathcal{R}(\mathcal{U}) \cup \mathcal{S}(\mathcal{U})$, where $\mathcal{U} = \mathcal{A}, \mathcal{B}$.

Let us now build two different graphs from the two cases considered. Such graphs $\tilde{\mathcal{G}}(\mathcal{U}) = (\tilde{\mathcal{V}}(\mathcal{U}), \tilde{\mathcal{E}}(\mathcal{U}), \tilde{W}(\mathcal{U}))$ – where $\mathcal{U} = \mathcal{A}, \mathcal{B}$ – will be built by taking the biggest graph, in this case $\mathcal{G}(\mathcal{B})$, and transforming the strategic nodes linked to the set $\mathcal{B} \setminus \mathcal{U}$ in regular ones, thus maintaining the same number of nodes of $\mathcal{G}(\mathcal{B})$. Summarizing, the two new graphs satisfy

$$\tilde{\mathcal{G}}(\mathcal{B}) : \begin{cases} \tilde{\mathcal{R}}(\mathcal{B}) = \mathcal{R}(\mathcal{B}) \\ \tilde{\mathcal{S}}(\mathcal{B}) = \mathcal{S}(\mathcal{B}) \end{cases}$$
$$\tilde{\mathcal{G}}(\mathcal{A}) : \begin{cases} \tilde{\mathcal{R}}(\mathcal{A}) = \mathcal{R}(\mathcal{B}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} \\ \tilde{\mathcal{S}}(\mathcal{A}) = \mathcal{R}(\mathcal{B}) \setminus \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} = \mathcal{S}(\mathcal{A}) \end{cases}$$

In such way, the number of nodes always stays the same while the discriminant between the two different cases becomes the set of type-one strategic nodes, unambiguously determined by $\mathcal{U} = \mathcal{A}, \mathcal{B}$ – set of nodes that are linked to the strategic node $+$ in the original graph.

In order to prove the monotonicity of the objective function, which is the sum of all the asymptotic opinions, let us remember that each asymptotic state $\bar{x}_i$ is proportional to the probability p_i that the associated random walk started in i ends up to the strategic node $+$. This holds true also for our equivalent graphs $\mathcal{G}(\mathcal{U})$ – which show the same Markov chain and probability value, after some modifications – and for the new graphs $\tilde{\mathcal{G}}(\mathcal{U})$ – even if with their differences – where $\mathcal{U} = \mathcal{A}, \mathcal{B}$.

Consequently, let us denote by $X^{\mathcal{U}}(t)$ the random walk with transition probability matrix $Q(\mathcal{U})$ – normalized version of $W(\mathcal{U})$ –, by $p_i^{\mathcal{U}}$ the probability that such random walk – started in i – is absorbed by one of the strategic nodes $+^{(p)}$, $p \in \mathcal{U}$, and by $+^{(\mathcal{U})} := \{+^{(p)}, p \in \mathcal{U}\}$ the set of strategic nodes linked to the node set $\mathcal{U}$, $\mathcal{U} = \mathcal{A}, \mathcal{B}$. Thus,

$$p_i^{\mathcal{U}} := \lim_{t \to +\infty} \mathbb{P}(X^{\mathcal{U}}(t) \in +^{(\mathcal{U})} | X^{\mathcal{U}}(0) = i) , \qquad i \in \mathcal{R},\, \mathcal{U} = \mathcal{A}, \mathcal{B}$$

On the other hand, let us define by $\tilde{X}^{\mathcal{U}}(t)$ the random walk with transition probability matrix $\tilde{Q}(\mathcal{U})$ – normalized version of $\tilde{W}(\mathcal{U})$ – and by $\tilde{p}_i^{\mathcal{U}}$ the probability that such random walk started in i hits one of the type-one strategic nodes $+^{(p)} \in +^{(\mathcal{U})}$ where $p \in \mathcal{U}$, $\mathcal{U} = \mathcal{A}, \mathcal{B}$, i.e.

$$\tilde{p}_i^{\mathcal{U}} := \lim_{t \to +\infty} \mathbb{P}(\tilde{X}^{\mathcal{U}}(t) \in +^{(\mathcal{U})} | \tilde{X}^{\mathcal{U}}(0) = i) , \qquad i \in \mathcal{R}, \mathcal{U} = \mathcal{A}, \mathcal{B}$$

which can be rewritten in terms of hitting times

$$\tilde{p}_i^{\mathcal{U}} = \mathbb{P}(T_{+^{(\mathcal{U})}} < T_{-} | \tilde{X}^{\mathcal{U}}(0) = i)$$

i.e. as the probability of hitting one of the type-one strategic nodes before hitting $-$.

At the same time, it is possible to evidence the equality between the value of $p_i^{\mathcal{U}}$ and $\tilde{p}_i^{\mathcal{U}}$ for the regular nodes of $\mathcal{G}(\mathcal{U})$. Indeed, while for $\mathcal{U} = \mathcal{B}$ we have $\tilde{\mathcal{G}}(\mathcal{B}) = \mathcal{G}(\mathcal{B})$, for $\mathcal{U} = \mathcal{A}$ the difference between $\tilde{\mathcal{G}}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A})$ is simply the addition of regular nodes of degree one, in place of the type-one strategic nodes of $\mathcal{G}(\mathcal{B})$ that are not linked to the nodes of $\mathcal{A}$ but only to the ones of $\mathcal{B} \setminus \mathcal{A}$. This means that such additional regular nodes – that originate by passing from $\mathcal{G}(\mathcal{U})$ to $\tilde{\mathcal{G}}(\mathcal{U})$ – are all of degree one, and, thus, are not affecting the asymptotic opinion of the other nodes. Indeed, by considering the equivalent electrical network, the nodes of degree one correspond to short-circuited nodes and do not affect the voltage of the overall network. Consequently we can state that

$$\tilde{\bar{x}}_i^{\mathcal{U}} = 2\tilde{p}_i^{\mathcal{U}} - 1 = 2p_i^{\mathcal{U}} - 1 = \bar{x}_i^{\mathcal{U}} , \qquad i \in \mathcal{R}$$

where I defined by $\tilde{\bar{x}}_i^{\mathcal{U}}$ the asymptotic opinion of agent i when the set of nodes linked to the strategic $+$ is $\mathcal{U}$, that is proportional to $\tilde{p}_i^{\mathcal{U}}$ in an analogous way to what has been found for $\bar{x}_i$ – which here specifies the set of nodes linked to + by the superscript $\mathcal{U}$. This leads to

$$\tilde{p}_i^{\mathcal{U}} = p_i^{\mathcal{U}}\,, \qquad i \in \mathcal{R}$$

Then, in order to prove the result, let us write

$$\begin{aligned}
\tilde{p}_i^{\mathcal{B}} &= \mathbb{P}(T_{+(\mathcal{B})} < T_- | \tilde{X}^{\mathcal{B}}(0) = i) \\
&\quad \text{by the partitioning theorem [32]} \\
&= \mathbb{P}(T_{+(\mathcal{B})} < T_- \cap T_{+(\mathcal{B})} < T_{+(\mathcal{A})} | \tilde{X}^{\mathcal{B}}(0) = i) + \\
&\quad + \mathbb{P}(T_{+(\mathcal{B})} < T_- \cap T_{+(\mathcal{B})} = T_{+(\mathcal{A})} | \tilde{X}^{\mathcal{B}}(0) = i) + \\
&\quad + \mathbb{P}(T_{+(\mathcal{B})} < T_- \cap T_{+(\mathcal{B})} > T_{+(\mathcal{A})} | \tilde{X}^{\mathcal{B}}(0) = i) \\
&\quad \text{where only the probabilities of hitting the strategic nodes linked to } \mathcal{B} \setminus \mathcal{A} \\
&\quad \text{or the ones linked to } \mathcal{A} \text{ survive} \\
&= \mathbb{P}(T_{+(\mathcal{B}\setminus\mathcal{A})} < T_- | \tilde{X}^{\mathcal{B}}(0) = i) + \mathbb{P}(T_{+(\mathcal{A})} < T_- | \tilde{X}^{\mathcal{B}}(0) = i) \\
&\quad \text{where the second term is equal to the probaility of hitting the strategic node} \\
&\quad \text{nodes linked to } \mathcal{A} \text{ with the Markov chain } \tilde{X}^{\mathcal{A}}(t) \\
&= \mathbb{P}(T_{+(\mathcal{B}\setminus\mathcal{A})} < T_- | \tilde{X}^{\mathcal{B}}(0) = i) + \mathbb{P}(T_{+(\mathcal{A})} < T_- | \tilde{X}^{\mathcal{A}}(0) = i) \\
&\geq \mathbb{P}(T_{+(\mathcal{A})} < T_- | \tilde{X}^{\mathcal{A}}(0) = i) \\
&= \tilde{p}_i^{\mathcal{A}}
\end{aligned}$$

proving that

$$\begin{aligned}
\tilde{p}_i^{\mathcal{B}} &\geq \tilde{p}_i^{\mathcal{A}} \\
p_i^{\mathcal{B}} &\geq p_i^{\mathcal{A}} \\
2p_i^{\mathcal{B}} - 1 &\geq 2p_i^{\mathcal{A}} - 1 \\
\bar{x}_i^{\mathcal{B}} &\geq \bar{x}_i^{\mathcal{A}}
\end{aligned}$$

which, by summing over $i \in \mathcal{R}$, leads to

$$\begin{aligned}
\frac{1}{N}\sum_{i\in\mathcal{R}} \bar{x}_i^{\mathcal{B}} &\geq \frac{1}{N}\sum_{i\in\mathcal{R}} \bar{x}_i^{\mathcal{A}} \\
F_+(\mathcal{B}) &\geq F_+(\mathcal{A})
\end{aligned}$$

□

4.2 Submodularity of $F_+(\,\cdot\,)$

Proposition 4.2.1. *Let $F_+ : 2^{\mathcal{R}} \to \mathbb{R}$ be the objective function of the optimization problem (2.8), which is a set function. Let $\mathcal{A}, \mathcal{B}$ be node sets such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{R}$ and let $v \in \mathcal{R} \setminus \mathcal{B}$ be a node. Then,*

$$F_+(\hat{\mathcal{A}}) - F_+(\mathcal{A}) \geq F_+(\hat{\mathcal{B}}) - F_+(\mathcal{B})$$

where $\hat{\mathcal{U}} = \mathcal{U} \cup \{v\}$, $\mathcal{U} = \mathcal{A}, \mathcal{B}$, i.e. F_+ is submodular.

Proof. To prove the proposition avoiding redundancies we relate as much as possible to the previous proof, by underlying the necessary differences when following analogous steps. However, for completeness, I will not omit any step, even if the construction of the proof is almost completely analogous to the previous one.

First of all, let us consider the generic sets $\mathcal{A}, \mathcal{B}$ and a node v such that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{R}$ and $v \in \mathcal{R} \setminus \mathcal{B}$. From these, let us build the four cases that I am going to compare: $\mathcal{A}, \hat{\mathcal{A}} = \mathcal{A} \cup \{v\}, \mathcal{B}, \hat{\mathcal{B}} = \mathcal{B} \cup \{v\}$. These correspond to the sets of nodes that are linked to the $+$ in each different case. Since their choice is arbitrary, proving that $F_+(\hat{\mathcal{A}}) - F_+(\mathcal{A}) \geq F_+(\hat{\mathcal{B}}) - F_+(\mathcal{B})$ will end the proof.

In (4.1) I showed that the asymptotic opinion $\bar{x}_i$ of each regular agent is proportional to the probability p_i that a random walk $X(t)$ with transition matrix Q is absorbed by the strategic node $+$, i.e.

$$\bar{x}_i = 2p_i - 1$$

In the same way as in the previous proof, in Chapter 3.3 I showed that, instead of asymptotic opinions, it is possible to reason in terms of voltages in the equivalent electrical network. In such way, it is possible to split the strategic node $+$ – described as a voltage source – into several identical sources, without affecting the network.

Then, let us build for each of the four cases $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$ a corresponding equivalent graph $\mathcal{G}(\mathcal{U}) = (\mathcal{V}(\mathcal{U}), \mathcal{E}(\mathcal{U}), W(\mathcal{U}))$ where the strategic node $+$ is no more unique in general, but there are as many type-one strategic nodes $+^{(p)}$ as the cardinality of $\mathcal{U}$, each one with degree of exactly one. Let us then rearrange the nodes in the following order: $\mathcal{A}$, $\mathcal{B} \setminus \mathcal{A}$, v, $\mathcal{R} \setminus \hat{\mathcal{B}}$, $\mathcal{S}$. Thus, by writing the node sets partitions as function of the case considered, we can summarize the four cases in

$$\begin{cases} \mathcal{R}(\hat{\mathcal{B}}) = \mathcal{R}(\mathcal{B}) = \mathcal{R}(\hat{\mathcal{A}}) = \mathcal{R}(\mathcal{A}) = \mathcal{R} \\ \mathcal{S}(\hat{\mathcal{B}}) = \mathcal{S}(\mathcal{B}) \cup \{+^{(|\mathcal{B}|+1)}\} = \mathcal{S}(\hat{\mathcal{A}}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} \\ \qquad = \mathcal{S}(\mathcal{A}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|+1)}\} = \{+^{(1)}, \dots, +^{(|\mathcal{B}|+1)}\} \end{cases}$$

where $\mathcal{R}$ is the set of regular nodes of $\mathcal{G}$, while $\mathcal{R}(\mathcal{U})$ and $\mathcal{S}(\mathcal{U})$ are the sets of regular and strategic nodes of $\mathcal{G}(\mathcal{U})$ respectively such that $\mathcal{V}(\mathcal{U}) = \mathcal{R}(\mathcal{U}) \cup \mathcal{S}(\mathcal{U})$, where $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$.

Let us now build four different graphs from the cases considered. Such graphs $\tilde{\mathcal{G}}(\mathcal{U}) = (\tilde{\mathcal{V}}(\mathcal{U}), \tilde{\mathcal{E}}(\mathcal{U}), \tilde{W}(\mathcal{U}))$ – where $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$ – will be built by taking the biggest graph, in this case $\mathcal{G}(\hat{\mathcal{B}})$, and transforming the strategic nodes linked to the set $\hat{\mathcal{B}} \setminus \mathcal{U}$ in regular ones, thus maintaining the same number of nodes of $\mathcal{G}(\hat{\mathcal{B}})$. Summarizing, the new graphs satisfy

$$\tilde{\mathcal{G}}(\hat{\mathcal{B}}) : \begin{cases} \tilde{\mathcal{R}}(\hat{\mathcal{B}}) = \mathcal{R}(\hat{\mathcal{B}}) \\ \tilde{\mathcal{S}}(\hat{\mathcal{B}}) = \mathcal{S}(\hat{\mathcal{B}}) \end{cases}$$

$$\tilde{\mathcal{G}}(\mathcal{B}) : \begin{cases} \tilde{\mathcal{R}}(\mathcal{B}) = \mathcal{R}(\mathcal{B}) \cup \{+^{(|\mathcal{B}|+1)}\} \\ \tilde{\mathcal{S}}(\mathcal{B}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus \{+^{(|\mathcal{B}|+1)}\} = \mathcal{S}(\mathcal{B}) \end{cases}$$

$$\tilde{\mathcal{G}}(\hat{\mathcal{A}}) : \begin{cases} \tilde{\mathcal{R}}(\hat{\mathcal{A}}) = \mathcal{R}(\mathcal{B}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} \\ \tilde{\mathcal{S}}(\hat{\mathcal{A}}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|)}\} = \mathcal{S}(\hat{\mathcal{A}}) \end{cases}$$

$$\tilde{\mathcal{G}}(\mathcal{A}) : \begin{cases} \tilde{\mathcal{R}}(\mathcal{A}) = \mathcal{R}(\mathcal{B}) \cup \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|+1)}\} \\ \tilde{\mathcal{S}}(\mathcal{A}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus \{+^{(|\mathcal{A}|+1)}, \dots, +^{(|\mathcal{B}|+1)}\} = \mathcal{S}(\mathcal{A}) \end{cases}$$

In such way, the number of nodes always stays the same while the discriminant between the two different cases becomes the set of type-one strategic nodes, unambiguously determined by $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$ – set of nodes that are linked to the strategic node $+$ in the original graph.

Now, it is possible to notice how the asymptotic opinion $\bar{x}_i$ can be written in terms of p_i also for the equivalent graphs $\mathcal{G}(\mathcal{U})$ – which show the same Markov chain and probability value, after some modifications – and for the new graphs $\tilde{\mathcal{G}}(\mathcal{U})$ where $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$.

Consequently, let us denote by $X^{\mathcal{U}}(t)$ the random walk with transition probability matrix $Q(\mathcal{U})$ – normalized version of $W(\mathcal{U})$ – , by $p_i^{\mathcal{U}}$ the probability that such random walk – started in i – is absorbed by one of the strategic nodes $+^{(p)}$, $p \in \mathcal{U}$, and by $+^{(\mathcal{U})} := \{+^{(p)}, p \in \mathcal{U}\}$ the set of strategic nodes linked to the node set $\mathcal{U}$, $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$. Thus,

$$p_i^{\mathcal{U}} := \lim_{t\to+\infty} \mathbb{P}(X^{\mathcal{U}}(t) \in +^{(\mathcal{U})} | X^{\mathcal{U}}(0) = i) \ , \qquad i \in \mathcal{R}, \mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$$

On the other hand, let us define by $\tilde{X}^{\mathcal{U}}(t)$ the random walk with transition probability matrix $\tilde{Q}(\mathcal{U})$ – normalized version of $\tilde{W}(\mathcal{U})$ – and by $\tilde{p}_i^{\mathcal{U}}$ the probability that such random walk started in i hits one of the type-one strategic nodes $+^{(p)}$ where $p \in \mathcal{U}$, $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$, i.e.

$$\tilde{p}_i^{\mathcal{U}} := \lim_{t\to+\infty} \mathbb{P}(\tilde{X}^{\mathcal{U}}(t) \in +^{(\mathcal{U})} | \tilde{X}^{\mathcal{U}}(0) = i) \ , \qquad i \in \mathcal{R}, \mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$$

which can be rewritten in terms of hitting times

$$\tilde{p}_i^{\mathcal{U}} = \mathbb{P}(T_{+^{(\mathcal{U})}} < T_- | \tilde{X}^{\mathcal{U}}(0) = i)$$

i.e. as the probability of hitting one of the type-one strategic nodes before hitting $-$.

At the same time, as it has been done in the previous proof, it is possible to show that $p_i^{\mathcal{U}} = \tilde{p}_i^{\mathcal{U}}$ for each regular node of $\mathcal{G}(\mathcal{U})$. Indeed, the difference between $\tilde{\mathcal{G}}(\mathcal{U})$ and $\mathcal{G}(\mathcal{U})$ is the addition of regular nodes of degree one – corresponding to the type-one strategic nodes of $\mathcal{G}(\mathcal{B})$ that are linked to the nodes of $\mathcal{B} \setminus \mathcal{U}$. Nevertheless, since such additional nodes have degree one, by building the analogous electrical network, they are all short-circuited with their only linked node, thus thet do not influence the other nodes' voltage. Hence,

$$\tilde{\bar{x}}_i^{\mathcal{U}} = 2\tilde{p}_i^{\mathcal{U}} - 1 = 2p_i^{\mathcal{U}} - 1 = \bar{x}_i^{\mathcal{U}} \ , \qquad i \in \mathcal{R},\ \mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$$

where $\tilde{\bar{x}}_i^{\mathcal{U}}$ is the asymptotic opinion of agent i when the set of nodes linked to $+$ is $\mathcal{U}$ and $\bar{x}_i$ specifies the set of nodes linked to $+$ by the superscript $\mathcal{U}$. Therefore,

$$\tilde{p}_i^{\mathcal{U}} = p_i^{\mathcal{U}} \ , \qquad i \in \mathcal{R},\ \mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$$

Now, let us prove that $\tilde{p}_i^{\mathcal{U}}$ – in terms of $\mathcal{U}$ – is submodular for each $i \in$

$\tilde{\mathcal{V}} \setminus \{-\}$, as it has been done by Yildiz et al. [11]. In order to do this, I need to prove the equivalent definition of submodularity, i.e.

$$\tilde{p}_i^{\mathcal{U}} \textit{ submodular } \iff \tilde{p}_i^{\mathcal{U}_1} + \tilde{p}_i^{\mathcal{U}_2} \geq \tilde{p}_i^{\mathcal{U}_1 \cup \mathcal{U}_2} + \tilde{p}_i^{\mathcal{U}_1 \cap \mathcal{U}_2}$$

for each nonempty set $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{R}$.

Since the Markov chain $\tilde{X}^{\mathcal{U}}(t)$ has a different absorbing set depending on $\mathcal{U}$, let us base our calculation on the random walk made by the Markov chain $\tilde{X}^{\emptyset}(t)$, i.e. with transition probability matrix $\tilde{Q}(\emptyset)$, whose only absorbing state is the strategic node $-$, independently from $\mathcal{U}$. In addition, let us denote by $\mathbb{P}_i(T_v < T_u)$ the probability that such random walk iniated at node i hits node v before hitting u. Thus, by using again the partitioning theorem [32], we can state that

$$\begin{aligned}\mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_-) = {} & \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} < T_{+^{(\mathcal{U}_2)}}) + \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} = T_{+^{(\mathcal{U}_2)}}) + \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} > T_{+^{(\mathcal{U}_2)}})\end{aligned}$$

$$\begin{aligned}\mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_-) = {} & \mathbb{P}_i(T_{+^{(\mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} < T_{+^{(\mathcal{U}_2)}}) + \mathbb{P}_i(T_{+^{(\mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} = T_{+^{(\mathcal{U}_2)}}) + \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} > T_{+^{(\mathcal{U}_2)}})\end{aligned}$$

$$\begin{aligned}\mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cup \mathcal{U}_2)}} < T_-) = {} & \mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cup \mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} < T_{+^{(\mathcal{U}_2)}}) + \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cup \mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} > T_{+^{(\mathcal{U}_2)}}) + \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cup \mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} = T_{+^{(\mathcal{U}_2)}}) \\ = {} & \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} < T_{+^{(\mathcal{U}_2)}}) + \mathbb{P}_i(T_{+^{(\mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} > T_{+^{(\mathcal{U}_2)}}) + \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cup \mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_{+^{(\mathcal{U}_1 \setminus \mathcal{U}_2)}} \cap T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_{+^{(\mathcal{U}_2 \setminus \mathcal{U}_1)}}) \\ = {} & \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} < T_{+^{(\mathcal{U}_2)}}) + \mathbb{P}_i(T_{+^{(\mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} > T_{+^{(\mathcal{U}_2)}}) + \quad (4.2) \\ & + \mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_- \cap T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_{+^{(\mathcal{U}_1 \setminus \mathcal{U}_2)}} \cap T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_{+^{(\mathcal{U}_2 \setminus \mathcal{U}_1)}}) \quad (4.3)\end{aligned}$$

where the two terms (4.2) are the probabilities of hitting $+^{(\mathcal{U}_1)}$ before hitting $(\{-\} \cup +^{(\mathcal{U}_2)})$ and of hitting $+^{(\mathcal{U}_2)}$ before hitting $(\{-\} \cup +^{(\mathcal{U}_1)})$, respectively, while (4.3) is the probability of hitting $+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}$ before hitting $(\{-\} \cup +^{(\mathcal{U}_1 \setminus \mathcal{U}_2)} \cup +^{(\mathcal{U}_2 \setminus \mathcal{U}_1)})$. On the other hand, since $T_{+^{(\mathcal{U}_1)}} = T_{+^{(\mathcal{U}_2)}}$ is the set of events where the hitting times to $+^{(\mathcal{U}_1)}$ and $+^{(\mathcal{U}_2)}$ are equal, it is possible to write

$$\mathbb{P}_i(T_{+^{(\mathcal{U}_1 \cap \mathcal{U}_2)}} < T_-) \leq \mathbb{P}_i(T_{+^{(\mathcal{U}_1)}} < T_- \cap T_{+^{(\mathcal{U}_1)}} = T_{+^{(\mathcal{U}_2)}})$$

Then, it follows that

$$\begin{aligned}
\tilde{p}_i^{\mathcal{U}_1} + \tilde{p}_i^{\mathcal{U}_2} &= \mathbb{P}_i(T_{+(\mathcal{U}_1)} < T_-) + \mathbb{P}_i(T_{+(\mathcal{U}_2)} < T_-) \\
&= \mathbb{P}_i(T_{+(\mathcal{U}_1)} < T_- \cap T_{+(\mathcal{U}_1)} < T_{+(\mathcal{U}_2)}) + \mathbb{P}_i(T_{+(\mathcal{U}_1)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) + \\
&\quad + \mathbb{P}_i(T_{+(\mathcal{U}_1)} < T_- \cap T_{+(\mathcal{U}_1)} > T_{+(\mathcal{U}_2)}) + \mathbb{P}_i(T_{+(\mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} < T_{+(\mathcal{U}_2)}) + \\
&\quad + \mathbb{P}_i(T_{+(\mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) + \mathbb{P}_i(T_{+(\mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} > T_{+(\mathcal{U}_2)}) \\
&= \mathbb{P}_i(T_{+(\mathcal{U}_1 \cup \mathcal{U}_2)} < T_-) + \\
&\quad - \mathbb{P}_i(T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_{+(\mathcal{U}_1 \setminus \mathcal{U}_2)} \cap T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_{+(\mathcal{U}_2 \setminus \mathcal{U}_1)}) + \\
&\quad + \mathbb{P}_i(T_{+(\mathcal{U}_1)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) + \mathbb{P}_i(T_{+(\mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) \\
&= \mathbb{P}_i(T_{+(\mathcal{U}_1 \cup \mathcal{U}_2)} < T_-) - \mathbb{P}_i(T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) \\
&\quad + \mathbb{P}_i(T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) + \mathbb{P}_i(T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_- \cap T_{+(\mathcal{U}_1)} = T_{+(\mathcal{U}_2)}) \\
&\geq \mathbb{P}_i(T_{+(\mathcal{U}_1 \cup \mathcal{U}_2)} < T_-) + \mathbb{P}_i(T_{+(\mathcal{U}_1 \cap \mathcal{U}_2)} < T_-) \\
&= \tilde{p}_i^{\mathcal{U}_1 \cup \mathcal{U}_2} + \tilde{p}_i^{\mathcal{U}_1 \cap \mathcal{U}_2}
\end{aligned}$$

which proves the submodularity of $\tilde{p}_i^{\mathcal{U}}$, for $i \in \tilde{V} \setminus \{-\}$.

On the other hand, having $\tilde{p}_i^{\mathcal{U}}$ submodular, it means that it is possible to use the classical definition of submodularity for the particular sets $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$ considered, i.e.

$$\tilde{p}_i^{\hat{\mathcal{A}}} - \tilde{p}_i^{\mathcal{A}} \geq \tilde{p}_i^{\hat{\mathcal{B}}} - \tilde{p}_i^{\mathcal{B}} , \qquad i \in \tilde{V} \setminus \{-\}$$

that since $\tilde{p}_i^{\mathcal{U}} = p_i^{\mathcal{U}}$ for each $i \in \mathcal{R}$, $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$, it leads to

$$p_i^{\hat{\mathcal{A}}} - p_i^{\mathcal{A}} \geq p_i^{\hat{\mathcal{B}}} - p_i^{\mathcal{B}} , \qquad i \in \mathcal{R}$$

which, by summing over $i \in \mathcal{R}$, it proves the submodularity of F_+

$$\sum_{i \in \mathcal{R}} p_i^{\hat{\mathcal{A}}} - \sum_{i \in \mathcal{R}} p_i^{\mathcal{A}} \geq \sum_{i \in \mathcal{R}} p_i^{\hat{\mathcal{B}}} - \sum_{i \in \mathcal{R}} p_i^{\mathcal{B}}$$

$$\frac{1}{N}\left(2\sum_{i \in \mathcal{R}} p_i^{\hat{\mathcal{A}}} - 1\right) - \frac{1}{N}\left(2\sum_{i \in \mathcal{R}} p_i^{\mathcal{A}} - 1\right) \geq \frac{1}{N}\left(2\sum_{i \in \mathcal{R}} p_i^{\hat{\mathcal{B}}} - 1\right) - \frac{1}{N}\left(2\sum_{i \in \mathcal{R}} p_i^{\mathcal{B}} - 1\right)$$

$$F_+(\hat{\mathcal{A}}) - F_+(\mathcal{A}) \geq F_+(\hat{\mathcal{B}}) - F_+(\mathcal{B})$$

□

Chapter 5

Optimal Targeting Analytical Results

In this chapter I will show the main results of this thesis work: the analytical solutions of the OTP and STP. Such results are calculated for specific meaningful graphs, also accounting for the number of nodes that each strategic agent can target.

In this regard, I will subdivide the work into two categories: single targeting on sparse graphs and multiple targeting on dense graphs, where I remember I talk of single targeting when $k_+, k_- = 1$, and multiple targeting when $k_+, k_- \geq 1$. In particular, in this chapter I will show the analytical results that I found for special graphs of both categories: line graph and tree graph for sparse graphs, and complete graph for dense graph. Then, from the results on these graphs I will build the main heuristics of Chapter 6, which can be applied to more general sparse and dense graphs, such as Tree-like Graphs and Erdos-Renyi Graphs, respectively.

5.1 Line Graph Single Targeting

One of the simplest settings from which to start when studying the optimal targeting problem (2.8) is an STP over a line graph – i.e. where the links available to agents $+$ and $-$ are $k_+ = 1$ and $k_- = 1$ respectively, while the graph made up by regular nodes is a Line Graph. Moreover, based on the position of node $-$, different complications arise. In this regard, let us denote by v^+ and v^- the nodes of the regular graph that are linked to the strategic nodes $+$ and $-$ respectively. In such way, the single targeting optimization problem becomes a problem in terms of such nodes, where v^- is given and F_+ depends on node v^+, as outlined in (2.9). Then, let us study the problem from the simplest case to the most general one.

$v^- = \ell$)

Let us see the case where the strategic node $-$ linked to a generic node ℓ, and let us split the problem in the two symmetric cases: $k \geq \ell$ and $k < \ell$.

- $k \geq \ell$)

  Starting with the former, the graph representation becomes

Figure 5.1: Line Graph with $v^- = \ell$, $v^+ = k$, $k \geq l$

Notice that the directedeness of the edges among strategic and regular nodes is draw to emphasize the strategic nature of $-$ and $+$. Then, by means of the electrical analogy, we can consider the strategic nodes $-$ and $+$ as voltage sources of value -1 and $+1$ respectively, while the nodes in the left and right tails of the line graph will be short-circuited with the nodes ℓ and k, respectively, i.e.

$$\begin{cases} V(-) = -1 \\ V(+) = +1 \\ V(1) = V(2) = \cdots = V(\ell) \\ V(k) = V(k+1) = \cdots = V(N) \end{cases}$$

represented by the analogous circuit

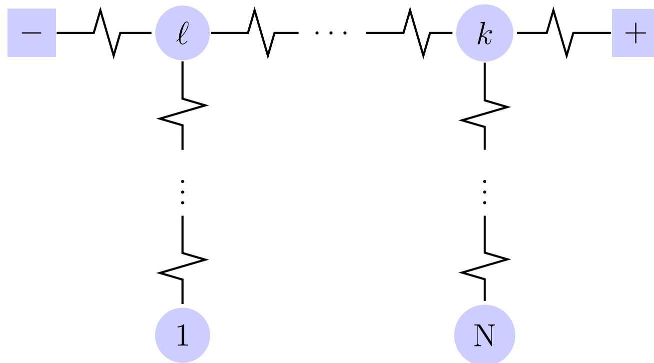
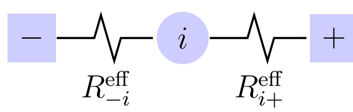


Figure 5.2: Circuit analogous line graph with $v^- = \ell$, $v^+ = k$, $k \geq l$

Since G is unweighted, i.e. $C_{ij} = 1$ for each $i, j \in \mathcal{V}$, it is possible to compute the voltage in each node $i = \ell, \ell+1 \ldots, k$ as the voltage drop in the voltage divider (as represented below) where the effective resistances are the summation of the resistances on the left and on the right of node i:



where

$$\begin{cases} R^{\text{eff}}_{-i} = i - \ell + 1 \\ R^{\text{eff}}_{i+} = k - i + 1 \end{cases}$$

which leads to

$$V(i) - V(-) = (V(+) - V(-))\frac{i - \ell + 1}{i - \ell + 1 + k - i + 1}$$
$$V(i) = 2\frac{i - \ell + 1}{k - \ell + 2} - 1$$

for each $i = \ell, \ell+1 \ldots k$, that is monotonically increasing with i, maximal in k and for $k = 1$ it implies $V(k) = V(1) = 0 \implies V(i) = 0 \ \ \forall i = 1 \ldots k$.

So, with a slight abuse of notation, it is possible to write the objective function as

$$
\begin{aligned}
F_+(k) &= (\ell-1)V(\ell) + \sum_{i=\ell}^{k} V(i) + (N-k)V(k) \\
&= \ell V(\ell) + \sum_{i=\ell+1}^{k} V(\ell) + (N-k)V(k) \\
&= \ell\left(2\frac{\ell-\ell+1}{k-\ell+2} - 1\right) + \left(\sum_{i=1}^{k} V(i) - \sum_{i=1}^{\ell} V(i)\right) + \\
&\quad + (N-k)\left(2\frac{k-\ell+1}{k-\ell+2} - 1\right) \\
&= \frac{1}{k-\ell+2}\left[-k^2 + (N+1)k - (N+1)\ell + \ell^2\right]
\end{aligned}
$$

which can be drawn for different values of ℓ as below

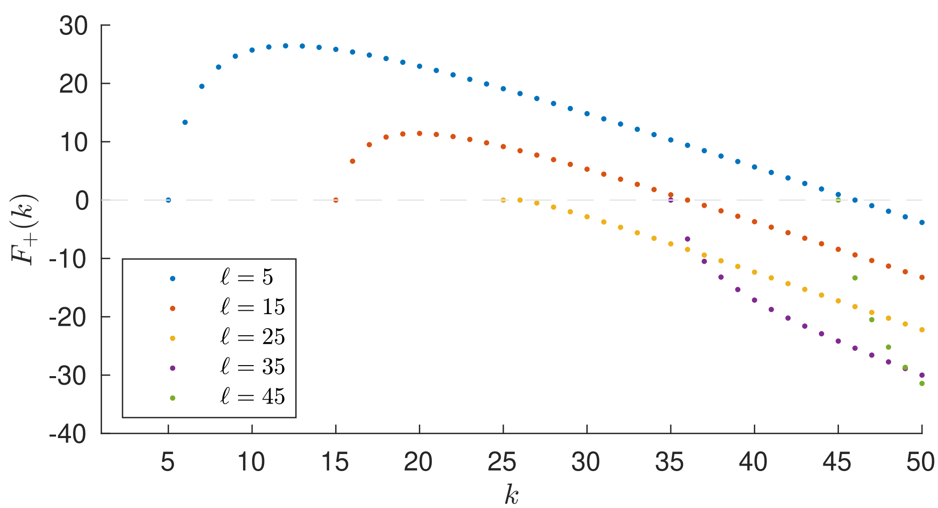


Figure 5.3: $F_+(k)$ of line graph with $v^- = \ell$, $k \geq l$, $N = 50$

By relaxing the objective function in the continuous domain, it is possible to compute the maximum of $F_+(k)$ by imposing $\frac{\partial F_+(k)}{\partial k} = 0$, i.e.

$$\begin{aligned}\frac{\partial F_+(k)}{\partial k} &= \frac{\partial}{\partial k}\left(\frac{1}{k-\ell+2}\Big[-k^2+(N+1)k-(N+1)\ell+\ell^2\Big]\right)\\&= -\frac{1}{(k-\ell+2)^2}\Big[-k^2+(N-1)k-(N+1)\ell+\ell^2\Big]+\\&\quad+\frac{1}{k-\ell+2}\Big[-2k+N+1\Big]\\&= \frac{1}{(k-\ell+2)^2}\Big[-k^2-2(2-\ell)k+(2N+2-\ell^2)\Big]\end{aligned}$$

so that, by studying the expression inside the square brackets, the roots can be written as

$$\begin{aligned}k_{1,2} &= \frac{(2-\ell)\pm\sqrt{(2-\ell)^2+(2N+2-\ell^2)}}{-1}\\&= (\ell-2)\mp\sqrt{2N+6-4\ell}\end{aligned}$$

In order to find a maximum inside the domain, I need $k_{1,2}$ to be greater than or equal to ℓ, leading to the existence of at most one maximum when the following inequality is satisfied

$$\begin{aligned}\ell-2+\sqrt{2N+6-4\ell} &\geq \ell\\2N+6-4\ell &\geq 4\\\ell &\leq \frac{N+1}{2}\end{aligned}\tag{5.1}$$

where condition (5.1) means that when $-$ is linked to a node ℓ in the first half of the Line Graph, there exists a maximum for F_+ for $k \geq l$. Otherwise, from what we know until here, the function is decreasing with k and the only option such that $k \geq \ell$ would be $k = \ell$, leading to $F_+ = 0$. However, let us also consider the symmetric case for which $k < \ell$ and let us see if there are other options for the case $\ell > \frac{N+1}{2}$

- $k < \ell$)

Let us repeat all the procedure for this symmetric case. The line graph can be represented as

Figure 5.4: Line Graph with $v^- = \ell$, $v^+ = k$, $k < l$

that can again be represented by the analogous circuit

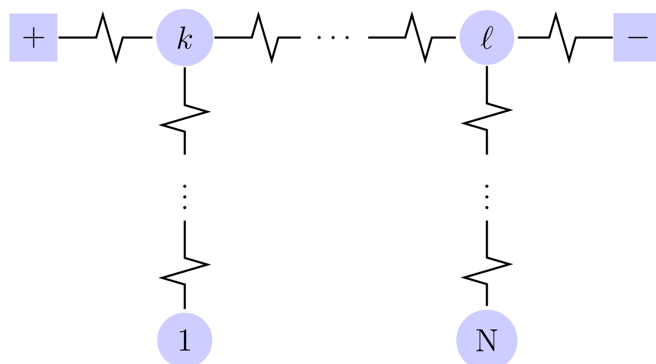
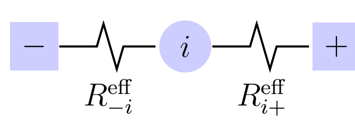


Figure 5.5: Circuit analogous line graph with $v^- = \ell$, $v^+ = k$, $k < l$

for which

$$\begin{cases} V(-) = -1 \\ V(+) = +1 \\ V(1) = V(2) = \cdots = V(k) \\ V(\ell) = V(\ell+1) = \cdots = V(N) \end{cases}$$

Then, the voltage divider for each node $i = \ell, \ell + 1 \ldots, k$, i.e.



where, now,

$$\begin{cases} R^{\text{eff}}_{-i} = \ell - i + 1 \\ R^{\text{eff}}_{i+} = i - k + 1 \end{cases}$$

it leads to

$$\begin{aligned} V(i) - V(-) &= (V(+) - V(-))\frac{\ell - i + 1}{i - \ell + 1 + k - i + 1} \\ V(i) &= 2\frac{\ell - i + 1}{\ell - k + 2} - 1 \end{aligned}$$

for each $i = \ell, \ell + 1 \ldots k$. Notice how in this case $V(i)$ is decreasing with i – since a smaller number means being closer to $+$.
So, with the same abuse of notation as before it is possible to write the objective function in terms of k, i.e.

$$\begin{aligned} F_+(k) &= kV(k) + \sum_{i=k+1}^{\ell} V(i) + (N - \ell)V(k) \\ &= \frac{1}{\ell - k + 2}\left[-k^2 + (N+1)k - (N+1)\ell + \ell^2\right] \end{aligned}$$

which is symmetrical to the previous expression of F_+. Indeed, we can see that the numerator is the same, whereas in the denominator k and ℓ are inverted. The similarity can be easily grasped by comparing Figure 5.6 with Figure 5.3, for which I maintained the same colors for corresponding ℓ values.

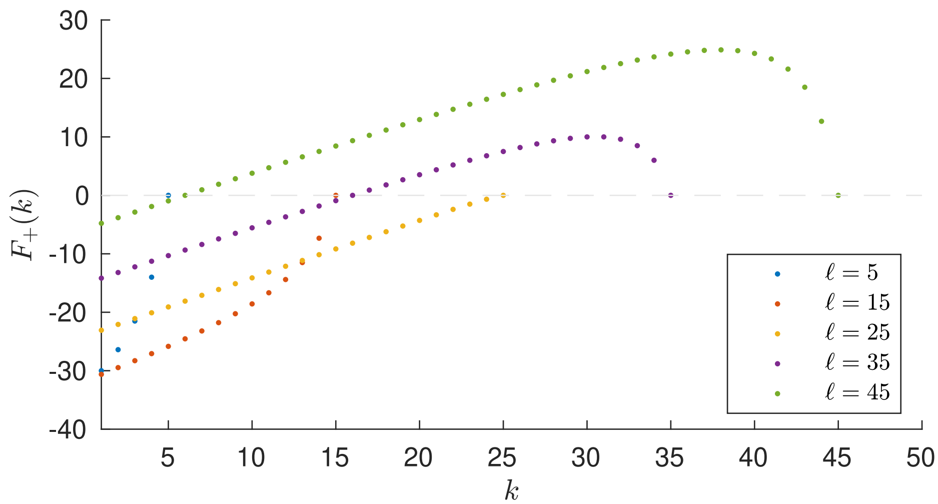


Figure 5.6: $F_+(k)$ of line graph with $v^- = \ell$, $k < l$, $N = 50$

Again, by relaxing the objective function in the continuous domain, it is possible to find the maximum of $F_+(k)$, by imposing $\frac{\partial F_+(k)}{\partial k} = 0$, i.e.

$$\begin{aligned}\frac{\partial F_+(k)}{\partial k} &= \frac{\partial}{\partial k}\left(\frac{1}{\ell - k + 2}\Big[-k^2 + (N+1)k - (N+1)\ell + \ell^2\Big]\right) \\ &= \frac{1}{(\ell - k + 2)^2}\Big[k^2 - 2(\ell+2)k + (\ell^2 + 2N + 2)\Big]\end{aligned}$$

so that, by studying the expression inside the square brackets, the roots of the polynomial are

$$k_{1,2} = (\ell + 2) \pm \sqrt{4\ell + 2 - 2N}$$

which are real for k smaller than ℓ, leading to the existence of at most one maximum when the following inequality is satisfied

$$\begin{aligned}\ell + 2 - \sqrt{4\ell + 2 - 2N} &< \ell \\ \ell &> \frac{N+1}{2}\end{aligned} \tag{5.2}$$

in perfect analogy with (5.1).

Then, by combining both conditions (5.1) and (5.2) we can tell that for the line graph it always exist an optimum value of $v^+ = k$, that is placed on the left or on the right of $v^- = \ell$ depending on the value of ℓ. This is quite trivial since, if the target of $-$ is not in the middle, by targeting an agent on the opposite side it is possible to influence a larger amount of individuals. However, the counterintuitive result is that in general it is not optimal to target an agent next to v^-, while it is more effective to target an agent slightly on the opposite side, with some nodes of distance. Such notion is probably due to the impact effect that also agent $-$ would have, being close to v^+ too, so that in order to be more impactful on other agents, it is better to stay with some distance from the opponent. Clearly, when the adversary targets optimally, i.e. in the middle, the only option for $+$ is to cancel out its effect by targeting the same note, kind of the same situation that happens in the famous game theoretical Hotelling model [33].

Concluding, let us represent the comprehensive result below, for different values of ℓ:

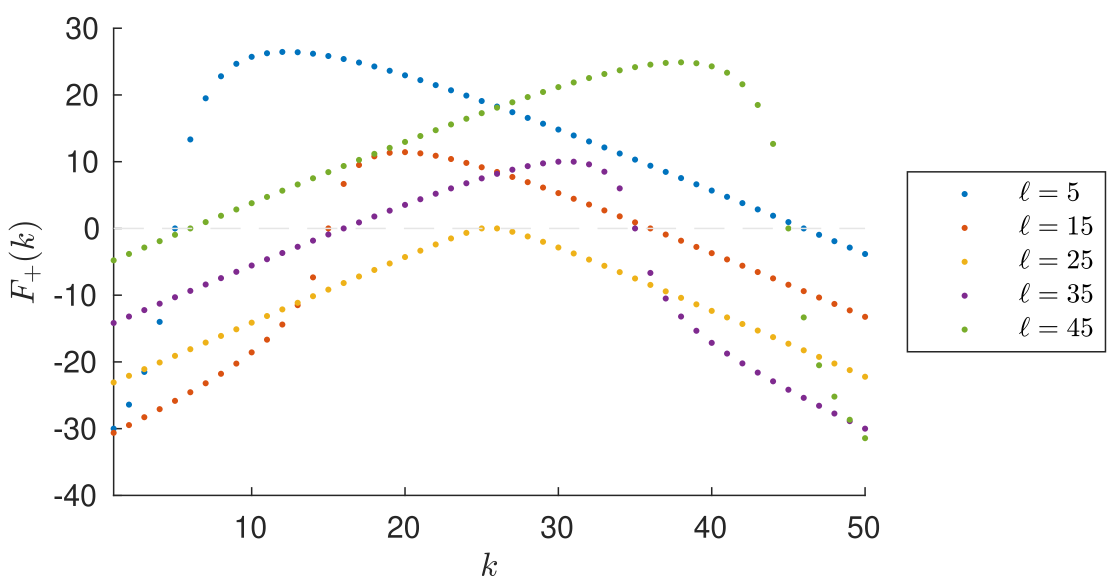


Figure 5.7: $F_+(k)$ of line graph with $v^- = \ell$, $N = 50$

5.2 Tree Graph Single Targeting

For the line graph we have seen an analytical solution for the problem (2.8), determining exactly the optimal position of v^+ in order to maximize the influence of $+$. In an analogous way, we could think of extending the reasoning behind the previous section to a generic tree. Indeed, given the position of v^- for each possible choice of v^+ it exists just one path connecting v^- to v^+, thus leading to a similar situation as before – by considering the corresponding electrical network it is easy to see that each node non-belonging to such path is short-circuited with one belonging to it, i.e. the only voltage drops happen along such line.

On the other hand, when talking of a generic tree, the situation gets more complicated, since the computation of $V(i)$ is not straightforward as before. Indeed, while for the line graph each intermediate node (between ℓ and k) caused an identical voltage drop, for the tree graph each node belonging to the path among v^- and v^+ contributes to such drop proportionally to the number of nodes of its subtree. This is better explained in Figure 5.8.

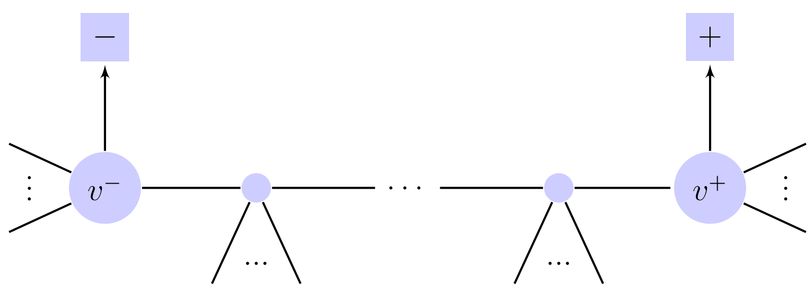


Figure 5.8: Tree Graph with $v_- = \ell$, $v_+ = k$

Because of this complication, in order to compute F_+ for a generic tree graph, it becomes necessary to have more information about the tree, making it unfeasible in general to have an analytical result as we had before.

In order to be more formal, let us introduce a few notations to describe the *subtrees* of a tree graph, i.e. the subsets of nodes of the original tree which form themselves a tree.

Let $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ be an unweighted tree graph. Then, given a pair of distinct nodes $i, j \in \mathcal{I}$, let denote by $\mathcal{I}^{<ij}$ the subtree rooted at node i that does not

contain node j, along with the path from i to j (apart from i), i.e.

$$\mathcal{I}^{<ij} = \{h \in \mathcal{I} | \text{path from } h \text{ to } j \text{ goes through } i\}$$

Such definition allows to describe all the needed subtrees in a formal way. Let us also notice that the intersection of two subtrees is also a subtree.

In addition, when studying the voltage drop happening on the path between v^- and v^+, a notion of interest is the cardinality of the subtrees generated under each node of such path, i.e. the subset of nodes short-circuited with each other node on the corresponding electrical network.
More formally, let denote by c_i the cardinality of the subtree rooted at node $i \in \{\text{path from } v^- \text{ to } v^+\}$ made up by the nodes $j \in \mathcal{I}^{<iv^-} \cap \mathcal{I}^{<iv^+}$, i.e.

$$c_i = \left|\mathcal{I}^{<iv^-} \cap \mathcal{I}^{<iv^+}\right|, \qquad i \in \{\text{path from } v^- \text{ to } v^+\}$$

Then, it is possible to write an expression for the objective function F_+:

$$F_+(k) = |I^{<v^-v^+}| V(v^-) + \sum_{\substack{i \in \{\text{path from} \\ v^- \text{ to } v^+\}}} c_i V(i) + |I^{<v^+v^-}| V(v^+)$$

which, however, cannot be used for solving the targeting problem without knowing the structure of the studied tree; apparently not allowing for better approaches than a *brute force* maximum search – i.e. by computing the objective function $F_+(v^+)$ for each possible node v^+ of the graph.

Despite this, the similarity among the line graph representation and the path among the two strategic nodes of a tree graph is strong, and suggests that the results on the former could be in some way transferred to the more general representation.

In particular, I will show in the following two important properties of the tree graph representation, that will make possible to solve the targeting problem in a smart way, by calculating the objective function just on a subset of the nodes – improving what would have been the brute force approach.

Branch Proposition

It has been shown in Chapter 5.1 how, for the line graph with a node ℓ on the first half of the line linked to $-$, i.e. $v^- = \ell < \frac{N}{2}$, by moving $v^+ = k$

from node ℓ towards the line's tail, the continuous version of the objective function $F_+(k)$ exhibits a concave behavior. In particular, this is true also for the case where $\ell = 1$, whose graph is represented below

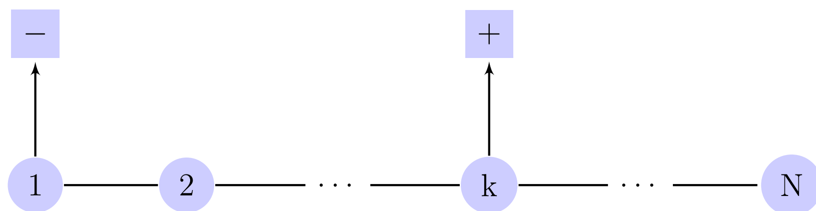


Figure 5.9: Line graph with $v_- = 1$, $v_+ = k$

In order to make use of this result for a generic tree, let us notice the similarity between the line graph and a branch of a tree. Indeed, when studying trees, it is always possible to consider each path from the root to one of the *leafs* – i.e. one of the nodes having degree one – as *pseudo*-line graphs where, now, each line node i has, in turn, another subtree generating from it of cardinality c_i. Notice that c_N is always equal to 1 by construction, while the line graph case can be seen as a special case of this where $c_i = 1 \ \ \forall i = 1, \ldots, N$.

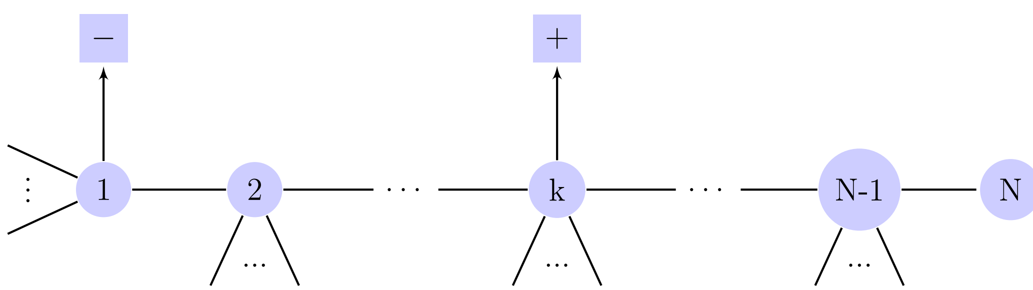


Figure 5.10: Generic branch of tree graph with $v^- = 1$, $v^+ = k$

By representing a generic tree in such form – that I will denote as *pseudo-line branch representation* of a tree – intuition suggests that, by considering v^- as the generic tree root as shown in Figure 5.10, each of its branches could behave as the line graph case of Figure 5.9, for some properties of F_+. Indeed, the most interesting feature of the objective function is about the concave behavior of its continuous version, that leads to a unique maximum or, at most, to two and adjacent ones in its discrete counterpart. Clearly, because of the loss of the subtrees' cardinality homogeneity in the extension, the objective function concave behavior would not hold anymore in general. However, the monotonicity property of such function prior to and following the maximum could intuitively hold.

This behavior, if transferred, would allow for a relevant performance improvement in the maximum search. Indeed, if this property successfully transfers to the general tree case, it could be used to just visit each branch until the objective function value starts decreasing on it, since the monotonicity of F_+ over the following tails would ensure that this procedure would not rule out any optimal solution.

Proposition 5.2.1. *Let us consider the single targeting problem (2.9) for an undirected tree graph $\mathcal{T} = (\mathcal{I}, \mathcal{E})$. Let the node linked to the strategic node $-$, v^-, be the root of the tree, and consider a tree branch as a path going from the root to one of the leafs. Then, the objective function $F_+(\,\cdot\,)$, on such branch is monotone prior to and following the maximum.*

Proof. Let us remember that in the line graph case with $v^- = 1$, represented in Figure 5.9, it is possible to write down the voltage equation as

$$V(i) = \frac{2i}{k+1} - 1 \qquad \forall i = 1, \dots, k$$

As it has been shown in Chapter 5.2 for the Tree optimal placement, this property still holds for its tree extension, since the subtrees eventually generating from each line node are short-circuited with their root, then assuming its voltage, i.e.

$$V(j) = V(i) \qquad \forall j \in \mathcal{I}^{<i1} \cap \mathcal{I}^{<ik}, \quad i \in \{1, \dots, k\}$$

It is then possible to write the objective function F_+ in terms of $k \in \{1, \dots, N\}$, that is the same expression computed for the tree graph with $\ell = 1$.

$$
\begin{aligned}
F_+(k) &= |\mathcal{I}^{<1k}|V(1) + \sum_{i=2}^{k-1} |\mathcal{I}^{<i1} \cap \mathcal{I}^{<ik}|V(i) + |\mathcal{I}^{<k1}|V(k) \\
&= |\mathcal{I}^{<1N}|V(1) + \sum_{i=2}^{k-1} |\mathcal{I}^{<i1} \cap \mathcal{I}^{<iN}|V(i) + |I^{<k1}|V(k) \\
&\quad \text{where } \ \mathcal{I}^{<ik} = \mathcal{I}^{<iN} \quad \forall k : i < k < N \\
&= \sum_{i=1}^{k-1} c_i V(1) + V(k) \sum_{j=k}^{N} c_j \\
&\quad \text{where } \ c_i = |\mathcal{I}^{<i1} \cap \mathcal{I}^{<iN}| \ \text{ and } \ \mathcal{I}^{<1N} = \mathcal{I}^{<11} \cap \mathcal{I}^{<1N} \\
&= \sum_{i=1}^{k-1} c_i \left(\frac{2i}{k+1} - 1 \right) + \left(\frac{2k}{k+1} - 1 \right) \sum_{j=k}^{N} c_j \\
&= \sum_{i=1}^{k} c_i \left(\frac{2i}{k+1} - 1 \right) + \left(\frac{2k}{k+1} - 1 \right) \sum_{j=k+1}^{N} c_j
\end{aligned}
$$

while

$$
F_+(k+1) = \sum_{i=1}^{k} c_i \left(\frac{2i}{k+2} - 1 \right) + \left(\frac{2k+2}{k+2} - 1 \right) \sum_{j=k+1}^{N} c_j
$$

so that

$$
\begin{aligned}
F_+(k+1) - F_+(k) &= \sum_{i=1}^{k} c_i \left(\frac{2i}{k+2} - \frac{2i}{k+1} \right) + \left(\frac{2k+2}{k+2} - \frac{2k}{k+1} \right) \sum_{j=k+1}^{N} c_j \\
&= \sum_{i=1}^{k} c_i \left(-\frac{2i}{(k+1)(k+2)} \right) + \frac{2}{(k+1)(k+2)} \sum_{j=k+1}^{N} c_j \\
&= \frac{2}{(k+1)(k+2)} \left(\sum_{j=k+1}^{N} c_j - \sum_{i=1}^{k} i c_i \right)
\end{aligned}
$$

Then, the proof reduces to the sign study of

$$\sum_{j=k+1}^{N} c_j - \sum_{i=1}^{k} i c_i$$

Yet, notice how this expression is always made up of a positive side (minuend) and a negative side (subtrahend) and, by increasing k of 1 unit, i.e. moving towards the line tail, one term moves from the positive side to the negative one, multiplied by $k+1$. Then, moving towards the tail always means decreasing the positive side and increasing the negative one, leading the increment $F_+(k+1) - F_+(k)$ to always decrease, as k grows. Such term, at some point, will necessarily reach a negative value, because of the multiplication factor, and along with this change of sign, if happening, such difference will continuously decrease heading towards the tail. This proves the result of having a monotone behavior prior to and following the maximum for each branch starting from v^- and, with it, the proposition. □

Offspring Proposition

Let us now focus on another intuition, considering the same pseudo-line branch representation of a generic tree. Indeed, by imagining to visit the tree nodes starting from the root node $v^- = 1$, and heading away from it, at each step k the strategic node $+$, currently linked to node k, could move to one of its children, looking for an increasing value of $F_+(\cdot)$. However, if we think about two competing possible choices m and n, by moving to m, $+$ would influence more the m's *subtree*, while she is losing influence over n's one. In the same way, choosing to move to n would mean losing influence over m's leg. Consequently, if some choice is optimal the other does not seem plausible to be either, whereas, if they are both equally important, staying on the parent should be a better choice. With such an intuition in mind, this would mean that, for each node k over a branch, at most one of its children can increase the objective function value, while the rest of them cannot – so that it would be useless to compare the rest of them, if an improving node has already been found. This, if true, would mean to improve once more the speed of the maximum search algorithm. This is what the proposition below tells.

Proposition 5.2.2. *Let us consider the single targeting problem (2.9) for an undirected tree graph $\mathcal{T} = (\mathcal{I}, \mathcal{E})$. Let the node linked to the strategic node $-$, $v^- \in \mathcal{I}$, be the root node, and let us consider a path from node v^- to a generic node $k \in \mathcal{I}$. Let $O(k)$ be the offspring of k, denoted as the set of nodes linked to k belonging to the subtree $\mathcal{I}^{<kv^-}$. If exists $m \in O(k)$ such that $F(m) > F(k)$, then $F(n) < F(k) \quad$ for each $n \in O(k) \setminus \{m\}$.*

Proof. Let us represent $\mathcal{T}$ in its pseudo-line branch representation, and let us denote the path from the root node v^- to k, as the length-k path shown in Figure 5.11. Let us assume that at least one node m in k's offspring $O(k)$ is such that $F(m) > F(k)$, where $|O(k)| \geq 2$, otherwise the proof would be trivial, and denote with n a generic node in such offspring different from m.

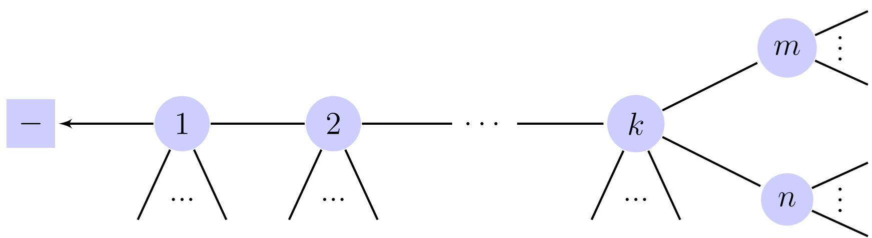


Figure 5.11: Generic path from 1 to k on a tree graph with $v^- = 1$

To be rigorous with previous notation, we should write the cardinality c_i of the subtree generating from each line node i with respect to the v^+ considered. Indeed, by stopping the line in k, m, n we are looking at different $c_k^{(v^+)}, c_m^{(v^+)}, c_n^{(v^+)}$ in terms of v^+, causing a very heavy notation, such as the relationship $c_k^{(k)} = c_k^{(m)} + c_m^{(m)} = c_k^{(n)} + c_n^{(n)}$. Consequently, in order to simplify the notation, let us use the following redefinition:

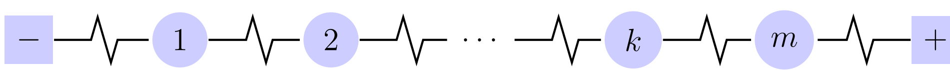
$$\begin{cases} c_m^{(m)} = |\mathcal{I}^{<m1}| := c_m \\ c_n^{(n)} = |\mathcal{I}^{<n1}| := c_n \\ c_k^{(k)} - c_m^{(m)} - c_n^{(n)} = |\mathcal{I}^{<k1} \cap \mathcal{I}^{<km} \cap \mathcal{I}^{<kn}| := M \end{cases}$$

Let us write the objective function F_+ for the three cases studied, along with their electrical analogy representation:

- $v^+ = k$

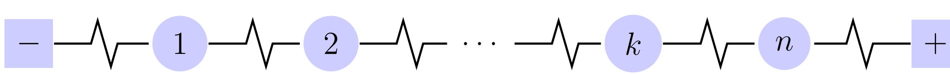
$$
\begin{aligned}
F_{+}(k) &= \sum_{i=1}^{k-1} c_i V_i^{(k)}(i) + c_k^{(k)} V_k^{(k)}(k) \\
&= \sum_{i=1}^{k-1} c_i \left(\frac{2i}{k+1} - 1 \right) + (M + c_m + c_n) \left(\frac{2k}{k+2} - 1 \right)
\end{aligned}
$$

- $v^{+} = m$



$$
\begin{aligned}
F_{+}(m) &= \sum_{i=1}^{k-1} c_i V_i^{(i)}(i) + c_k^{(m)} V_k^{(m)}(k) + c_m^{(m)} V_m^{(m)}(m) \\
&= \sum_{i=1}^{k-1} c_i \left(\frac{2i}{k+2} - 1 \right) + (M + c_n) \left(\frac{2k}{k+2} - 1 \right) + c_m \left(\frac{2k+2}{k+2} - 1 \right)
\end{aligned}
$$

- $v^{+} = n$



$$
\begin{aligned}
F_{+}(n) &= \sum_{i=1}^{k-1} c_i V_i^{(i)}(i) + c_k^{(n)} V_k^{(n)}(k) + c_n^{(n)} V_n^{(n)}(n) \\
&= \sum_{i=1}^{k-1} c_i \left(\frac{2i}{k+2} - 1 \right) + (M + c_m) \left(\frac{2k}{k+2} - 1 \right) + c_n \left(\frac{2k+2}{k+2} - 1 \right)
\end{aligned}
$$

So that

$$
\begin{aligned}
F_+(m) - F_+(k) &= \sum_{i=1}^{k-1} c_i \left(\frac{2i}{k+2} - \frac{2i}{k+1} \right) + (M + c_n) \left(\frac{2k}{k+2} - \frac{2k}{k+1} \right) + \\
&\quad + c_m \left(\frac{2k+2}{k+2} - \frac{2k}{k+1} \right) \\
&= \frac{2}{(k+1)(k+2)} \left[- \sum_{i=1}^{k-1} i c_i - k(M + c_n) + c_m \right] \\
&= \frac{2}{(k+1)(k+2)} [-f - k c_n + c_m] \\
&\quad \text{where } f := \sum_{i=1}^{k-1} i c_i + kM > 0
\end{aligned}
$$

while, analogously

$$
F_+(n) - F_+(k) = \frac{2}{(k+1)(k+2)} [-f - k c_n + c_m]
$$

By hypothesis, we know

$$
\begin{aligned}
F_+(m) - F_+(k) &> 0 \\
\frac{2}{(k+1)(k+2)} [-f - k c_n + c_m] &> 0 \\
c_m &> k c_n + f
\end{aligned}
$$

and we want to prove that

$$
\begin{aligned}
F_+(n) - F_+(k) &< 0 \\
\frac{2}{(k+1)(k+2)} [-f - k c_m + c_n] &< 0 \\
c_m &> \frac{1}{k} c_n - \frac{f}{k}
\end{aligned}
$$

but

$$\begin{aligned} c_m &> kc_n + f \\ &\quad \text{since } k \geq 1 \\ &\geq \frac{1}{k}c_n + \frac{f}{k} \\ &\quad \text{since } c_n, f > 0 \\ &> \frac{1}{k}c_n - \frac{f}{k} \end{aligned}$$

proving the proposition. □

Tree Graph 1-edge Optimal Targeting Algorithm

Concluding, the two propositions presented above can be summarized like this: when studying the single targeting problem (2.9) for a generic tree, by considering v^- as the root node, and moving v^+ from v^- to one node in its offspring, at most one of them will make F_+ increase. Then, this is true also for the offspring of such node, and so on until no improving son is found.This holds true because F_+ increases at most on one of its sons, and because it is monotonically increasing until the maximum is found, and then decreasing on the rest of the branch. This notion has been exploited in the algorithm below to solve the STP.

Theorem 5.2.3. *Given a generic tree graph, the Algorithm (1) solves the single targeting problem (2.9).*

Proof. The proof simply comes as a result of both Proposition 5.2.1 and 5.2.2, since the former tells that $F_+(\cdot)$ is monotone prior to and following the maximum of each branch, while the latter tells that there is at most one initial node from which a monotonically increasing branch can start. □

The power of this algorithm is the fact that, in order to solve STP, instead of applying the objective function onto each node of the graph to find the maximum, it is possible to reduce the number of computations to a smaller subset of nodes. In this way, it is possible to reduce enormously the computational expensiveness of the problem.

Algorithm 1 Tree Graph Single Targeting Algorithm

Require: $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ tree graph, node v^-
  Initialization:
    Root node $r = v^-$
    Number of visited nodes $s = 0$
    Evaluate $F_+(r)$
    Flag $f = 0$
  while $f = 0$ **do**
    $f = 1$
    for $\ell \in \mathcal{O}(r)$ **do**
      $s = s + 1$
      Evaluate $F_+(\ell)$
      if $F_+(r) < F_+(\ell)$ **then**
        $r = \ell, F_+(r) = F_+(\ell), f = 0$
        break for
      end if
    end for
  end while
  return $v^+ = r,\ F_+(r),\ s$

Numerical Experiment 1. *As a first experiment, let us consider a tree obtained starting from a root node and generating two nodes from each other node, until reaching the desired number of nodes N, for $N = 11, 101, 1001$. For all cases, by linking $-$ to the last generated leaf and solving the STP, we find that the optimal choice for the strategic agent $+$ is not the root, but a node in the first or second generation, in the same branch of $-$. Let us show the result for the tree with $N = 101$, in Figure 5.12. There, each node is colored in terms of its asymptotic opinion, as reported in the color-bar on the right, while the strategic nodes are marked by squares..*

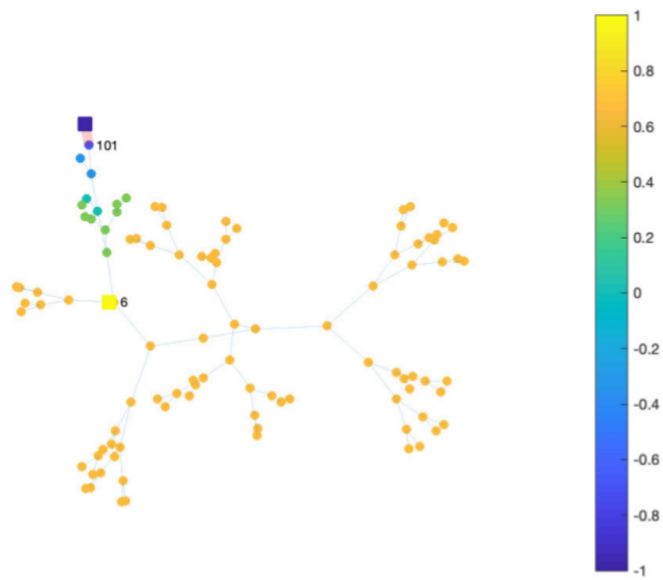


Figure 5.12: Asymptotic opinion of agents in the tree graph, when solving STP, with $v^- = 101$, $N = 101$

We can easily see how, by solving STP, the agent $+$ made all the agents in the other side of the tree, with respect to $-$, contribute positively. The optimal choice consists then in making a trade-off between having more agents contributing with a positive net contribution – by moving towards the opposite agent – and increasing the contribution of each agent – by moving in the opposite direction.

Now, in order to test the power of Algorithm 1, we apply it to random generated trees. In particular, I will make this simulations in order to test the number of steps required by the algorithm to find the optimum. This is done for a certain number of simulations, for four different kind of trees, each with increasing density, and for increasing sizes of the trees.

Numerical Experiment 2. *Let us generate a Branching Process by starting from a root node and generating its offspring according to a truncated Poisson distribution of parameter* λ *and with probability of having no sons equal to* 0*. Then, by treating each node of the previous generation offspring as the roots of their own subtree, the process is repeated until the number of nodes becomes the desired* N*.*

Now, let us generate 50 *trees for each* $\lambda \in \{3, 6, 9, 12\}$*, and* $N = 100, 200, \ldots, 800$*: this leads to the generation of* 1600 *trees. For each of these trees we solve the STP and we save the number of steps that each algorithm needs to find the optimum. We show the average percentage of visited nodes over the* 50 *simulations as function of number of nodes* N*, for different* λ*, in Figure 5.13.*

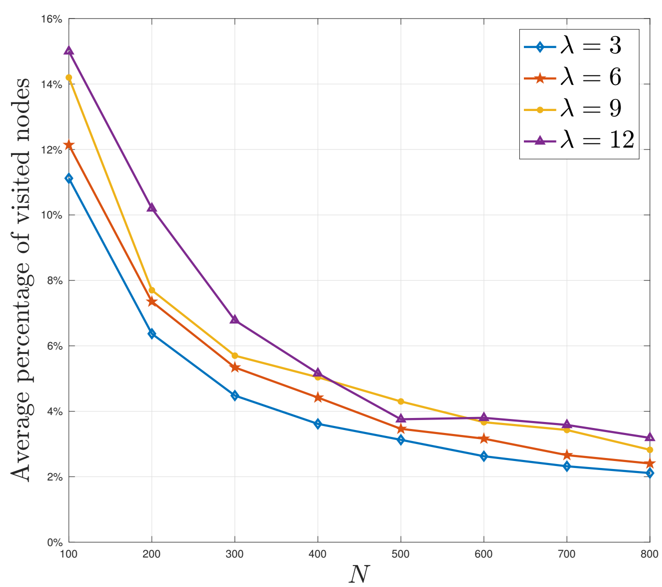


Figure 5.13: Average percentage of visited nodes averaged over 50 simulations for different offspring distribution generated as truncated Poisson(λ)

From the figure above it becomes evident how Algorithm 1 reduces the computational expensiveness of the STP. Moreover, not only the percentage of visited nodes is much smaller than the number of nodes (assuming values in the range $[11\%, 15\%]$ *with N=100), but also that as N increases (which means when complexity reduction is most needed), such percentage is decreasing, reaching values in the range* $[2\%, 4\%]$*, allowing for a reduction of time expensiveness of two orders of magnitude.*

In Chapter 6.2 I will discuss how this notion can also be extended to more general cases, such as locally tree-like graphs or more in general sparse graphs.

5.3 Complete Graph Multiple Targeting

One of the most powerful results of this work is the analytical solution of the multiple targeting problem (2.8) for the complete graph. Such result is immediate, whichever k_- and k_+ there are. But, more importantly, being valid for complete graphs, also allows for similar considerations on relaxations of the complete graph case – as the Erdos-Renyi graph could be, or any other dense graph. Let us then analyze the multiple targeting problem (2.8) for the complete graph.

Definitions and Objective Function

In order to compute the objective function for the complete graph, let us exploit its anonymity property – the fact that each node shares the neighborhood with all the others. When the $k_- + k_+ + k_+^0$ edges originating from the strategic nodes are placed, we can simply distinguish among four kinds of nodes, in terms of possible strategic nodes linked: the ones linked to $+$ but not to $-$, the ones linked to $-$ but not to $+$, the ones linked to both, and the ones linked to none of them. In particular, let us split these N nodes into

- p nodes linked to $+$ but not to $-$
- q nodes linked to $-$ but not to $+$
- r nodes linked to both $+$ and $-$
- $N-p-q-r$ nodes linked to none of them

where $k_+ + k_+^0 = p+r$, $k_- = q+r$. In addition, let us indicate by v^p,v^q,v^r and v^0 a generic node of each category, respectively, as shown in the example of Figure 5.14.

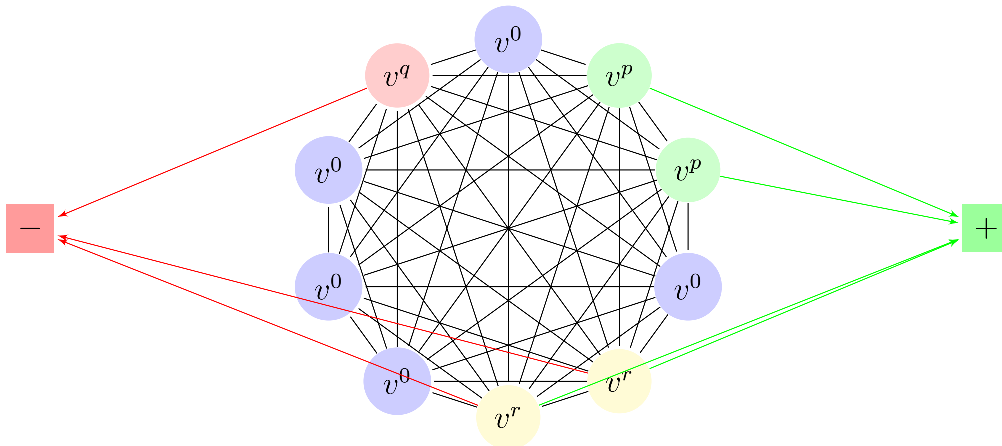


Figure 5.14: Complete graph with $N = 10$, $p = 2$, $q = 1$, and $r = 2$

This subdivision turns out to be extremely useful when describing the asymptotic opinion of agents. Indeed, asymptotically, each agent's opinion can be described just by the geometry of the graph, without considering the agent's initial opinion. This means that within a type each individual is indistinguishable, making the description of the asymptotic opinion easy: the asymptotic opinion state vector corresponds to the fixed point satisfying the system of four equations below.

$$\begin{cases} \bar{x}_{v^p} = \frac{p-1}{N}\bar{x}_{v^p} + \frac{q}{N}\bar{x}_{v^q} + \frac{r}{N}\bar{x}_{v^r} + \frac{N-p-q-r}{N}\bar{x}_{v^0} + \frac{1}{N} \\ \bar{x}_{v^q} = \frac{p}{N}\bar{x}_{v^p} + \frac{q-1}{N}\bar{x}_{v^q} + \frac{r}{N}\bar{x}_{v^r} + \frac{N-p-q-r}{N}\bar{x}_{v^0 - \frac{1}{N}} \\ \bar{x}_{v^r} = \frac{p}{N+1}\bar{x}_{v^p} + \frac{q}{N+1}\bar{x}_{v^q} + \frac{r-1}{N+1}\bar{x}_{v^r} + \frac{N-p-q-r}{N+1}\bar{x}_{v^0} + \frac{1}{N+1} - \frac{1}{N+1} \\ \bar{x}_{v^s} = \frac{p}{N-1}\bar{x}_{v^p} + \frac{q}{N-1}\bar{x}_{v^q} + \frac{r}{N-1}\bar{x}_{v^r} + \frac{N-p-q-r-1}{N-1}\bar{x}_{v^0} \end{cases}$$

leading to

$$\begin{aligned} \tilde{F}_+(p,q,r) &= p\bar{x}_{v^p} + q\bar{x}_{v^q} + r\bar{x}_{v^r} + (N-p-q-r)\bar{x}_{v^0} \\ &= \frac{N(N+2)(p-q)}{(N+2)(p+q) + 2(N+1)r} \end{aligned}$$

where $\tilde{F}_+(p,q,r) = F_+(\mathcal{A})$ is the objective function of the OTP (2.8) such that $k_+ = |\mathcal{A}| = p + r, k_- = q + r$, i.e. it is the objective function rewritten in terms of p, q, r.
Notice here that $\tilde{F}_+$ is symmetric with respect to $p = q$, which gives 0 value,

as expected. Moreover, let us notice that the more connections there are – greater p, q, r – the smaller $\tilde{F}_+$ would be, under the same $p - q$. This means that, when there are a lot of links coming from the strategic nodes, an increment of one edge to p would lead to a smaller variation of $\tilde{F}_+$ with respect to a less connected situation: the basic intuition behind submodularity. In particular, it is possible to prove the submodularity of the objective function for the complete graph by simply comparing different configurations of $\tilde{F}_+$, as reported in Appendix B.1.

Optimal Targeting

Let us now present the analytical solution of the OTP on complete graphs, and let us prove the result by making use of the notations above presented.

Theorem 5.3.1 (OTP on complete graph)**.** *Let p, q, r be the number of regular nodes initially linked to strategic agent $+$ but not to $-$, to $-$ but not to $+$, and to both, respectively. Then, the optimal objective function $F_+^\star$ of the OTP (2.8) is given by*

$$F_+^\star := \max_{\mathcal{A} \subseteq \mathcal{R} : |\mathcal{A}| \leq k_+} F_+(\mathcal{A})$$

where

- *if $k_+ < q - p$, then*
$$F_+^\star = \tilde{F}_+(p + k_+, q, r)$$
- *if $k_+ = q - p$, then*
$$F_+^\star = \tilde{F}_+(p + m, q - (k_+ - m), r_0 + (k_+ - m))$$
for any m such that $\max\{0, k_+ - q\} \leq m \leq \min\{k_+ - 1, N - p - q - r\}$
- *if $k_+ > q - p$, then*
$$F_+^\star = \tilde{F}_+(p + \max\{0, k_+ - q\}, q - \min\{k_+, q\}, r + \min\{k_+, q\})$$

solving the OTP for the complete graph.

Proof. In order to compute the solution of the OTP for the complete graph, we need to fully understand the setting here considered: each of the N regular nodes is linked to all the others; the k_- nodes linked to $-$ and the k_+^0 nodes initially linked to $+$ are given. Then, we can split these nodes as we previously did, where now $k_+ = 0$: i.e. $k_- = q+r$, $k_+^0 = p+r$.

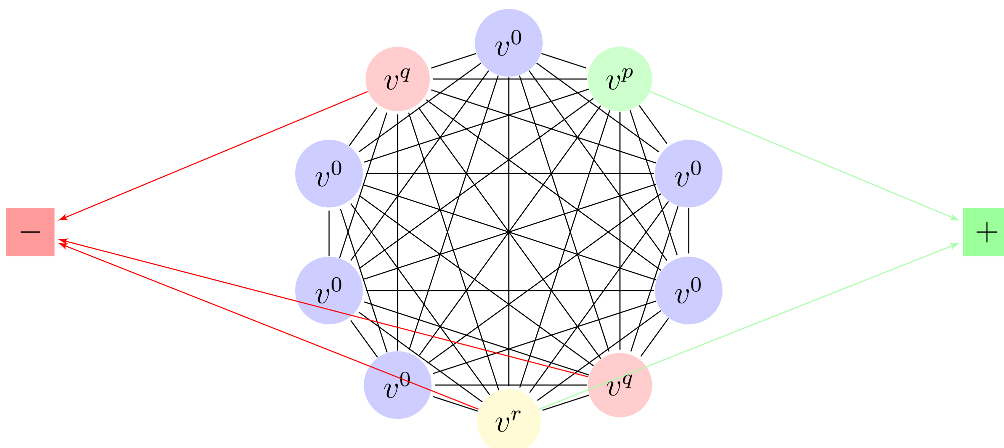


Figure 5.15: Complete graph with $N = 10$, $p = 1$, $q = 2$, and $r = 1$

This means that before $+$ placing its available k_+ edges, the objective function can be written as

$$\tilde{F}_+(p, q, r) = \frac{N\ (N+2)\ (p-q)}{(N+2)\,(p+q) + 2\,(N+1)\,r}$$

Starting from this, we are interested in the computation of the optimal placement of these k_+ edges available. In this context, agent $+$ has simply two options: the targeting of nodes not linked to $-$, or the targeting of nodes already linked to $-$ – the former causing the increase of p, the latter causing both the increase of r, and the decrease of q. It is then convenient to calculate $\tilde{F}_+$ in terms of the number of nodes that agent $+$ decides to place to nodes not linked to $-$, that I will denote as m. For simplicity, let us also rewrite k_+ as z, so that m is the number of nodes not linked to $-$ targeted by $+$, whereas $z - m$ is the number of nodes already linked to $-$ targeted by $+$.

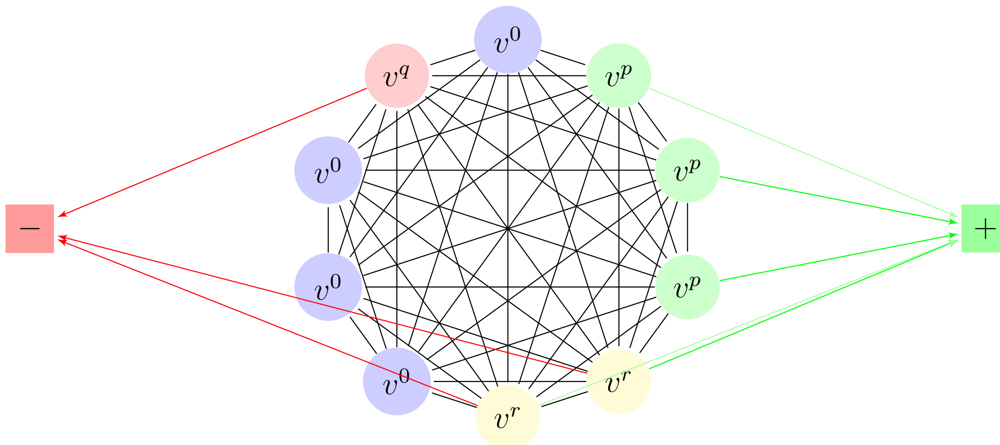


Figure 5.16: Complete graph with $N=10$, $z=3$, $m=2$, $p+m=3$, $q-(z-m)=1$, and $r+(z-m)=2$

In addition, let us notice that agent + can target at most q nodes already linked to $-$, so that the maximum $z-m$ value is q. This means that if $q<z$, the minimum number of nodes not linked to $-$ that + can target is $z-q$. On the other hand, agent + cannot target more than $N-p-q-r$ nodes not linked to $-$. Then, for m such that $\max\{0, z-q\} \leq m \leq \min\{z, N-p-q-r\}$,

$$F_+(p+m, q-(z-m), r+(z-m)) = \\ = \frac{N(N+2)(p+m-q+(z-m))}{(N+2)(p+m+q-(z-m))) + 2(N+1)(r+(z-m))} \\ = \frac{N(N+2)(p+z-q)}{D_m}$$

where $D_m = (N+2)(p+m+q-(z-m))) + 2(N+1)(r+(z-m))$. Then, for m such that $\max\{0, z-q\} \leq m \leq \min\{z-1, N-p-q-r\}$, if + changes one of the targeted nodes: from one already linked to $-$ to one not linked to

$-$, the objective function becomes

$$\begin{aligned}
&\tilde{F}_+(p+(m+1), q-(z-(m+1)), r+(z-(m+1))) = \\
&= \frac{N(N+2)(p+(m+1)-q+(z-(m+1)))}{(N+2)(p+(m+1)+q-(z-(m+1)))+2(N+1)(r+(z-(m+1)))} \\
&= \frac{N(N+2)(p+z-q)}{(N+2)(p+m+q-(z-m))+2(N+2)+2(N+1)(r+(z-m)))-2(N+1)} \\
&= \frac{N(N+2)(p+z-q)}{(N+2)(p+m+q-(z-m))+2(N+1)(r+(z-m)))+2} \\
&= \frac{N(N+2)(p+z-q)}{D_m+2}
\end{aligned}$$

for which we can easily see how the numerator is the same for both expressions, while the denominator is larger for the latter, so that

$$\begin{aligned}
\tilde{F}_+(p+(m+1), q-(z-(m+1)), r+(z-(m+1))) > \tilde{F}_+(p+m, q-(z-m), r+(z-m)) &\iff p+z<q \\
\tilde{F}_+(p+(m+1), q-(z-(m+1)), r+(z-(m+1))) = \tilde{F}_+(p+m, q-(z-m), r+(z-m)) &\iff p+z=q \\
\tilde{F}_+(p+(m+1), q-(z-(m+1)), r+(z-(m+1))) < \tilde{F}_+(p+m, q-(z-m), r+(z-m)) &\iff p+z>q
\end{aligned}$$

for each m such that $\max\{0, z-q\} \leq m \leq \min\{z-1, N-p-q-r\}$.

This means that the objective function $\tilde{F}_+$ is increasing or decreasing with m, depending on the inequality $p+z<q$. Hence the maximum value of $\tilde{F}_+$ will be one of the two extreme cases, i.e. where $m = z-1$ or $m = \max\{0, q-z\}$, depending on the values of p, q, z, proving the theorem. □

Let us present graphically the interpretation of these results, from the + agent perspective:

- $p + k_+ < q$: if the sum of links available and placed, not targeted by the opponent, does not surpass the number of nodes hit by $-$ but not by $+$, the optimal strategy is to use all the available budget to target the nodes not already linked to $-$; leading to

  $$\tilde{F}_+(p+k_+, q, r)$$

  This means that when the adversary has a bigger budget, the optimal tactic is to target nodes different from the opponent.

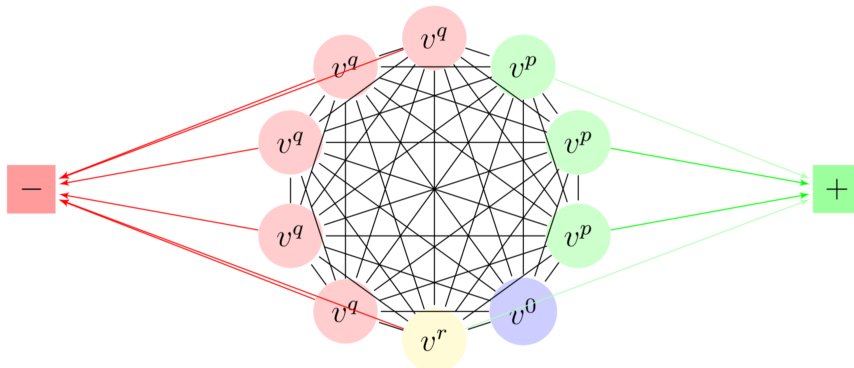


Figure 5.17: Complete graph with $N=10$, $k_+=2$, $m=2$, and before targeting $p=1$, $q=5$, $r=1$

- $p+k_+=q$: if the sum of links available and placed, not targeted by the opponent, is equal to the number of nodes hit by $-$ but not by $+$, each strategy is optimal; leading, for each m such that $\max\{0, k_+ - q\} \leq m \leq \min\{k_+ - 1, N-p-q-r\}$, to

$$\tilde{F}_+(p+m, q-(k_+-m), r+(k_+-m))$$

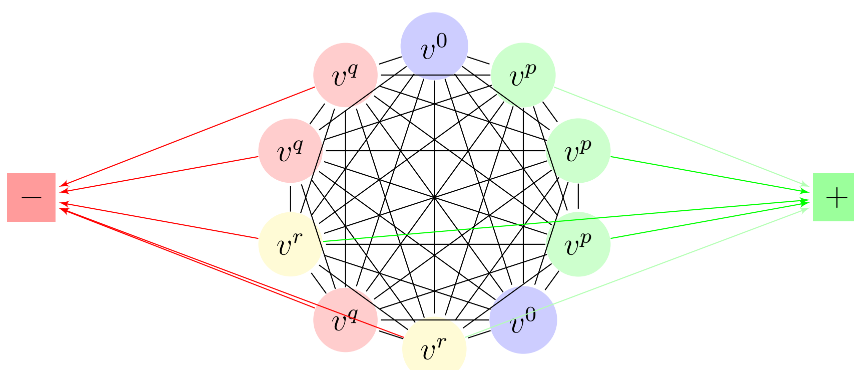


Figure 5.18: Complete graph with $N=10$, $z=3$, $m=2$, and before targeting $p=1$, $q=4$, $r=1$

- $p+k_+>q$: if the sum of links available and placed, not targeted by the opponent, surpasses the number of nodes hit by $-$ but not by $+$, the optimal strategy is to use a portion of the budget to target all the nodes linked to $-$, and the extra budget to influence the rest; leading to

$$\tilde{F}_+(p+\max\{0,k_+-q\}, q-\min\{k_+,q\}, r+\min\{k_+,q\})$$

This means that when the budget of the adversary is lower, the optimal tactic is to target all the opponent's nodes and next to target the rest.

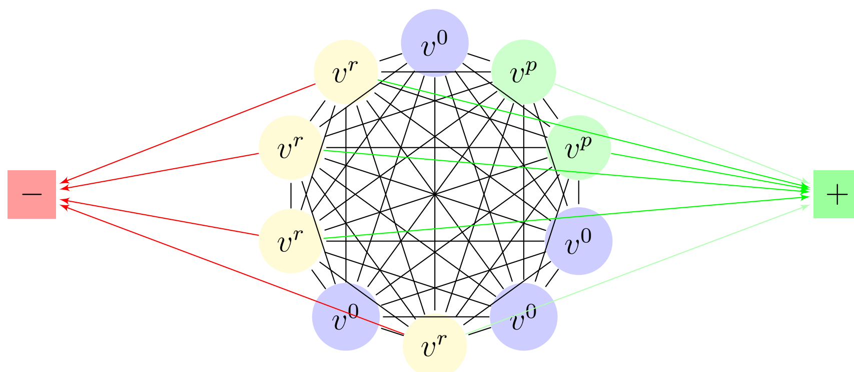


Figure 5.19: Complete graph with $N=10$, $k_+=4$, $m=1$, and before targeting $p=1$, $q=3$, $r=1$

In Chapter **??** I will show how these results can be exploited to build powerful heuristics for dense graphs, by treating such graphs as a relaxation of the complete graph model.

Chapter 6

Optimal Targeting Heuristics

In the previous chapter we have seen how the optimal targeting problem can be tackled for the cases of tree graphs and complete graphs. However, in order to get the optimum for more general cases, the only option is to use a brute-force approach – i.e. computing the objective function for every subset of nodes agent + could link to, given the budget k_+. This means computing F_+ for each possible combination of nodes agent + could be connected to, that consists in solving the linear system $(I - Q^{11})\bar{x}^{\mathcal{R}} = Q^{12}x^{\mathcal{S}}$, where the unknown is $\bar{x}^{\mathcal{R}}$, for every possible combination. This leads the problem to a complexity of $O(N^{3+k_+})$, that for large graphs becomes easily unfeasible.

Since the OTP is too expensive to be optimally solved, in this section I will show, depending on the underlying graph, how it is possible to find some heuristics giving the suboptimum of the problem, i.e. an approximation of the optimal solution that gives a result that is close to the real optimum. I then discuss the complexity and the assessments that can be made in order to get the result in a feasible time, discussing the possible trade-offs that can be made in terms of expensiveness and accuracy. Finally, we compare these strategies with zero-cost heuristics that can be built by simply targeting agents in a smart way, and we put together all the strategies to get a scheme of work for building the best possible heuristic, depending on the underlying graph.

More specifically, I will start presenting a simple greedy algorithm, that is supported by the submodularity result in Chapter 4.2. Then, I will provide two heuristics that are built upon the results of the previous chapter: one that combines a generalization of the STP algorithm over tree graphs, with a greedy approach, the other that exploits the complete graph result. These are motivated by looking at more general graphs as relaxations of tree graphs and complete graphs respectively. In this way, it has been possible to built algorithms that are a relaxation of the techniques presented in Chapters 5.2 and 5.3. Then again, a heuristic is built from the intuition that high degree nodes are more relevant, as a modification of the greedy result. Ultimately, two zero-cost heuristics are provided for comparison. All of these are then combined and summed up in a scheme that suggests the best heuristic depending on the underlying graph.

6.1 Standard Greedy Heuristic

To avoid the computations of the objective function for all the $\binom{N}{k_+}$ combinations of the subset of nodes with maximal cardinality, one of the most common approaches is to solve the problem in a greedy manner. This means that we look for a solution by solving the STP one target at a time, until all the k_+ edges available are placed. Let us denote by $\Delta(v|\mathcal{A})$ the discrete derivative, defined as $\Delta(v|\mathcal{A}) := F_+(\mathcal{A} \cup \{v\}) - F_+(\mathcal{A})$. Then, let us define the *Greedy Heuristic* algorithm as follows.

Algorithm 2 Greedy algorithm for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, number of available links k_+
  Initialization:
    $\mathcal{A}_0 = \emptyset$
  for $i \in \{1, \ldots, k_+\}$ **do**
    $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \operatorname*{argmax}_{v \in \mathcal{R}} \{\Delta(v|\mathcal{A}_{i-1})\}$
  end for
  return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

Since the objective function F_+ is monotone and submodular, we know from [24] that

$$F_+(\mathcal{A}_{k_+}) \geq (1 - 1/e) F_+^\star$$

where $F_+^\star = \max_{\mathcal{A}:|\mathcal{A}<k_+|} F_+(\mathcal{A})$ and $F_+(\mathcal{A}_{k_+})$ is the result returned by the algorithm. This means that by using Algorithm 2 it is possible to find a bounded approximation of the real optimum, while reducing the complexity from $O(N^{3+k_+})$ to $O(N^4 k_+)$.

6.2 Tree-like Heuristic

Thanks to the results in Chapter 5.2, it is possible to additionally reduce the number of computations of the objective function when applying a greedy heuristic. To show this, I will first present how it is possible to extend the STP Algorithm 1 over tree-like/sparse graphs. Then, I will generalize it to the cases where $-$ has more than one placed edge. Finally, I will apply it repetitively in a greedy manner to produce a heuristic for the OTP.

Tree-like Heuristic for STP

When trying to solve the STP for the tree graph case, we have seen that an analytical solution is not feasible in general. However, we have seen that when moving v_+ (the node selected by agent $+$) from the root v_- (the node linked to agent $-$) to the extremes of the branches, it is possible to infer the behavior of F_+. In particular, as shown in Numerical Experiment 2, an algorithm that is able to reduce the computations of almost two order of magnitude when N is large is built, by reducing the objective function computations to a subset of the nodes. For locally tree-like graphs, it is possible to imagine that F_+ follows approximately the same behavior, and that one can build an algorithm exploiting such result. In particular, since such graph can be seen as a relaxed version of the tree graph, we need to build the relaxed version of the algorithm, denoted as *Tree-like Single Targeting Algorithm*. This new algorithm, when looking at the root's offspring, does not stop looking at the first increasing F_+ value found, but it saves each *improving node* (i.e. a node leading to an increasing value of F_+) and uses all of them as roots for the next iterations. In plain words, we are assuming that, being the graph not a tree, it is possible to have more values in the nearest neighborhood leading to increasing values of F_+. Then, we can decide to visit all of the branches starting from them. In such a way, while for the tree the sequence of improving nodes consists of a path (made up of increasing values of F_+), now the sequence of visited nodes consists of a tree: this means that the number of iterations is much larger, but this is needed to avoid stopping the algorithm at non-optimal values.

The results of the algorithm over Regular Graphs follow below.

Numerical Experiment 3. *For each* $\mu \in \{3, 6, 9, 12\}$*, and* $N = 100, 200, \dots, 800$*, let us generate* 50 *Regular Graphs of degree* μ *and number of nodes* N*. Regu-*

Algorithm 3 Tree-like Single Targeting Algorithm

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node v^-
  Initialization:
    Set of root nodes $r = \{v^-\}$
    Number of visited nodes $s = 0$
    $w = \emptyset$ set of visited nodes
  while $r \neq \emptyset$ **do**
    $v = \emptyset$ set of improving nodes
    for $\hat{r} \in r$ **do**
      for $\ell \in \mathcal{O}(\hat{r}) \setminus w$ **do**
        $s = s + 1$
        Evaluate $F_+(\ell)$
        if $F_+(\hat{r}) < F_+(\ell)$ **then**
          $v = v \cup \{\ell\}$
        end if
      end for
    end for
    $r = v \setminus w$
    $w = w \cup r$
  end while
  $m = \underset{\hat{w} \in w}{\text{argmax}} \left(F_+(\hat{w})\right)$
  return $v^+ = m,\ F_+(m),\ s$

lar graphs are known to be locally tree-like, so let us use Algorithm 3 to solve the STP over such graphs.

For each simulation, let us plot the suboptmal value of F_+ that is found by the algorithm versus the optimal value (computed by a brute-force maximum search). This is shown below for $\mu = 3$ and $\mu = 9$

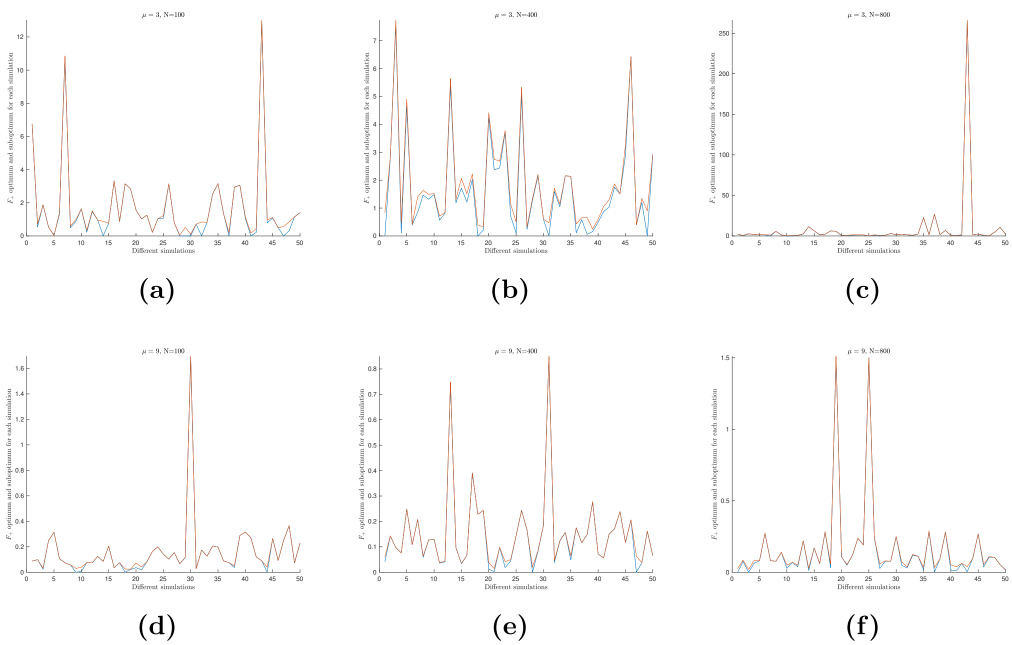


Figure 6.1: F_+ found by Algorithm 3 versus optimal value, for each simulation, for $N = 100, 400, 800$, for $\mu = 3$ (a)(b)(c) and $\mu = 9$ (d)(e)(f)

While the average percentages of visited nodes by the algorithm, for each N and μ, are shown below

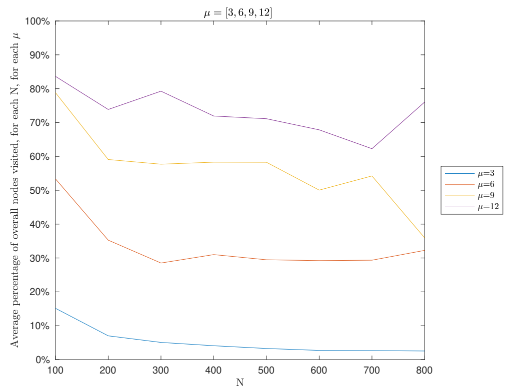


Figure 6.2: Average percentage of visited nodes averaged over 50 simulations for different degree of the regular graphs nodes μ

From Figure 6.1 we see that the value reached by the algorithm are usually close to the optimal values. On the other hand, from Figure 6.2 we see that as μ increases, the percentage of nodes visited increases as well. This means that the algorithm becomes much less efficient as the graph becomes more dense. At the same time, by comparing the upper plots of Figure 6.1 with the lower ones, in the latter cases by visiting a larger portion of the graph, F_+ tends to be more accurate.

As it has been shown in the above experiment, Algorithm 3, depending on the graph, could be too much expensive or too much strict. On the one hand, when the graph is more dense (it has more connections), the number of nodes that will be visited can grow almost exponentially: if each node for m steps has, on average, k improving sons, this would lead to an order of k^m nodes that have to be visited. Clearly we do not evaluate F_+ for the same node twice, but it is easy to see how this algorithm could not be effective in reducing computational complexity, leading to the visit of most of the graph. On the other hand, when the graph is more sparse, the condition of proceeding along *improving paths* (i.e. paths made up of improving nodes) is really strict and can stop the algorithm before finding the optimum, or at least a suboptimum.

For these two reasons, if we have more information about the graph on which we are studying the problem, let us build the following two modified version of Algorithm 3, denoted as *Strict version of Tree-like Single Targeting Algorithm* and *Relaxed version of Tree-like Single Targeting Algorithm*.

- *Strict version of Tree-like Single Targeting Algorithm*:
  The strict version of Algorithm 3 should be used on graphs that we think are not much sparse, in order to save computational time. The modification to the original version is the following: of all the nodes visited, save each improving node, but use only the node leading to the maximum improvement as the root for the next iteration. In this way the sequence of improving nodes of the algorithm consists again in a path, as for the tree original algorithm.

Algorithm 4 Strict Version of Tree-like Single Targeting Algorithm

---

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node v^-
  Initialization:
    Root node $r = v^-$
    Number of visited nodes $s = 0$
  while $r \neq \emptyset$ **do**
    $v = \emptyset$ empty set of improving nodes
    for $\ell \in \mathcal{O}(r)$ **do**
      $s = s + 1$
      Evaluate $F_+(\ell)$
      if $F_+(r) < F_+(\ell)$ **then**
        $v = v \cup \{\ell\}$
      end if
    end for
    $r = \operatorname{argmax}_{\hat{v} \in v} F_+(\hat{v})$
  end while
  return $v^+ = r$, $F_+(r)$, s

- *Relaxed version of Tree-like Single Targeting Algorithm*:
  The strict version of Algorithm 3 should be used on graphs that we think are very sparse, in order to find a sufficiently optimal result. The modification to the original version is the following: instead of saving and iterating the research only on improving nodes, we select the new roots of the algorithm also to non-improving nodes, with a certain tolerance. In this case we set the tolerance as 0.95 of the parent

F_+ value, but different values can be implemented depending on the application: if it is more important to look for the best result or saving time.

Algorithm 5 Relaxed Version of Tree-like Single Targeting Algorithm

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node v^-
  Initialization:
    Set of root nodes $r = \{v^-\}$
    Number of visited nodes $s = 0$
    $w = \emptyset$ set of visited nodes
  while $r \neq \emptyset$ **do**
    $v = \emptyset$ set of improving nodes
    for $\hat{r} \in r$ **do**
      for $\ell \in \mathcal{O}(\hat{r}) \setminus w$ **do**
        $s = s + 1$
        Evaluate $F_+(\ell)$
        if $F_+(\hat{r}) < 0.95 * F_+(\ell)$ **then**
          $v = v \cup \{\ell\}$
        end if
      end for
    end for
    $r = v \setminus w$
    $w = w \cup r$
  end while
  $m = \underset{\hat{w} \in w}{\operatorname{argmax}} (F_+(\hat{w}))$
  return $v^+ = m$, $F_+(m)$, s

Numerical Experiment 4. *Let us now test the strict version of Algorithm 3 on a real large-scale online social network: the Facebook ego-network, retrieved from Stanford Large Network Dataset Collection (https://snap.stanford.edu/data/egonets-Facebook.html). This dataset contains anonymized personal networks of connections between friends and the size of the graph associated is* $|\mathcal{V}| = 4039$*, while the number of links is equal to* $|\mathcal{E}| = 88234$*. Such graph is extremely sparse, since the number of nonzero elements* $|\mathcal{E}|/|\mathcal{V}|^2 \approx 5 * 10^{-3}$*. We then set 10 different STP over the graphs to test the performance of both Algorithm 3 and Algorithm 4. The 10 STP are generated by linking agent* $-$ *to 10 random different agents.*

We find that both the algorithms reach the optimum (computed by using the solving STP by means of the brute-force approach), so that the strict version is the most convenient algorithm to use. The distribution of the numbers of visited nodes for each algorithm are plotted in the boxplot below:

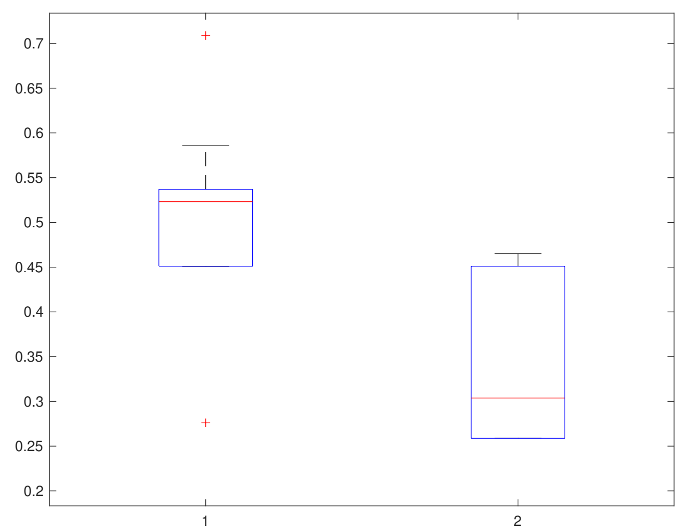


Figure 6.3: Algorithms 3 (on the left) and 4 (on the right) over Facebook ego-network: Proportion of visited nodes over 10 experiments

By comparing the median, and the 25th and 75th percentiles, we can see how the strict version improves the computational complexity of the algorithm.

Tree-like Heuristic for single targeting with $k_- > 1$

In an analogous way, it is possible to produce an algorithm for the cases where the single targeting is done when the number of links placed by strategic agent $-$ is greater than one. In this case, we use a generalized version of the tree-like heuristic, as summarized in Algorithm 6. In particular, among the nodes linked to $-$, we select the one with smallest degree as the root v^- from which Algorithm 6 is started. The reasoning behind this algorithm, supported by experimental results, is that in sparser graphs it is easier to move away from less relevant nodes rather than vice versa. Indeed, by starting the algorithm from high degree nodes, the first steps would generally be more affected by the influence of $-$, compromising the accuracy.

Algorithm 6 Tree-like Single Targeting Algorithm, $|\mathcal{V}_-| > 1$

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node set $\mathcal{V}_-$

- **Initialization:**
  - Root node $r \in \operatorname{argmin}_{v^- \in \mathcal{V}_-} d(v^-)$
  - Number of visited nodes $s = 0$
- **while** $r \neq \emptyset$ **do**
  - $v = \emptyset$ set of improving moves
  - **for** $\ell \in \mathcal{O}(r)$ **do**
    - $s = s + 1$
    - Evaluate $F_+(\ell)$
    - **if** $F_+(r) < F_+(\ell)$ **then**
      - $v = v \cup \{\ell\}$
    - **end if**
  - **end for**
  - $r = \operatorname{argmax}_{w \in v} F_+(w)$
- **end while**
- **return** $v^+ = r$, $F_+(r)$, s

Tree-like Heuristic for OTP

Thanks to these results, it is then possible to construct the *Tree-like Heuristic* for the OTP. In particular, I propose an algorithm that applies the singular targeting version of the Tree-like Heuristic in a greedy manner, in order to approximate the Greedy Heuristic. Specifically, in order to increase the accuracy, this is done by selecting at each step a different root node among the ones linked to $-$, as summarized in Algorithm 7.

Algorithm 7 Tree-like Optimal Targeting Algorithm

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node set $\mathcal{V}_-$, budget k
 Initialization:
  Node set $\mathcal{V}_-$
  Empty sets $\mathcal{V}_+ = \emptyset, \mathcal{F}_+ = \emptyset$
  Number of visited nodes $s = 0$
 for $i = 1, \ldots, k$ **do**
  $r =$ (mod$(i-1, k)+1$)-th element of $\mathcal{V}_-$
  while $r \neq \emptyset$ **do**
   $v = \emptyset$ set of improving moves
   for $\ell \in \mathcal{O}(r)$ **do**
    $s = s + 1$
    Evaluate $F_+(\ell)$
    if $F_+(r) < F_+(\ell)$ **then**
     $v = v \cup \{\ell\}$
    end if
   end for
   $r = \operatorname*{argmax}_{w \in v} F_+(w)$
  end while
  $\mathcal{V}_+ = \mathcal{V}_+ \cup \{r\}$
  $\mathcal{F}_+ = \mathcal{F}_+ \cup \{F_+(r)\}$
 end for
 return $\mathcal{V}_+$, $\mathcal{F}_+$, s

Numerical Experiment 5. *Let us now compare two Tree-like Heuristics by running them on random generated Erdos-Renyi graphs of parameters* $N = 200$ *and* $p = 0.1$. *We generate* 15 *random graphs and we link* $k_- = 3$ *nodes randomly selected to the strategic agent* $-$. *Then, we perform one Tree-*

like Heuristic using a strict version of the Tree-like algorithm, one Tree-like Heuristic using the standard version, and Greedy Heuristic. The results are reported below:

	Average F_+	Average number of computations
Tree-like Heuristic (strict)	51.09	54.53
Tree-like Heuristic	51.25	103.87
Greedy Heuristic	51.31	200

We can easily see how the F_+ values are really close to each other, while the average number of computations, determining the speed of the algorithm, is halved thanks to the tree-like algorithm, and halved again if the strict version is used.

6.3 Blocking Heuristic

In Chapter 5.3 we solved the OTP for the complete graph, finding that if $p+k_+ > q$, it is optimal to block the influence of agent $-$ placing $\min\{k_+, q\}$ edges to the nodes linked to $-$, otherwise it is not. On the other hand, if we wanted to find a suboptimum of the OTP for the complete graph by using the greedy Algorithm 2, at each iteration the algorithm would have tested the condition $p^{(k)} + 1 > q^{(k)}$, where $p^{(k)}$, $q^{(k)}$ are p, q at iteration $k = 1, \dots, k_+$.

This suggests that when relating to the greedy algorithm, if the graph is dense enough we are losing at each iteration some amount of the optimal result, and this is more true as the graph is closer to a complete graph. We then exploit this intuition, coming from the result on the complete graph, to build the following algorithm. Let us denote by $\mathcal{A}^-$ the set of nodes linked to $-$, while by $\mathcal{A}^{(0)}$ the set of initial nodes linked to $+$.

Algorithm 8 Blocking Heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, set node $\mathcal{A}^{(0)}$, number of available links k_+
  Initialization:
    $\mathcal{A}_0 = \emptyset$, $s = 0$
  if $k_+ > |\mathcal{A}^- \setminus \mathcal{A}^{(0)}| - |\mathcal{A}^{(0)} \setminus \mathcal{A}_-|$ **then**
    $\mathcal{A}_s = \mathcal{A}^- \setminus \mathcal{A}^{(0)}$, $s = |\mathcal{A}^- \setminus \mathcal{A}^{(0)}|$
  end if
  if $k_+ > s$ **then**
    for $i \in \{s+1, \dots, k_+\}$ **do**
      $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \underset{v \in \mathcal{R}}{\operatorname{argmax}} \{\Delta(v|\mathcal{A}_{i-1})\}$
    end for
  else
    $\mathcal{A}_{k_+} = \mathcal{A}_s$
  end if
  return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

Algorithm 8, in practice, compares the $|\mathcal{A}^{(0)} \setminus \mathcal{A}_-| + k_+$ overall edges of agent $+$ not linked to $-$, with the $|\mathcal{A}^- \setminus \mathcal{A}^{(0)}|$ edges not linked to $+$ of agent $-$. If the comparison tells that the former il larger, the k_+ edges of $+$ yet to be placed are used until possible to target the nodes in $\mathcal{A}^- \setminus \mathcal{A}^{(0)}$. Then, if some edges are still available to agent $+$, they are placed by following the greedy Algorithm 2. On the other hand, if the first condition is not satisfied, it simply reduces to the Greedy Heuristics.

In such a way it is possible to improve both the accuracy and the cost of the heuristic. Indeed, if the first condition is satisfied, the complexity passes from $O(N^3 k_+)$ to $O(N^4(k_+ - \min\{k_+, q\})$.

However, the more the degrees are heterogeneous among the nodes, the more the case will be far from the complete graph scenario. Then, the opponent could be linked to nodes that have a small influence on the overall graph, and blocking them by targeting the same nodes could not be optimal in such cases. For this reason, a modification to the algorithm is considered where the only nodes that are considered for the blocking targeting are the ones with a larger degree. In this way, the decision of blocking or not the opponent targets is determined by comparing the number k_-^* of opponent's high degree nodes with $k_+ + k_+^{0*}$, where k_+^{0*} is the number of high degree previously placed links, rather than comparing k_- with $k_+ + k_+^0$.

Algorithm 9 Modified Blocking Heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, set node $\mathcal{A}^{(0)}$, number of available links k_+
  Initialization:
    $\mathcal{A}_0 = \emptyset$, $s = 0$
    $\mathcal{D}$ set containing the $N/10$ distinct nodes with highest degree
    $\mathcal{A}^{-*} = \mathcal{A}^- \cap \mathcal{D}$, $\mathcal{A}^{(0)*} = \mathcal{A}^{(0)} \cap \mathcal{D}$
  if $k_+ > |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}| - |\mathcal{A}^{(0)*} \setminus \mathcal{A}^{-*}|$ **then**
    $\mathcal{A}_s = \mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}$, $s = |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}|$
  end if
  if $k_+ > s$ **then**
    for $i \in \{s+1, \dots, k_+\}$ **do**
      $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \underset{v \in \mathcal{R}}{\operatorname{argmax}} \{\Delta(v|\mathcal{A}_{i-1})\}$
    end for
  else
    $\mathcal{A}_{k_+} = \mathcal{A}_s$
  end if
  return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

The results of this modified version of the Blocking Heuristic are much better than the standard version ones when the opponent is not placed optimally.

6.4 Degree Heuristic

I now build the *Degree Heuristic*, a heuristic that makes use of the node degrees of the graph. Indeed, when the graph has a heterogeneous degree distribution, the nodes with highest degrees are also the most influential ones. This is both intuitive and empirically found in various performed experiments. From this notion, I build a greedy algorithm that instead of looking for the optimum over the entire node set, it performs the optimum search restricted to the nodes with highest degrees. In this way, it is possible to find an approximation of the Greedy Heuristics results, additionally reducing the OTP complexity.

Algorithm 10 Degree Heuristic (Greedy Search Version) for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, number of available links k_+, vector of nodes degree d

Initialization:

$\mathcal{A}_0 = \emptyset$

$M = \min\{10 * k_+, |\mathcal{R}|\}$

$\mathcal{M}$ set containing the M distinct nodes with highest degrees

for $i \in \{1, \dots, k_+\}$ **do**

$\mathcal{A}_i = \mathcal{A}_{i-1} \cup \underset{v \in \mathcal{M}}{\operatorname{argmax}} \{\Delta(v|\mathcal{A}_{i-1})\}$

end for

return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

In this version of the Degree Heuristic the algorithm selects the $10k_+$ nodes with highest degree. Then, it performs the greedy algorithm over that nodes to find a suboptimum of the problem. In this way, the computational complexity goes from the $O(N^4 k_+)$ of the greedy algorithm to $O(N^3 10 k_+^2)$.

Example 1 (Blocking + Degree Heuristic). *Let us combine the Blocking Heuristic with the Degree Heuristic in order to show how these heuristics can be easily mixed together, in order to reach better accuracy or to reduce complexity. The algorithm of this mixed heuristic is shown below.*

Algorithm 11 Blocking + Degree Heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, number of available links k_+
  Initialization:
    $\mathcal{A}_0 = \emptyset$, $s = 0$
    $M = \min\{10 * k_+, |\mathcal{R}|\}$
    $\mathcal{M}$ set containing the M distinct nodes with highest degrees
    $\mathcal{D}$ set containing the $N/10$ distinct nodes with highest degree
    $\mathcal{A}^{-*} = \mathcal{A}^- \cap \mathcal{D}$, $\mathcal{A}^{(0)*} = \mathcal{A}^{(0)} \cap \mathcal{D}$
  if $k_+ > |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}| - |\mathcal{A}^{(0)*} \setminus \mathcal{A}_{-*}|$ **then**
    $\mathcal{A}_s = \mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}$, $s = |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}|$
  end if
  if $k_+ > s$ **then**
    for $i \in \{s+1, \dots, k_+\}$ **do**
      $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \operatorname*{argmax}_{v \in \mathcal{M}} \{\Delta(v|\mathcal{A}_{i-1})\}$
    end for
  else
    $\mathcal{A}_{k_+} = \mathcal{A}_s$
  end if
  return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

This algorithm, combining the Blocking Heuristic with the Degree Heuristics, states that if the condition $|\mathcal{A}^{(0)} \setminus \mathcal{A}_-| + k_+ > |\mathcal{A}^- \setminus \mathcal{A}^{(0)}|$ is satisfied, the nodes in $\mathcal{A}^-$ have to be chosen before running the greedy algorithm. Then, if some edges are left, the greedy algorithm is applied among the $10 * k_+$ nodes with highest degree. This leads to a complexity of $O(N^3 10 k_+(k_+ - \min\{k_+, q\})$, if the condition above is satisfied, and $O(N^3 10 k_+^2)$ if not.

6.5 Zero-cost Heuristics

I now present two *Zero-cost Heuristics* that can be used to approximate the OTP solution without performing any computation. These heuristics can be used as a comparison for the other heuristics, in order to understand when these are needed to improve accuracy, or when their cost is not justified.

- The first one consists in selecting the k_+ nodes of highest degree (if there are more subsets with this property, select one of them randomly).

Algorithm 12 Zero-cost Standard Heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, number of available links k_+
Initialization:
$\mathcal{D}$ set of nodes with top-k_+ degree
$\mathcal{A}_{k_+} = \mathcal{D}$
return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

- The second one, motivated by the complete graph and the Blocking Heuristic results, compares the available budget with the number of high degree nodes that are already placed in a similar manner, and places the remaining links to the highest degree nodes.

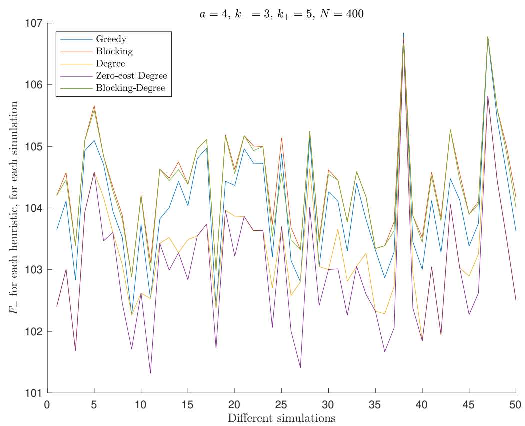
Algorithm 13 Zero-cost Blocking Heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node $\mathcal{A}^-$, set node $\mathcal{A}^{(0)}$, number of available links k_+
Initialization:
$\mathcal{A}_0 = \emptyset$, $s = 0$
$\mathcal{D}$ set of nodes with top-10% degree
$\mathcal{A}^{-*} = \mathcal{A}^- \cap \mathcal{D}$, $\mathcal{A}^{(0)*} = \mathcal{A}^{(0)} \cap \mathcal{D}$
if $k_+ > |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}| - |\mathcal{A}^{(0)*} \setminus \mathcal{A}^{-*}|$ **then**
$\mathcal{A}_s = \mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}$, $s = |\mathcal{A}^{-*} \setminus \mathcal{A}^{(0)*}|$
end if
if $k_+ > s$ **then**
$\mathcal{D}'$ set containing the $k_+ - s$ distinct nodes with highest degree
$\mathcal{A}_{k_+} = \mathcal{A}_s \cup \mathcal{D}'$
else
$\mathcal{A}_{k_+} = \mathcal{A}_s$
end if
return $\mathcal{A}_{k_+}$, $F_+(\mathcal{A}_{k_+})$

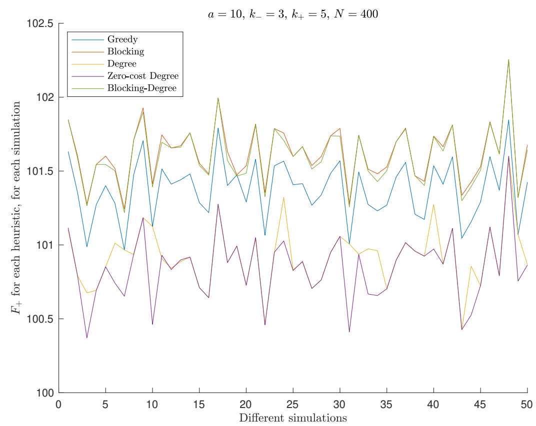
Numerical Experiment 6 (Greedy Heuristics Comparison)**.** *Let us now compare some of these heuristics by running them on random generated Erdos-Renyi graphs of parameters* $N = 400$ *and* $p = a\frac{logN}{N}$*. For each parameter* $a \in \{10, 4\}$*, we generate* 50 *random graphs and we link* $k_- = 3$ *nodes randomly selected to the strategic agent* $-$ *(so that it becomes conve-*

nient to use the Blocking algorithms). Then, we perform the five heuristics over such graphs and we compare the results.

Let us plot the results for each simulation, for different a values:



(a)



(b)

Figure 6.4

For $a = 10$ *the generated graphs are the ones that are the closest to be complete graphs, with an average number of edges of* $\mathbb{E}[|\mathcal{E}|] = a(N-1)logN = 23896$ *and a nonzero elements percentage about* 15%. *Here, the Blocking Heuristic advantage should be more evident. We can se how in all the experiments, the best performing solution seems to be the Blocking Heuristic, followed by the Blocking+Degree Heuristic and then by the Greedy Heuristic, whereas the worst solutions are given by the two Degree-based Heuristics: in particular, the zero-cost one is the one that performs the worst.*

However, to better see the actual advantage of such heuristic over the others, let us repeat the experiment above, by running the same heuristics to random generated Erdos-Renyi graph for a larger number of a values and let us average the responses, over each a, to get a curve of average performance of each algorithm. By taking $a \in \{1.5, 1.6032, 1.7316, 1.8911, 2.0893, 2.3357, 2.6419, 3.0226, 3.4957, 4, 4.0838, 4.8147, 5.7231, 6.8522, 8.2557, 10\}$ *we get:*

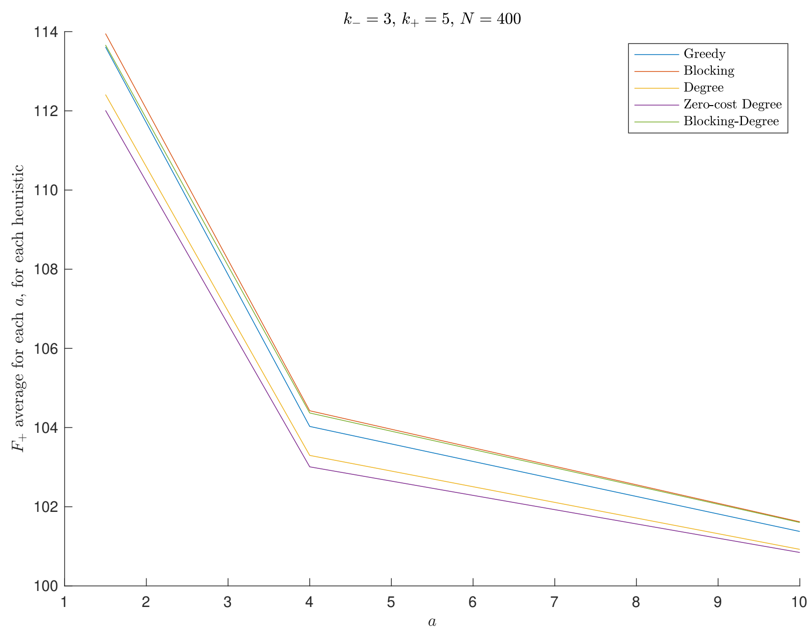


Figure 6.5

we can see how the performances of the Blocking and Blocking+Degree heuristics are the ones that give the best results, whereas the zero-cost approach here performs the worst.

6.6 Heuristic Choice Scheme

Let us now summarize all previous heuristics for the OTP by combining them in the visual scheme of Figure 6.6. In this scheme, the composition of a heuristic is performed moving from the black central node towards the leafs. The circles are colored in terms of the computational complexity of the choices. In particular, the greedy heuristics are colored in blue, and they are graduated towards lighter colors when the complexity is reduced. Conversely, zero-cost heuristics are colored in orange.

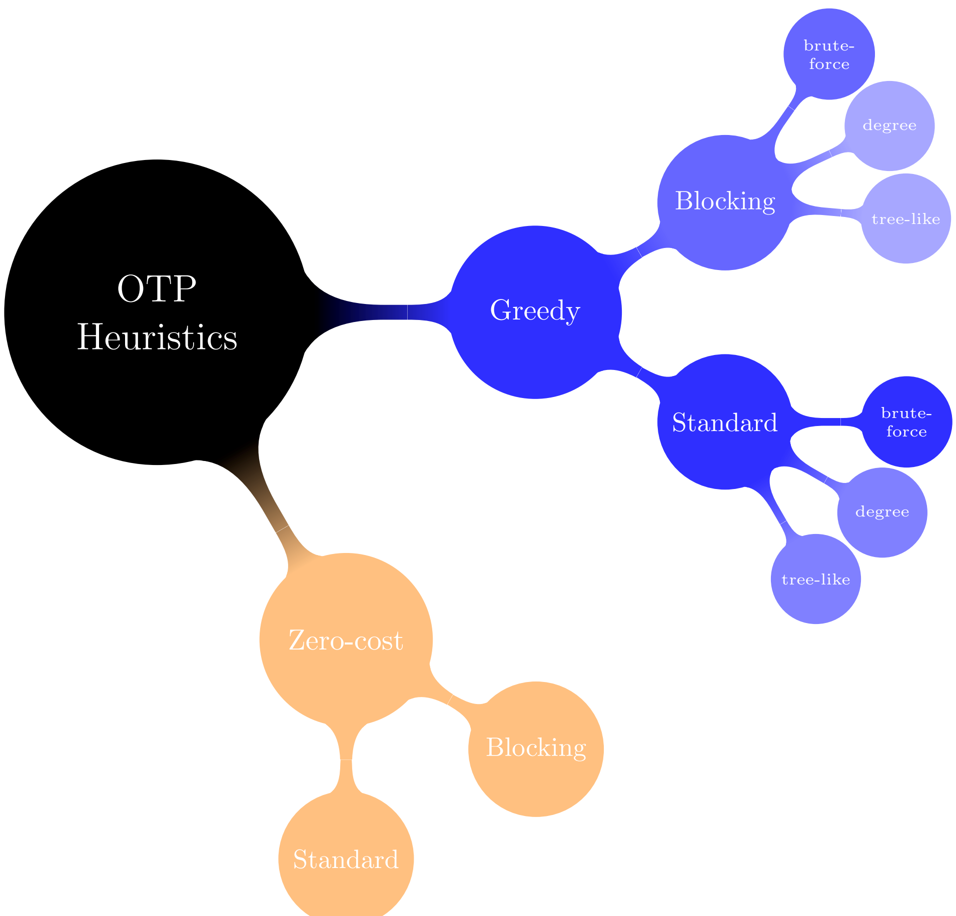
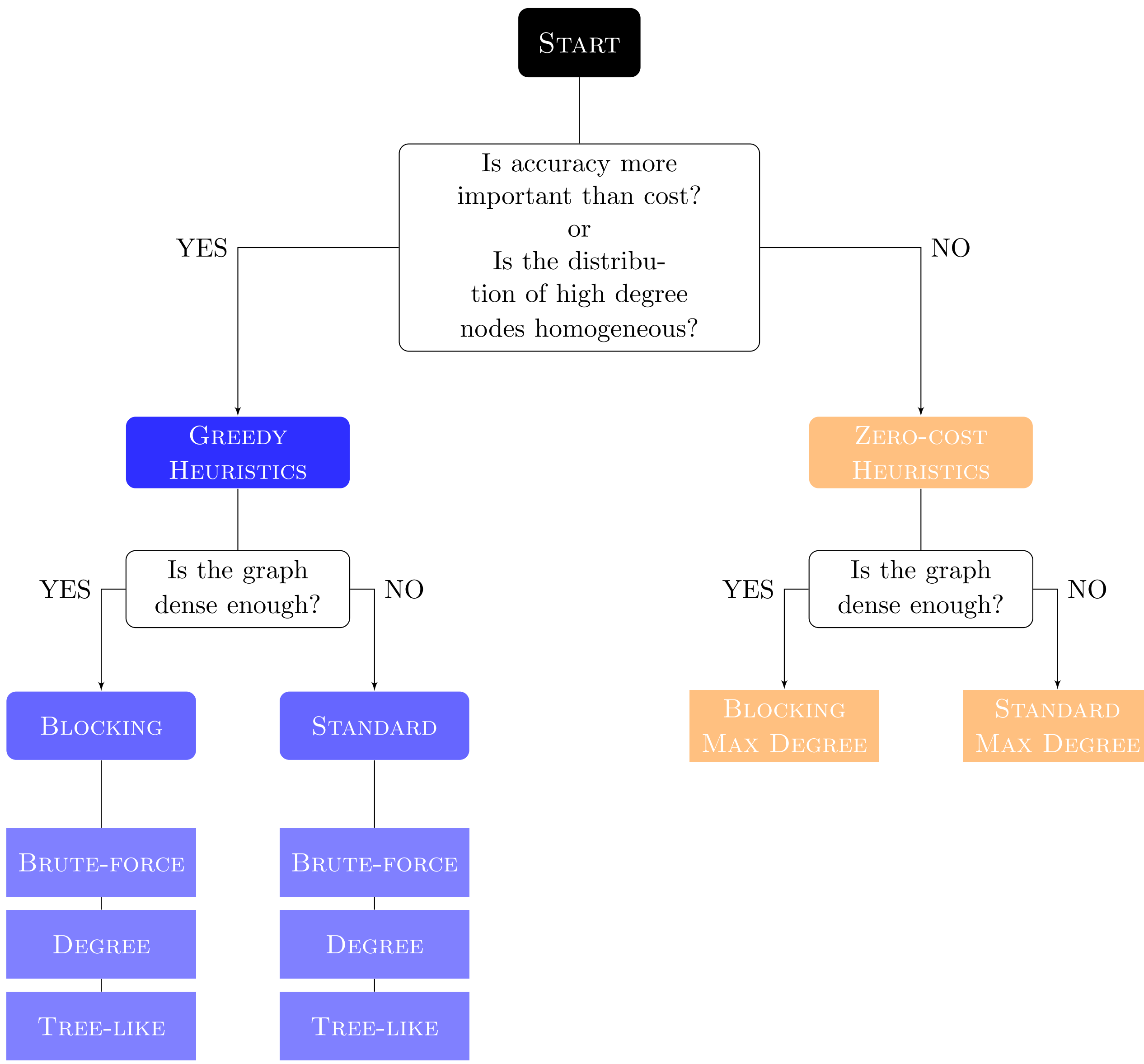


Figure 6.6: Scheme of OTP heuristics, colored in terms of computational complexity. Dark blue: more expensive heuristics; light blue: less expensive; orange: zero-cost

Another level of interpretation is determined by the accuracy of the heuristics. Specifically, they have been ordered clockwise in terms of generally better performance. However, notice that this is much less rigorous, since heuristic accuracy is strongly dependent on the underlying graph.

In particular, a simple scheme of work providing some of the main reasons behind different choices can be summarized as follows



The first decision split the heuristics between greedy and zero-cost ones. Here the choice is done depending mostly on the accuracy. Indeed, the complexity difference is extremely large between the two options. On the other hand, a zero-cost heuristic could be extremely accurate if the distribution of high degree nodes is heterogeneous, and some extremely high degree nodes are present.

Then, the second decision is among blocking or standard heuristics. While the general results, show a generally better performance for Blocking Heuristics, let us highlight that the modified version of the algorithm is always suggested. Indeed, when the opponent is not linked to high degree nodes, it is not convenient to block her in such nodes. On the other hand, this heuristics sometimes performs worse than the greedy one, when the graph is sparser.

Concluding, the reasons behind the three last greedy options (brute-force, degree, and tree-like) are mainly based on the graph structure. Indeed, if the graph is locally tree-like or sparse, a tree-like approach would be well-performing, improving the complexity of the heuristic. In addition, in this case, one has also to choose among the three modified versions of the Tree-like algorithm in the STP steps (standard, strict, and relaxed). On the other hand, if the graph is well split between high and low degree nodes, a degree-based approach would be a good choice. Conversely, if none of these notions are true or no information is known about the graph, a brute-force approach is the safest option.

Nevertheless, notice that a good approach would probably be to study a subset of the social network to compare different heuristics, so as to better understand their accuracy and cost before performing the computations with the entire network.

Chapter 7

Conclusions

In this thesis, I provided different methods for solving the Optimal Targeting Problem in a smart way.

First, I presented the theoretical tools thanks to which one can describe a network and its properties. By formalizing the graph theoretical notions with well known algebraic tools, it is possible to derive the main properties of the graph by looking at the spectral properties of the matrices involved. In this context, the French-DeGroot opinion dynamics model is presented and the competition between two agents is set. Such competition involves two strategic agents of opposite opinion who try to influence the network providing the same opinion to the population. However, the consequent optimization problem is computationally hard and feasible heuristics are needed to solve the problem.

Therefore, I proved the monotoncity and submodularity of the average opinion, making use of the Markov Chain representation of opinion dynamics over graphs. This justified the main greedy approach upon which my heuristics are built.

Then, I exploited the equivalent electrical representation of the graph to solve the singular targeting problem for the line graph. Next, I extended this result to general tree graphs, thanks to which I built an algorithm allowing to drastically reduce the complexity of a brute-force approach for STP over tree-graphs. Upon this, I successively extended the algorithm for STP over tree-like/sparse graphs, and then to the more general OTP, by combining STP solutions in a greedy manner.

Moreover, the solution of the OTP for the complete graph is provided. This result is quite simple and powerful since it provides the answer to the question whether it is convenient or not to target to same nodes of the opponent to block her influence, by simply looking at the available and placed links of the two strategic nodes. Upon this result, another heuristic is built, consisting in the simple strategy to block the opponent or not, depending on a similar criteria.

Then, motivated by intuition, two kinds of degree based heuristics are built. One to reduce the complexity of the greedy algorithms, one to provide a zero-cost solution.

A scheme of all the possible heuristics combinations is then provided to better understand which approach one should have depending on the accuracy vs cost trade-off, and the underlying graph.

Appendices

Appendix A

Alternative Proofs

A.1 Alternative Monotonicity Proof

Proof.

$$\begin{aligned} F_+(\mathcal{B}) &= \sum_{i\in\mathcal{R}} \bar{x}_i^{(\mathcal{B})} \\ &= \sum_{i\in\mathcal{R}} [(I - Q^{11,\mathcal{B}})^{-1} Q^{12,\mathcal{B}} x^{\mathcal{S}}]_i \\ &= \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} \left[(I - Q^{11,\mathcal{B}})^{-1}\right]_{in} Q^{12,\mathcal{B}}_{nm} x^{\mathcal{S}}_m \right) \end{aligned}$$

analogously

$$F_+(\mathcal{A}) = \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} [(I - Q^{11,\mathcal{A}})^{-1}]_{in} Q^{12,\mathcal{A}}_{nm} x^{\mathcal{S}}_m \right)$$

where the superscripts $\mathcal{A}$ and $\mathcal{B}$ on the matrices Q^{11} and Q^{12} are used to identify the two matrices in the different cases. Indeed each new connection with the strategic node $+$ of the nodes that are in $\mathcal{B}$ but not in $\mathcal{A}$ has an impact on the whole matrix Q, even if on the weight matrix W the only block affected is the W^{12} one, having 1 values on the corresponding nodes linked to the strategic agents: in particular $\mathcal{B}$ will have all the 1 elements of

$\mathcal{A}$, and the ones corresponding to the elements in $\mathcal{B} \setminus \mathcal{A}$, since $\mathcal{A} \subseteq \mathcal{B}$.

So

$$F_+(\mathcal{B}) - F_+(\mathcal{A}) = \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} \left([(I - Q^{11,\mathcal{B}})^{-1}]_{in} Q^{12,\mathcal{B}}_{nm} - [(I - Q^{11,\mathcal{A}})^{-1}]_{in} Q^{12,\mathcal{A}}_{nm} \right) x^{\mathcal{S}}_m \right)$$

$$= \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} \left(\Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{B}} W^{11})^k \Big]_{in} [D^{-1}_{\mathcal{B}}]_{nn} W^{12,\mathcal{B}}_{nm} - \Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{A}} W^{11})^k \Big]_{in} [D^{-1}_{\mathcal{A}}]_{nn} W^{12,\mathcal{A}}_{nm} \right) x^{\mathcal{S}}_m \right)$$

$$\geq \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} \left(\Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{B}} W^{11})^k \Big]_{in} [D^{-1}_{\mathcal{B}}]_{nn} W^{12,\mathcal{B}}_{nm} - \Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{B}} W^{11})^k \Big]_{in} [D^{-1}_{\mathcal{B}}]_{nn} W^{12,\mathcal{B}}_{nm} \right) x^{\mathcal{S}}_m \right)$$

$$= \sum_{i\in\mathcal{R}} \left(\sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{R}} \Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{B}} W^{11})^k \Big]_{in} [D^{-1}_{\mathcal{B}}]_{nn} \left(W^{12,\mathcal{B}}_{nm} - W^{12,\mathcal{A}}_{nm} \right) x^{\mathcal{S}}_m \right)$$

$$\geq 0$$

where the first inequality comes from the fact that $[D^{-1}_{\mathcal{B}}]_{nn} \leq [D^{-1}_{\mathcal{A}}]_{nn}$ for each n, indeed the matrix D with the degrees of the nodes on the diagonal counts how many links each node has, then having a *plus one* for each row corresponding to a node in $\mathcal{B} \setminus \mathcal{A}$. The last inequality holds since $\Big[\sum_{k\geq 0} (D^{-1}_{\mathcal{B}} W^{11})^k \Big]_{in} \geq 0$, $[D^{-1}_{\mathcal{B}}]_{nn} \geq 0$ and $\left(W^{12,\mathcal{B}}_{nm} - -W^{12,\mathcal{A}}_{nm} \right) \in \{0, 1\}$ for each i, n, m in their domain, while $x^{\mathcal{S}}_m$ takes only 1 values in correspondence to each $\left(W^{12,\mathcal{B}}_{nm} - -W^{12,\mathcal{A}}_{nm} \right) \neq 0$, since the strategic node that influences the matrices $W^{12,\mathcal{B}}$ and $W^{12,\mathcal{A}}$ is just the $+$ one, i.e. all the -1 contributions are canceled out. □

A.2 Alternative Submodularity Proof

Proof. In order to prove the proposition, we relate to the work made by Yildiz et al. [11] where they studied the optimal placement of strategic agents in a voter model dynamics. In this work they successfully proved the submodularity of what is the analogous of our agent's asymptotic opinion, so that we can build our proof as a corollary of their result.

In order to do so let us present the analogies between the two problems. First of all, the asymptotic opinion of a regular agent following the DeGroot dynamics, that we have seen as being representable as a convex hull of the strategic agent's opinion, can be formulated in a probabilistic context. Consequently, for a regular agent i, the coefficient ahead the opinion x_s of each generic strategic node s can be seen as the probability that the simple random walk $Z(t)$ initiated at node i with transition probability matrix Q will be absorbed by the absorbing state s, i.e.

$$\begin{aligned}
\bar{x}_i &= \sum_{s\in\mathcal{S}} (HQ^{12})_{is} x_s \\
&= \sum_{s\in\mathcal{S}} \lim_{t\to+\infty} \mathbb{P}(Z(t)=s|Z(0)=i)x_s \\
&= \lim_{t\to+\infty} \mathbb{P}(Z(t)=+|Z(0)=i) - \lim_{t\to+\infty} \mathbb{P}(Z(t)=-|Z(0)=i) \\
&= p_i - (1-p_i) \\
&\quad \text{where } p_i = \lim_{t\to+\infty} \mathbb{P}(Z(t)=+|Z(0)=i) \\
&= 2p_i - 1
\end{aligned}$$

Let us now present the analogies with the Yildiz et al. model. In their work they studied the optimal placement of k strategic nodes in a slightly different setting, where all the agents' opinion can assume values in $\{0,1\}$, by copying the other agents' opinion as a continuous-time Markov process, according to the transition rate matrix Λ, while the strategic nodes are collected in the non-empty set $\mathcal{V}_0$, if type-zero, and in $\mathcal{V}_1$, if type-one. Additionally, such optimal strategic agent placement (OSAP) for the type-one agent corresponds to the choice of the optimal $k > 0$ regular nodes to be added to $\mathcal{V}_1$ in order to maximize the sum of the asymptotic opinions' expected value, i.e. solving

$$\max_{\ell_1,\dots\ell_k\in\mathcal{V}\setminus(\mathcal{V}_0\cup\mathcal{V}_1)}\sum_{i\in\mathcal{V}}\mathbb{E}[\bar{x}_i]^{(\ell_1\cup\dots\cup\ell_k)} \tag{A.1}$$

where the superscript of $\mathbb{E}[\bar{x}_i]$ is just a notation to specify the set of nodes to be added to $\mathcal{V}_1$.

On the other hand, analogously to our case, the expected value of the asymptotic opinion of a regular agent i can be written as the probability that the continuous-time random walk $\tilde{Z}(t)$ initiated at node i with transition rate matrix Λ and jump chain $\tilde{Q}$ will be absorbed by the set $\mathcal{V}_1$, i.e.

$$\begin{aligned}\mathbb{E}[\bar{x}_i] &= \lim_{t\to+\infty}\mathbb{P}(\tilde{Z}(t)\in\mathcal{V}_1|\tilde{Z}(0)=i)\\ &= \tilde{p}_i\end{aligned}$$

It is now evident how, by studying the same graph $\mathcal{G}$ with $W_{ij}=\Lambda_{ij}$ for $i\neq j$, it follows that $Q=\tilde{Q}$ (since Q_{ii} can be put to 0 without changing the asymptotic opinion, for each i) [*o da definire meglio o da omettere*] and then

$$p_i=\tilde{p}_i$$

Successively we will simply treat the two problems as equivalent taking for granted these necessary conditions.

Let us now consider the Lemma proved by Yildiz et al. that we want to exploit. Such lemma successfully proved the submodularity of $\sum_{i\in\mathcal{V}}\tilde{p}_i^{(\ell_1\cup\dots\cup\ell_k)}$ but actually a more general result has been implicitly proved, i.e. that the very same $\tilde{p}_i$ is submodular, not only the sum over all the agents. Hence, we try to exploit such result by connecting our optimal targeting problem to the optimal strategic agent placement.

First of all, let us take a generic $\mathcal{B}\subseteq\mathcal{V},\mathcal{A}\subseteq\mathcal{B}$ and $v\in\mathcal{V}\setminus\mathcal{B}$, that will form the four cases $\mathcal{A},\hat{\mathcal{A}}=\mathcal{A}\cup\{v\},\mathcal{B},\hat{\mathcal{B}}=\mathcal{B}\cup\{v\}$ in terms of nodes directly connected to + that we need to compare to prove the submodularity. Then, let us build for each case $\mathcal{U}$ a corresponding graph $\mathcal{G}(\mathcal{U})$ where the strategic node + is not unique in general, but there are as many type-one strategic nodes $+^{(i)}$ as the cardinality of $\mathcal{U}$, each one with degree of exactly one. By also writing the node sets as function of the case considered, we can summarize the four cases in

$$\begin{cases} \mathcal{R}(\mathcal{A}) = \mathcal{R}(\hat{\mathcal{A}}) = \mathcal{R}(\mathcal{B}) = \mathcal{R}(\hat{\mathcal{B}}) = \mathcal{R} \\ \mathcal{S}(\hat{\mathcal{B}}) = \mathcal{S}(\mathcal{B}) \cup \{+^{(v)}\} = \mathcal{S}(\hat{\mathcal{A}}) \cup \{+^{(1)}, \dots, +^{(m)}\} = \mathcal{S}(\mathcal{A}) \cup \{+^{(v)}\} \cup \{+^{(1)}, \dots, +^{(m)}\} \end{cases}$$

where $+^{(v)}$ is the strategic node directly connected to v while $+^{(1)}, \dots, +^{(m)}$ are the strategic nodes connected to the m nodes belonging to $\mathcal{B} \setminus \mathcal{A}$.
Such new representation doesn't modify the previous dynamics and in particular the asymptotic opinions, as we can see by studying the electrical network representation, since the strategic nodes can be considered as voltage sources.

Let us now build other four different graphs $\mathcal{G}'(\mathcal{U})$, one for each case, where $\mathcal{V}'(\mathcal{U}) = \mathcal{V}(\hat{\mathcal{B}})$, $\mathcal{E}'(\mathcal{U}) = \mathcal{E}(\hat{\mathcal{B}})$ for $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$ and such that

$$\begin{cases} \mathcal{R}'(\hat{\mathcal{B}}) = \mathcal{R} \\ \mathcal{S}'(\hat{\mathcal{B}}) = \mathcal{S}(\hat{\mathcal{B}}) \end{cases}$$

$$\begin{cases} \mathcal{R}'(\mathcal{B}) = \mathcal{R} \cup \{+^{(v)}\} \\ \mathcal{S}'(\mathcal{B}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus \{+^{(v)}\} = \mathcal{S}(\mathcal{B}) \end{cases}$$

$$\begin{cases} \mathcal{R}'(\hat{\mathcal{A}}) = \mathcal{R} \cup \{+^{(1)}, \dots, +^{(m)}\} \\ \mathcal{S}'(\hat{\mathcal{A}}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus \{+^{(1)}, \dots, +^{(m)}\} = \mathcal{S}(\hat{\mathcal{A}}) \end{cases}$$

$$\begin{cases} \mathcal{R}'(\mathcal{A}) = \mathcal{R} \cup \{+^{(1)}, \dots, +^{(m)}\} \cup \{+^{(v)}\} \\ \mathcal{S}'(\mathcal{A}) = \mathcal{S}(\hat{\mathcal{B}}) \setminus (\{+^{(1)}, \dots, +^{(m)}\} \cup \{+^{(v)}\}) = \mathcal{S}(\mathcal{A}) \end{cases}$$

where we built such new four configurations in such a way that the numbers of nodes and edges always stay the same while the discriminant between the different cases is just the set of type-one strategic nodes, unambiguously determined by the nodes directly connected to the strategic + in the original graph.

Now, by using the stronger result coming from Yildiz et al. on $\mathcal{G}'(\mathcal{A})$, combined with the previously proved equivalence of the two settings, it is possible to say that

$$p_i'^{(\hat{\mathcal{A}})} - p_i'^{(\mathcal{A})} \geq p_i'^{(\hat{\mathcal{B}})} - p_i'^{(\mathcal{B})} \qquad \forall i \in \mathcal{R} \cup \{+^{(1)}, \dots, +^{(m)}\} \cup \{+^{(v)}\}$$

At the same time we have that, for $\mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}$, the main difference between $\mathcal{G}'(\mathcal{U})$ and $\mathcal{G}(\mathcal{U})$ is that the former presents regular nodes connected to $\hat{\mathcal{B}} \setminus \mathcal{U}$ where in $\hat{\mathcal{B}}$ there are strategic nodes, while the latter simply doesn't have such connected nodes. Nevertheless, such regular nodes have all degree one, by construction, so that they do not influence the asymptotic opinion of each other node. indeed, we can easily see how, by building the electrical network analogous, such nodes are all short-circuited with their only connected node, hence not influencing the other nodes' voltage. Consequently we can say that

$$p_i^{(\mathcal{U})} = p_i'^{(\mathcal{U})} \qquad \forall i \in \mathcal{R},\ \mathcal{U} = \mathcal{A}, \hat{\mathcal{A}}, \mathcal{B}, \hat{\mathcal{B}}$$

hence

$$\begin{aligned} p_i^{(\hat{\mathcal{A}})} - p_i^{(\mathcal{A})} &\geq p_i^{(\hat{\mathcal{B}})} - p_i^{(\mathcal{B})} \qquad \forall i \in \mathcal{R} \\ \sum_{i \in \mathcal{R}} (2p_i^{(\hat{\mathcal{A}})} - 1) - \sum_{i \in \mathcal{R}} (2p_i^{(\mathcal{A})} - 1) &\geq \sum_{i \in \mathcal{R}} (2p_i^{(\hat{\mathcal{B}})} - 1) - \sum_{i \in \mathcal{R}} (2p_i^{(\mathcal{B})} - 1) \qquad \forall i \in \mathcal{R} \\ F_+(\hat{\mathcal{A}}) - F_+(\mathcal{A}) &\geq F_+(\hat{\mathcal{B}}) - F_+(\mathcal{B}) \end{aligned}$$

and since the choice of $\mathcal{B} \subseteq \mathcal{V}, \mathcal{A} \subseteq \mathcal{B}$ and $v \in \mathcal{V} \setminus \mathcal{B}$ has been generic, it follows that F_+ is submodular. $\square$

Appendix B

Additional Results

B.1 Complete Graph Objective Function Submodularity

In order to prove the submodularity of F_+ for the complete graph we need to compare four different configurations of p, q, r, in terms of nodes connected to $+$ but not to $-$, nodes connected to $+$ but not to $-$, and nodes connected to both of them. We need to prove that adding one edge-to-strategic in a greater set is less convenient than adding it in a smaller one. In particular, it is sufficient to prove that adding one edge to a general configuration determined by p, q, r is more convenient than adding two of them, i.e. we just need to compare the four possible configurations

	configuration 1	configuration 2	configuration 3	configuration 4
$\mathcal{A}, q$	p, q, r	p, q, r	p, q, r	p, q, r
$\hat{\mathcal{A}}, q$	$p+1, q, r$	$p+1, q, r$	$p, q-1, r+1$	$p, q-1, r+1$
$\mathcal{B}, q$	$p+1, q, r$	$p, q-1, r+1$	$p+1, q, r$	$p, q-1, r+1$
$\hat{\mathcal{B}}, q$	$p+2, q, r$	$p+1, q-1, r+1$	$p+1, q-1, r+1$	$p, q-2, r+2$

Table B.1: Possible configuration by adding two edges to $\mathcal{A}, q = p, q, r$

Then, by comparing $F_+(\hat{\mathcal{A}}) - F_+(\mathcal{A})$ with $F_+(\hat{\mathcal{B}}) - F_+(\mathcal{B})$ for each configuration we find

- $(F_+(k+1,l,p) - F_+(k,l,p)) - (F_+(k+2,l,p) - F_+(k+1,l,p)) =$

$$\frac{4\,N\,(N+2)\,(4\,l+2\,p+4\,N\,l+3\,N\,p+N^2\,l+N^2\,p)}{(2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\,(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+2)\,(2\,N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+4)}$$
$$\geq 0$$

- $(F_+(k+1,l,p) - F_+(k,l,p)) - (F_+(k+1,l-1,p+1) - F_+(k,l-1,p+1)) =$

$$\frac{\begin{aligned}&2\,N\,(N+2)\,(2\,N^3\,k\,l+2\,N^3\,k\,p+2\,N^3\,l^2+6\,N^3\,l\,p+2\,N^3\,l+4\,N^3\,p^2+2\,N^3\,p+\\&+N^2\,k^2+10\,N^2\,k\,l+10\,N^2\,k\,p+N^2\,k+9\,N^2\,l^2+22\,N^2\,l\,p+7\,N^2\,l+12\,N^2\,p^2+\\&+6\,N^2\,p+4\,N\,k^2+16\,N\,k\,l+16\,N\,k\,p+4\,N\,k+12\,N\,l^2+24\,N\,l\,p+8\,N\,l+\\&+12\,N\,p^2+8\,N\,p+4\,k^2+8\,k\,l+8\,k\,p+4\,k+4\,l^2+8\,l\,p+4\,l+4\,p^2+4\,p)\end{aligned}}{\begin{aligned}&(2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\,(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\\&(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+2)\\&(2\,N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+2)\end{aligned}}$$
$$\geq 0$$

- $(F_+(k,l-1,p+1) - F_+(k,l,p)) - (F_+(k+1,l-1,p+1) - F_+(k+1,l,p)) =$

$$\frac{\begin{aligned}&2\,N\,(N+2)\,(2\,N^3\,k\,l+2\,N^3\,k\,p+2\,N^3\,l^2+6\,N^3\,l\,p+2\,N^3\,l+4\,N^3\,p^2+2\,N^3\,p+\\&+N^2\,k^2+10\,N^2\,k\,l+10\,N^2\,k\,p+N^2\,k+9\,N^2\,l^2+22\,N^2\,l\,p+7\,N^2\,l+12\,N^2\,p^2+\\&+6\,N^2\,p+4\,N\,k^2+16\,N\,k\,l+16\,N\,k\,p+4\,N\,k+12\,N\,l^2+24\,N\,l\,p+8\,N\,l+\\&+12\,N\,p^2+8\,N\,p+4\,k^2+8\,k\,l+8\,k\,p+4\,k+4\,l^2+8\,l\,p+4\,l+4\,p^2+4\,p)\end{aligned}}{\begin{aligned}&(2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\,(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\\&(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+2)\\&(2\,N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p+2)\end{aligned}}$$
$$= F_+(k+1,l,p) - F_+(k,l,p)) - (F_+(k+1,l-1,p+1) - F_+(k,l-1,p+1))$$
$$\geq 0$$

- $(F_+(k,l-1,p+1) - F_+(k,l,p)) - (F_+(k,l-2,p+2) - F_+(k,l-1,p+1)) =$

$$\frac{4\,N\,(N+2)\,(N\,k+N\,l+N\,p+N^2\,l+N^2\,p)}{(2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\,(N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)\,(2\,N+2\,k+2\,l+2\,p+N\,k+N\,l+2\,N\,p)}$$
$$\geq 0$$

which can be summarized as

$$\begin{cases} [F_+(p+1,q,r) - F_+(p,q,r)] - [F_+(p+1,q-1,r+1) - F_+(p,q-1,r+1)] \geq 0 \\ [F_+(p,q-1,r+1) - F_+(p,q,r)] - [F_+(p+1,q-1,r+1) - F_+(p+1,q,r)] \geq 0 \\ [F_+(p,q-1,r+1) - F_+(p,q,r)] - [F_+(p,q-2,r+2) - F_+(p,q-1,r+1)] \geq 0 \end{cases}$$

then proving the submodularity of F_+.

Bibliography

[1] P. Iosifidis and M. Wheeler. *Modern Political Communication and Web 2.0 in Representative Democracies*. Javnost - The Public, 25:1-2, 110-118, DOI: 10.1080/13183222.2018.1418962, 2018

[2] P. Metaxas and S. T. Finn. *The infamous #Pizzagate conspiracy theory: Insight from a TwitterTrails investigation*. Conference proceeding from Computation + Journalism Symposium 2017, Northwestern University, Chicago, IL, 2017.

[3] M. H. DeGroot. *Reaching a consensus*. J. Am. Stat. Assoc. 69, 118-121, 1974.

[4] J.R.P. French. *A formal theory of social power*. Psychological Review 63, 181=194, 1956.

[5] F. Harari. *A criterion for unanimity in french's theory of social power*. vol. Studies in Social Power, Institute for Social Research, 1959.

[6] N. E. Friedkin and E. C. Johnsen. *Social influence networks and opinion change*. Advances in Group Processes, 16(1), 1-29, 1999.

[7] N. E. Friedkin. *A formal theory of reflected appraisals in the evolution of power*. Administrative Science Quarterly, 56(4), 501-529, 2011.

[8] J. Ghaderi, and R. Srikant. *Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate*. Automatica, 50(12), 3209-3215, 2014.

[9] D. Acemoglu, A. Ozdaglar, A. Parandehgheibi. *Spread of misinformation in social networks*. Dynamic Games Appl. 1, 1, 3–49, 2011.

[10] Y. Yi, T. Castiglia, S. Patterson. *Shifting Opinions in a Social Network Through Leader Selection*. arXiv:1910.13009 [cs.SI], 2019

[11] E. Yildiz, D. Acemoglu, A. Ozdaglar, A. Saberi, A. Scaglione. *Binary Opinion Dynamics with Stubborn Agents*. ACM Tran. Econ. Comput., vol. 1, no. 4, 2013.

[12] L. Vassio, F. Fagnani, P. Frasca, A. Ozdaglar. *Message Passing Optimization of Harmonic Influence Centrality*. IEEE Transactions on Control of Network Systems 1 (1), 109–120, 2014.

[13] D. Kempe, J. Kleinberg, and E. Tardos. *Maximizing the Spread of Influence through a Social Network*. Proceedings of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. 137-146, 2003.

[14] A. Gionis, E. Terzi, and P. Tsaparas. *Opinion Maximization in Social Networks*. 10.1137/1.9781611972832.43, 2013

[15] V. Mai, and E. Abed. *Optimizing Leader Influence in Networks Through Selection of Direct Followers* IEEE Transactions on Automatic Control. PP. 1-1, 2018.

[16] S. Dhamal, W. Ben-Ameur, T. Chahed, E. Altman. *Optimal investment strategies for competing camps in a social network: A broad framework*. IEEE Transactions on Network Science and Engineering, in press, 2018.

[17] S. Dhamal, W. Ben-Ameur, T. Chahed, E. Altman. *Manipulating opinion dynamics in social networks in two phases*. in: The Joint International Workshop on Social Influence Analysis and Mining Actionable Insights from Social Networks, 2018.

[18] S. Dhamal, W. Ben-Ameur, T. Chahed, E. Altman. *A two phase investment game for competitive opinion dynamics in social networks*. Information Processing & Management, Volume 57, Issue 2, 2020

[19] M. Grabisch, A. Mandel, A. Rusinowska, E. Tanimura. *Strategic influence in social networks*. Mathematics of Operations Research 43 (1), 29–50, 2018.

[20] A. Rusinowska, A. Taalaibekova. *Opinion formation and targeting when persuaders have extreme and centrist opinions*. Université Paris1 Panthéon-Sorbonne (Post-Print and Working Papers) halshs-01720017, HAL, 2018.

[21] A. Mandel, X. Venel. *Dynamic competition over social networks*. European Journal of Operational Research 280 597–608, 2020.

[22] N. Tsakas. *Optimal Influence under Observational Learning*. SSRN Working Paper, 2016.

[23] N. Tsakas. *Diffusion by imitation: The importance of targeting agents*. Journal of Economic Behavior & Organization 139, 118–151, 2017.

[24] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher. *An analysis of approximations for maximizing submodular set functions*. Mathematical programming, vol. 14, no. 1, pp. 265–294, 1978.

[25] O. Perron. *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*. Mathematische Annalen vol. 64, 11–76, 1907.

[26] G. Frobenius. *Uber Matrizen aus nicht negativen Elementen*. Sitzungsberichte Preussische Akademie der Wissenschaft, Berlin, 456–477, 1912.

[27] O. Perron. *Zur Theorie der Matrices*. Mathematische Annalen vol. 64, 248–263, 1907.

[28] S. Karlin. *Positive operators*. Journal of Mathematics and Mechanics vol. 8, 6, 907-937, 1959.

[29] H. Wielandt. *Unzerlegbare, nicht negative Matrizen*. Mathematische Zeitschrift vol. 52, 642–648, 1950.

[30] A. Borobia and U. R. Trías. *A geometric proof of the Perron-Frobenius theorem*. Revista Matemática de la Universidad Complutense de Madrid 5.1 (1992), pp. 57–63.

[31] C. R. MacCluer. *The many proofs and applications of Perron's theorem*. SIAM Review, 42(3): 487–498, 2000.

[32] J. Jacob, P. Protter. *Probability Essentials*. Springer, 2000.

[33] Hotelling, Harold. *Stability in competition.* Economic Journal 39, 41–57, 1929.

[34] A. Bekessy, P. Bekessy, and J. Komlos. *Asymptotic Enumeration of Regular Matrices.* Studia Scientiarum Mathematicarum Hungarica, 7, 343–353, 1972.

[35] E. A. Bender E. A. and E. R. Canfield. *The Asymptotic Number of Labeled Graphs with Given Degree Sequences.* J. Comb. Theory A, 24, 296–307 (1978).

[36] M. Molloy and B. Reed. *A Critical Point for Random Graphs with a Given Degree Sequence.* Random Structures and Algorithms 6 161-180, 1995.