POLITECNICO DI TORINO

Master's Degree in Physics of complex systems

Master's Degree Thesis

Spatially extended touch sensation

Supervisors

Prof. Andrea PAGNANI

Prof. Massimo VERGASSOLA

Prof. Alessandro SANZENI

Candidate

Georgi Hristov SPASOV

Abstract

The ability to sense external mechanical loads (touch), movements within muscles (proprioception) and internal organs (visceral sensation) is shared by all animals. Their ability stems from specialized neurons encapsulated in the skin, muscle, joints, and internal organs, which sense mechanical loads. Those sensations are crucial for communication, controlled locomotion, bodily homeostasis; in humans, their disruption by disease or chemotherapy, can lead to acute or chronic peripheral sensory neuropathy, gastrointestinal distress, and cardiovascular dysfunction. Touch sensation depends on the coupling between mechanoreceptor neurons and the material properties of the embedding tissues, which provides for a most appealing link between biology and physics, namely elasticity and viscoelasticity.

Although much has been learned separately on mechanosensitive channels and on tissue mechanics, our understanding of their integration in the sense of touch is still in its infancy. In particular, our grasp of how mechanical stimuli are conveyed through the tissues and affect the neural sensory processing is still rudimentary. Underlying this knowledge gap is the lack of integrated approaches that combine experimental devices for the delivery and the quantitative assay of controlled mechanical stimuli with predictive physical models for the transmission and the sensing of those stimuli. The main goal of this work is to develop models and biomechanical simulations.

We specifically consider the roundworm *Caenorhabditis elegans*, which is ideal for touch sensation studies because it allows to record currents in identified mechanoreceptor neurons in their native tissue environment. The pore-forming subunits of the mechano-electrical transduction (MeT) channels that convert mechanical stimuli into electrical signals are known and the MEC-4-dependent, ASIC-like sodium channel of *C. elegans* was the first MeT complex to be identified in any animal.MeT channels decorate six Touch Receptor Neurons (TRNs), which run along the body of the worm. The TRNs extend long (ca. $500\mu m$), straight sensory dendrites that are encapsulated within the epidermis and are positioned within 200nm of the skin. Like other *C. elegans* neurons, the TRNs express voltage-gated potassium and calcium channels, but lack voltage-gated sodium channels. They generate neither evoked nor spontaneous action potentials. Because the TRNs are nearly isopotential, action potentials are not needed to transmit signals along their length. This feature

also enables high-quality voltage-clamp recordings of currents activated along the length of the TRN sensory dendrites.

Previous work ([1],[2]) developed a new theoretical model for the coupled process by which loads applied to the skin of the worm are transduced through the tissue and TRNs are activated. The worm is modeled as a cylindrical shell under internal pressure, which is indented by a spherical ball, as in the experimental set-up. The transmission of mechanical stimuli through the tissue is obtained via simulations of nonlinear equations for elastic media. As for the gating of the neuron, the gating mechanism is supposed to be provided by tangential stimuli on MeT channels, which are read out as an increased probability of channel activation. This differs from previous models for hair cell bundles, where the gating is perpendicular. The model accounts for the observed adaptation and the symmetry of the response between onset and offset of the applied force, which was a long-standing mystery, and reproduces existing response data to various stimuli.

The current state of the model is that only spatially localized stimuli have been considered so far. In reality, the worm is subject to stimuli that are spatially extended over its entire body and involve the response of multiple neurons. Furthermore, experimental possibilities of extended stimulation are becoming available with ultrasound techniques poised to stimulate the entire body of the worm.

We built upon the existing model and generalized it by considering those more realistic conditions and stimulations. As the first analysis, we established the necessity to simulate the nonlinear equations for elastic media. We investigated, in particular, a stimulus directly coming from the nematode's environment, such as constricting rings which are used by carnivorous fungi that feed on the worm and other kinds of stimuli. The expected fast adaptation and symmetry of the neural response was recovered again. The importance of the position of the trapping ring-like stimulus was established and the multiple activations of different neurons linked to it.

Contents

List of Figures

Chapter 1

Introduction

Animals feel themselves and the environment around them through mechanoreceptor neurons that are embedded in their body and transform mechanical stimuli into electrical ones.

The human body has thousands of these neurons that have different purpose, e.g. there are the ones that are used to detect sound, the ones used to feel the position and the movement of our body, there are others that allows us to feel strong air flows when they impact on our body. These are just a small numbers of examples of what we can perceive. The human body is complex and has many different types and experimental manipulations are hampered by ethical reasons.

In order to reduce the reduce the complexity of the number of neurons and the different types of receptors, one looks for a organism which has a small number of mechanoreceptor neurons, which is suited for molecular biology. One of these organisms is the nematode *Caenorhabditis elegans* which has around 300 neurons that are completely mapped.

Different kinds of mechanical stimuli are processed by the organism with different kinds of receptors. All kinds of stimuli need to be converted and transduced into electrical signals in order to be elaborated. The nature of transmission of the signals depends on the nature of the stimuli. In the case of mechanical stimuli the transduction is very rapid compared with the other senses where the transducton is chemical, which is a signature of mechanical activation.

The various stimuli that the organism receives elicit responses on different time scales, such as adaptation and habituation. Different stimuli are processed by a large variety of receptor cells, and in the case of touch sensation the identification of these receptors is a non trivial problem. This is due mainly to the fact that the same structure is involved in different functions, their role in sensory transduction is difficult to distinguish from other functions. Those that have been identified differ in the range of force they respond to, the frequencies of these forces, the structure and their position inside the organisms' body.

We will focus on the adaptation phenomenon which is defined as the receptors property of not responding to an external stimulus after its being applied for some time. The time by which the receptor stops responding classifies the receptors as rapidly adapting or slowly adapting.

1.1 Caenorhabditis elegans

Caenorhabditis elegans is a small nematode with fast life cycle. The size of a grown nematode is around $1mm$ and that of a larva is around $0.25mm$, [3]. In nature, the nematode is present either as hermaphrodite or male, which differ from each other by the presence of a specialised tail in the male. Its lifespan, size and easily modifiable features make it a perfect candidate for molecular biology studies.

The nervous system of the nematode is completely mapped and an adult hermaphrodite has 302 neurons, around 30 of which have been classified as MechanoReceptor Neurons(MRNs). The MRNs of the *C.elegans* differ from each other mainly in the dendrites' structure and the kind of force they are specialised to detect [3]. These mechanoreceptor neurons contain, as a subclass, the Touch Receptor Neurons (TRNs) which are six, ALML[1], ALMR[2], PLML[3], PLMR[4],AVM[5] and PVM[6], [3], and extend approximatively half the length of the body of the nematode. These TRNs are in close proximity of the nematode's skin(cuticle) and are specialized to detect external stimuli.

The TRNs of the *C. elegans* are rapidly adapting, analogous to the mammalian Pacinian corpuscles[7] which are not present in the worm.

A widely-studied behaviour of *C. elegans*, e.g [4], analyses how the nematode escapes the traps of carnivorous fungi, e.g *D. doedycoides*. This study suggested that pressures imposed by this predators on the nematode have shaped their evolution, since both of these organism share the same habitat [5]. The nematode tries to escape backward if touched on the nose or the anterior part of the body, and escapes forward quite quickly when touched on the posterior part of the body. What actually is quite astonishing is that the head movements of *C. elegans* are present when touched on the nose or posterior

[1]Anterior lateral microtubule cell,left
[2]Anterior lateral microtubule cell,right
[3]Posterior lateral microtubule cell,left
[4]Posterior lateral microtubule cell,right
[5]Anterior ventral microtubule cell
[6]Posterior ventral microtubule cell
[7]Pacinian corpuscles are one of the main mechanoreceptor cell in mammals, they are surrounded by a fluid filled with lamellae. They are rapidly adapting and have symmetric response.

part of the body, but they are not if touched on the rest of its body. These head movements are used as escaping techniques when the fungi try to catch them. Nematophagous fungi use different kinds of traps, such as constricting and non-constricting rings, adhesive nets and others. The ring uninflated has an average diameter between $10\mu m$ and $25\mu m$. When the worm passes through the ring and sets the trap on, the ring starts to inflate. The ring completely inflates in a time around $0.1s$ [4], but the time between the worm entering the ring, which sets on the trap ring, and the inflation of the ring is around $5s$. This leaves the possibility to the worm to escape.

1.2 The mathematical model

We will use the mathematical model developed in [1] and [2]. It is based on the hypothesis that the dynamic connection between the Mechanoeletrical Transduction channels (MeT) and their surrounding tissue is brought by a visco-elastic element, featuring an elastic filament and viscous damping. The cuticle itself is seen as a viscoelastic medium in which one of the extremes of the filament, connected to a mass point, moves in. The effect of the filament on the channel can be seen as an effective coarse-grained model of the interaction between the channel and the phospholipid bilayer.
When the worm's skin deforms under the application of an external force, this causes a relative displacement of the filament, which favours the opening of the channels. This strongly depends on the velocity of application since the mass point moves in a viscoelastic medium which brings it back at equilibrium. This model explains the symmetry in the experimental observation of the on/off response of the channels.

From a mathematical modelling point of view, we see the worm's skin as a pressurized cylindrical membrane. We simulate the deformations using [6], which solves numerically the three-dimensional elastic problem [7] :

$$\frac{\partial}{\partial x_j}\left(\frac{\partial(x_k+u_k)}{\partial x_i}\sigma_{ij}\right)+\rho g_k=0\,, \tag{1.1}$$

where σ_{ij} is the Piola-Kirchoff stress tensor of the second type, see Chapter(2), u_i is the displacement. In [2] it was observed that a linear stress-strain relation, from Hookean elasticity, was enough to describe the deformations

$$\sigma_{ij}=\frac{E}{1+\nu}\left(\epsilon_{ij}+\frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)\,, \tag{1.2}$$

and also the strain-displacement relation is given by the nonlinear Green-Lagrange expression

$$2\epsilon_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\,. \tag{1.3}$$

We use the same notation as [2], and denote by $\overline{r}^{f,c}$ respectively the undeformed position of the channel and the endpoint of the filament connected to the mass point. In the same way, $\Delta\overline{r}^{f,c}$ will be the respective displacement. On the mass point acts the elastic force due to the filament, with potential

$$V(x) = \frac{k}{2}x^2 + \frac{k_4}{4}x^4 + \cdots ,$$

where $\overline{x} = \Delta\overline{r}^f - \Delta\overline{r}^c$. As done in [1] and [2], we retain only the first term. Moreover, the mass point moves in the viscoelastic medium, so it feels a friction of the following form

$$\overline{F}_{friction} = -\gamma\frac{d(\Delta\overline{r}^f - \overline{u}(\overline{r}'))}{dt} ,$$

where $\overline{r'}$ is the initial position of the mass point, and $\overline{u}(r')$ is the deformation from Equation (1.1). Since the neural tissue is inside the hypodermis, near the cuticle, it is reasonable to assume that it deforms in the same way as the membrane, so this yields $\Delta\overline{r}^c = \overline{u}(\overline{r}^c)$. Defining $\overline{r}'$ as the position of the material point that coincides with the location of the tip, i.e $\overline{r}^f + \Delta\overline{r}^f = \overline{r}' + \overline{u}(\overline{r}')$. Assuming that the gradients of $\overline{u}$ are small, one can approximate $\overline{u}(\overline{r}') \simeq \overline{u}(\overline{r}^f)$. This allows us to replace $\overline{u}(\overline{r}')$ with $\overline{u}(\overline{r}^f)$ in the friction force. Since the overdamped approximation is valid at the microscopic scale, the equation of motion for the elongation of the filament becomes

$$\frac{d\overline{x}}{dt} + \frac{1}{\tau}\overline{x} = \frac{d\overline{\Gamma}}{dt} = \frac{d(\overline{u}(\overline{r}^f) - \overline{u}(\overline{r}^c))}{dt} , \tag{1.4}$$

where τ is the relaxation time. The dynamics of $\overline{\Gamma}$ is determined by the deformation $\overline{u}$ from Equation(1.1). The left-hand side of this equation is the Kelvin viscoelastic model, i.e. a spring in parallel with a dashpot. The adaptation and symmetry features, observed experimentally, are captured by this model. Indeed, the left-hand side is proportional to the time derivative of the deformation, so for a constant deformation, which will result in a $\Gamma = const$, we will have that x goes to zero, which is adaptation.

The movement of the filament is constrained due the fact that it can move only in the vertical direction, i.e. on top of the channel. This constraint is implemented by defining a local basis on top of the channel, $\hat{w}_i$. This local basis is defined as a function of the neural membrane's local basis $\hat{e}'_i$ in such a way that $\hat{e}'_y$ coincides with the deformed local basis of the cylinder, running from head to tail, and $\hat{e}'_z$ is orthogonal to the neural membrane, and $\hat{e}'_x = \hat{e}'_y \wedge \hat{e}'_z$. Then the local basis of the channel is a rotation of the $\hat{e}'_i$ in such a way that if the channel is rotated around $\hat{e}'_y$, it becomes $\hat{w}'_1 = \cos(\theta)\hat{e}'_x - \sin(\theta)\hat{e}'_z$, $\hat{w}_2 = \hat{e}'_y$ and $\hat{w}'_3 = \sin(\theta)\hat{e}'_x + \cos(\theta)\hat{e}'_z$. So the constraint is imposed as $\hat{w}_3 \cdot \overline{x} \geq 0$.

Concerning the opening and the closing mechanism of the channels, it has been observed [8], that there are several sub-conducting states. Since the introduction of a single sub-conduction state is enough to capture the effects, [2], we will introduce only one as well, in order to minimise the free parameters of the model. In this way we have a 3-state Markov chain,

$$C \rightleftharpoons S \rightleftharpoons O\,, \tag{1.5}$$

with the following master equation

$$\frac{d}{dt}\begin{pmatrix} P_c \\ P_s \\ P_o \end{pmatrix} = \begin{pmatrix} -R_{cs} & R_{sc} & 0 \\ R_{sc} & -(R_{sc}+R_{so}) & R_{os} \\ 0 & R_{so} & -R_{os} \end{pmatrix}\begin{pmatrix} P_c \\ P_s \\ P_o \end{pmatrix}\,, \tag{1.6}$$

where one puts $R_{oc} = R_{co} = 0$ in order to reduce the number or free parameters. We want that the channels open only due to mechanical stimuli, we require that they work at equilibrium and detailed balance is satisfied, which implies

$$\frac{P_s^{eq}}{P_c^{eq}} = \frac{R_{cs}}{R_{sc}} = e^{-\beta \Delta G_{sc}}\,, \qquad \frac{P_o^{eq}}{P_s^{eq}} = \frac{R_{sc}}{R_{os}} = e^{-\beta \Delta G_{os}}\,, \tag{1.7}$$

where one uses a statistical physics description, where $\beta^{-1} = k_k T$ and ΔG_{ij} is the free energy difference between the state i and j. With this assumption, the equilibrium transition probabilities become

$$P_o^{eq} = \frac{1}{1 + e^{-\beta \Delta G_{co}} + e^{-\beta \Delta G_{so}}}\,, \qquad P_s^{eq} = \frac{1}{1 + e^{-\beta \Delta G_{os}} + e^{-\beta \Delta G_{cs}}}\,, \tag{1.8}$$

with $P_c^{eq} = 1 - P_o^{eq} - P_s^{eq}$ from the normalization condition. The deformation of the neural membrane is the same as the deformation of the elastic membrane of the worm (the skin), in the proposed model, the free energy is a function of the displacement, $\Delta G(\overline{x})$. The expression of the free energy proposed in [2], is

$$\beta \Delta G_{oc} = g_0 - g_1 \mathcal{F}\,, \tag{1.9}$$

where $\mathcal{F}$ is the modulus of the tangential components of the elastic force, with $\mathcal{F}_1 = \overline{F}_{elas} \cdot \hat{w}_1$ and $\mathcal{F}_2 = \overline{F}_{elas} \cdot \hat{w}_2$. For the free energy of the transition $o \to s$ and $c \to s$, one assumes that the free energy is of the form

$$\Delta G_{os} = a \Delta G_{oc}, \qquad \Delta G_{sc} = (1-a)\Delta G_{oc}, \tag{1.10}$$

with one additional parameter $a \in [0, 1]$.

The distribution of the channels is in spots along the neural membrane and the spacing between two successive channels is log-normal [9]. As in [2], we assume that each spot contains a single channel. With this description the mean current along the TRN is the sum of the currents of each channel, i.e.

$$\langle I \rangle = i_o \sum_k P_o(k) + i_s \sum_k P_s(k)\,, \tag{1.11}$$

with i_o the current that passes through the channel when it is open and i_s when it is in the sub-conducting state. The current of the single channel in the open state was measured experimentally in [10], and its value is $i_o = -1.6 \pm 0.2 \ pA$. Conversely, the i_s in one of the free parameters.

The model parameters were inferred from numerical data in [2]. As stated before, we will use the same model, with the same parameters. The transition rates are written as

$$R_{sc} = r_{cs} e^{(1+b)(1-a)\beta\Delta G_{oc}}, \quad R_{cs} = r_{cs} e^{b(1-a)\beta\Delta G_{oc}}, \tag{1.12}$$

$$R_{os} = r_{so} e^{(1+d)\beta\Delta G_{oc}}, \quad R_{so} = r_{so} e^{da\beta\Delta G_{oc}}, \tag{1.13}$$

in which r_{cs} and r_{so} control respectively the rate of transition between the closed and sub-conducting and open and sub-conducting state. The parameters $d, b \in [-1, 0]$ control the global shift of the transition rates.

The free parameters are obtained from different realisations of the channels' position and the different direction of the filament, [2], then they were averaged and the values obtained are

$$\begin{gathered} \tau = 1.4ms, \quad g_h = 1.4 \cdot 10^{-3}, \quad \frac{g_s}{g_h} = 0.09, \quad r_{cs} = 1/69.5ms \\ b = -0.75, \quad r_{so} = 1/18ms, \quad d = 0.56, \quad \frac{i_s}{i_o} = 0.71 \end{gathered}, \tag{1.14}$$

where $g_0 = g_h/g_s$ and $g_2 = g_s^{-1}$.

Chapter 2

Rudiments of Elasticity

Continuum mechanics theory sees physical objects as continuous bodies without taking into consideration their atomic structure. In this frame all the physical quantities are space and time varying fields. Our physical body is defined as a differentiable submanifold, $\mathbb{B}$, whose elements are called (material) particle, and one assumes that there is a differentiable embedding of the manifold in a bounded, open, connected subspace $\Omega \subset \mathbb{R}^3$, with sufficiently smooth boundaries.

Since $\mathbb{B}$ is a differentiable manifold, for each $X \in \mathbb{B}$ there exist a local chart (U, ϕ) such that $\phi(X) = (x^1(X), x^2(X), x^3(X))$ gives the local coordinates of the chart. Through this mapping one can define a measure on $\mathbb{B}$, such that

$$\mu(U) = \int_{\phi(U)} \rho(\overline{x}) dV \, , \tag{2.1}$$

where ρ is the density, i.e. the mass per unit volume.

The main ingredient in our theory is the deformation field. We fix an orthonormal basis $\{\overline{e}_i\}$ in $\mathbb{R}^3$, and identify $\mathbb{B}$ with Ω. A deformation field of the reference configuration $\overline{\Omega}$ is a vector field $\overline{\phi} : \overline{\Omega} \to \mathbb{R}^3$, that is smooth enough, injective and orientation preserving. This last condition translates into the request that $det(\overline{\nabla\phi}) > 0$ for $\forall x \in \overline{\Omega}$. From the deformation field one defines the displacement field, $\overline{u}$, which is the more-frequently used and defined as $\overline{u} : \overline{\Omega} \to \mathbb{R}^3$ such that $\overline{\phi} = id_{|\overline{\Omega}} + \overline{u}$.

All that we defined until now is in the so-called reference configuration. But when a body gets deformed, its shape changes through the deformation field. The image of the body $\overline{\Omega}$ when one applies the deformation field is called the deformation configuration, i.e. $\overline{\Omega}^{\phi} = \overline{\phi}(\overline{\Omega})$ [1]. Clearly one can define a new local coordinated system through $\{\partial_i \overline{\phi}\}$. When a body deforms, the volume and surface elements get deformed too, the relation between the deformed and

[1]We will use the same notation in [11], since the author of this work finds it clearer than any other notation.

undeformed volume and surface elements are given by

$$dV^\phi = det\left(\overline{\nabla\phi}\right)dV, \quad d\sigma^\phi = |Cof\left(\overline{\nabla\phi}\right)\hat{n}|d\sigma\,, \tag{2.2}$$

where $\hat{n}$ is the outward normal to the surface element[2] .

We can derive two equivalent equations of equilibrium for deformable continuum bodies, one stated in the reference configuration and one in the deformed configuration. From a mathematical point of view these two formulations are completely equivalent, they are connected by a pull-back (or push-forward) between the two manifolds defined by the reference and deformed configuration.

Before going into the mathematical formulation of the theory we are building up, let us introduce the basic concepts of the different kind of forces that can act on a continuous body. These forces are divided in two groups, internal and external forces. The external forces are those such that their sources are located outside the considered body. The external forces acting on the deformed body $\overline{\Omega}^\phi$ are of two types

a. Body forces defined by a density of force per unit volume in the deformed configuration, i.e.

$$\overline{f}^\phi : \overline{\Omega}^\phi \to \mathbb{R}^3\,, \tag{2.3}$$

or similarly if ρ^ϕ is the density of the body in the deformed configuration one can define $\overline{h}^\phi$ a density of force per unit mass, in such a way that $\overline{f}^\phi = \overline{h}^\phi\rho^\phi$.

b. Surface forces defined by a force density per unit surface in the deformed configuration, i.e.

$$\overline{g}^\phi : \Gamma^\phi \to \mathbb{R}^3\,, \tag{2.4}$$

with Γ^ϕ a measurable subset of $\partial\Omega^\phi$.

Let us consider now a partition of the body $\overline{\Omega}^\phi$ in elementary volumes. Let V_1^ϕ and V_2^ϕ be two neighbouring elementary volumes. On the sub-volume V_1^ϕ, acts the external forces and also the reaction force of V_2^ϕ on V_1^ϕ. It is assumed that this interaction between an elementary volume, at point $\overline{x}$, and its neighbours are statistically equivalent to a force $\overline{t}^\phi(\overline{x}, \hat{n}^\phi)d\sigma^\phi$, applied on the surface with the normal to the surface in the outward direction. Mathematically this force is formulated by saying that for each oriented surface in the continuum, there is a force $\overline{t}^\phi(\overline{\phi})d\sigma^\phi$ as defined above. This means that, for each point of a given surface in $\overline{\Omega}^\phi$ there exists a linear[3] mapping from $\mathbb{R}^3$ to itself. This is

[2]We will indicate with σ and σ^ϕ the stress tensor respectively in the reference and deformed configuration and the distinction between the surface element will be clear from the context. If eventually the distinction will not be clear, we will clarify explicitly.

[3]The fact that this mapping is linear is shown in [12], we will not replicate the proof.

expressed as $\overline{t}^\phi(\overline{n}^\phi) = \sigma^\phi\left[\overline{n}^\phi\right]$ where σ is a second order tensor.
In order to show that this mapping exists and it is unique, we need to put down some axioms.

Axiom 1. Each arbitrary part of a continuum body is at equilibrium if and only if the whole body is at equilibrium.

Axiom 2. The necessary condition for the equilibrium of an arbitrary part of the continuum are the ones the rigid body.

Axiom 3. *Stress principle of Euler and Cauchy*
Let us consider a a deformed body $\overline{\Omega}^\phi$ subjected to external forced per unit volume and surface, $\overline{f}^\phi$ and $\overline{g}^\phi$. Then there exists a vector field

$$\overline{t}^\phi : \overline{\Omega}^\phi \times S^2 \to \mathbb{R}^3\,, \tag{2.5}$$

called the *Cauchy stress vector field* such that

1. for $\forall V^\phi \subset \overline{\Omega}^\phi$ and for $\forall \overline{x} \in \partial\overline{\Omega}^\phi \cap \partial V^\phi$ where the unit other normal vector $\overline{n}$ to $\partial\overline{\Omega}^\phi \cap \partial V^\phi$ exists and

$$\overline{t}^\phi(\overline{x}, \overline{n}) = \overline{g}^\phi(\overline{x})\,. \tag{2.6}$$

2. For any subset $V^\phi \subset \overline{\Omega}^\phi$ the force balance holds, i.e.

$$\int_{V^\phi} \overline{f}^\phi dV^\phi + \int_{\partial V^\phi} \overline{t}^\phi(\overline{x}, \overline{n}^\phi) d\sigma^\phi = 0\,, \tag{2.7}$$

where $\overline{n}^\phi$ denotes the normal to the surface.

3. For any subset $V^\phi \subset \overline{\Omega}^\phi$ the moment balance holds, i.e.

$$\int_{V^\phi} \overline{r} \wedge \overline{f}^\phi dV^\phi + \int_{\partial V^\phi} \overline{r} \wedge \overline{t}^\phi(\overline{x}, \overline{n}^\phi) d\sigma^{\overline{\phi}} = 0\,. \tag{2.8}$$

Observation: In the stress principle, it is assumed that the element surface force depends on the element via the normal $\overline{n}^\phi$. One should not rule out the possibility that there is some dependence on the curvature of ∂V^ϕ or other geometrical characteristics.
Noll, [13], when he builds the mathematical theory of elasticity, assumes that the the stress depends on boundary of the subdomain under consideration, without considering explicitly properties of this boundary such as normal vector or curvature. Then he showed [13], that the dependence on the curvature or any other local property of the boundary at given point is impossible.

In order to derive the equilibrium equations, we need to show that the linear mapping σ^ϕ exists and it is unique. This mapping is called the stress tensor, and in order to prove its existence, thanks to the axioms we are adopting, we need the following few theorems.[4]

Theorem 1. *Let $\overline{f}^\phi$,$\overline{g}^\phi$ and $\overline{t}^\phi$ be the external and internal forces acting on $\overline{\Omega}^\phi$, with $\overline{\Omega}^\phi$ bounded, then for any unit vector $\overline{n}^\phi$ the following holds*

$$\overline{t}^\phi(\hat{n}^\phi) = -\overline{t}^\phi(-\hat{n}^\phi)$$

Proof. From the balance of linear momentum we have

$$\int_{\partial V^\phi} \overline{t}(\hat{n}^\phi) d\sigma^\phi + \int_{V^\phi} \overline{f}^\phi dV^\phi = 0 \qquad \forall V^\phi \subset \overline{\Omega}^\phi . \tag{2.9}$$

Since $\overline{\Omega}^\phi$ is bounded, and from the continuity of $\overline{f}^\phi$ we have that

$$C = \sup_{x \in \Omega} \mid \overline{f}^\phi(\overline{x}) \mid \tag{2.10}$$

is finite. From this observation we have that

$$\mid \int_{\partial V^\phi} \overline{t}^\phi(\hat{n}^\phi) d\sigma^\phi \mid \leq C\mu(V^\phi) , \tag{2.11}$$

where μ is the Lebesgue measure on $\mathbb{R}^3$. Now let us fix $\overline{x}_0 \in \overline{\Omega}^\phi$, a normal vector $\hat{k}$, and $P \equiv P(\epsilon)$ where $P(\epsilon)$ is a rectangular parallelepiped centred in $\overline{x}_0$ with height ϵ^2,the other two dimension equal to ϵ, and $\hat{k}$ parallel to the sides of the parallelepiped. The top and bottom sides, Σ^+ and Σ^-, have normal $\hat{k}$ and $-\hat{k}$. The sides of $P(\epsilon)$ can be written as $\partial P(\epsilon) = \Sigma^+ \cup \Sigma^- \cup \Sigma_\epsilon$ with respective measures equal to $\mu(P(\epsilon)) = \epsilon^4$, $\mu(\Sigma^\pm) = \epsilon^2$ and $\mu(\Sigma_\epsilon) = 4\epsilon^3$. Applying this result to the previous observations, we have

$$\frac{1}{\epsilon^2} \mid \int_{\partial P(\epsilon)} \overline{t}^\phi(\hat{n}^\phi) d\sigma^\phi \mid = C\epsilon^2 \xrightarrow{\epsilon \to 0} 0 \tag{2.12}$$

Since $\overline{t}^\phi(\hat{n}^\phi)$ is continuous for a fixed $\hat{n}^\phi$, we conclude that

$$\frac{1}{\epsilon^2} \int_{\Sigma^\pm} \overline{t}^\phi(\pm\hat{k}) d\sigma^\phi \to \overline{t}^\phi(\overline{x}_0, \pm\hat{k}) , \tag{2.13}$$

and thus we have

$$\overline{t}^\phi(\hat{k}) + \overline{t}^\phi(-\hat{k}) = 0 , \tag{2.14}$$

which completes the proof since $\hat{k}$ and $\overline{x}_0$ are arbitrary. □

[4] These theorems are the stationary cases from [14].

Theorem 2. *Let σ be a class C^1 tensor field on $\overline{\Omega}^\phi$ with σ and $div^\phi(\sigma)$ continuous on $\overline{\Omega}^\phi$. Then $\forall V^\phi \in \overline{\Omega}^\phi$ the following holds*

$$\int_{\partial V^\phi} \overline{r} \wedge \left(\sigma^\phi[\hat{n}^\phi]\right) d\sigma^\phi = \int_{V^\phi} \overline{r} \wedge div^\phi \left(\sigma^\phi\right) dV^\phi + 2\int_{V^\phi} skew(\sigma^\phi) dV^\phi \quad (2.15)$$

Now we present the main theorem, that links the forces that act on a body with a tensor field called the stress tensor, this tensor is the linear application we have been referring to in the past. We present in a certain sense a weaker version of the *Cauchy-Poisson theorem*, just to point out that there exists, [14], a stronger version of this theorem of the form "if and only if".

Theorem 3. *Let $\overline{f}^\phi$ and $\overline{t}^\phi$ be the volume forces and the Cauchy stress vector field acting on the deformed body $\overline{\Omega}^\phi$.*
If $\overline{f}^\phi$ is continuous on $\overline{\Omega}^\phi$ and

$$\overline{t}^\phi : \overline{\Omega}^\phi \times S_1 \to \mathbb{R}^3 \quad (2.16)$$

is C^1 on $\overline{\Omega}^\phi$ and C^0 on S_1 then there exists

$$\sigma^\phi : \overline{\Omega}^\phi \to \mathbb{R}^3 \otimes \mathbb{R}^3 \quad (2.17)$$

a $C^1(\overline{\Omega}^\phi$ tensor field such that

1. $\overline{t}^\phi(\overline{x}, \hat{n}) = \sigma^\phi(\overline{x})[\hat{n}]$ *for* $\forall x \in \overline{\Omega}^\phi$ *and* $\forall \hat{n} \in S_1$,
2. $-div^\phi \sigma^\phi = \overline{f}^\phi$ *for* $\forall \overline{x} \in \Omega^\phi$,
3. $\sigma^\phi = (\sigma^\phi)^T$ *for* $\forall \overline{x} \in \overline{\Omega}^\phi$,
4. $\sigma^\phi \hat{n} = g^\phi$ *for* $\forall \overline{x} \in \Gamma^\phi$

Proof. Let $\langle \overline{e}_i \rangle$ be an orthonormal basis for $\mathbb{R}^3$, let $\hat{k}$ be a unit vector different from the elements of the basis and $\overline{x}_0$ an arbitrary point of Ω^ϕ. Consider the tetrahedral $P(h) \subset \overline{\Omega}^\phi$ with sides Σ and Σ_i having linear dimensions h and normals respectively $\hat{k}$ and $\hat{k}_i = -\left[sgn(\overline{e}_i \cdot \hat{k})\right] \overline{e}_i$, and whose faces intersect at $\overline{x}_0$. The areas of the sides on the tetrahedral are dependent on the linear dimension h and will be denoted as $\mu(h)$ and $-\mu(h)\hat{k} \cdot \hat{k}_i$.

By hypothesis, $\overline{t}^\phi(\overline{x}, \overline{n})$ is continuous on $\overline{\Omega}^\phi$ for $\forall \overline{n} \in S_1$, and in the limit of $h \to 0$, and by using Theorem 1, we have

$$\frac{1}{\mu(h)} \int_\Sigma \overline{t}^\phi(\overline{x}, \overline{n}) d\sigma^\phi \to \overline{t}^\phi(\overline{x}_0, \hat{k}) \, , \quad (2.18)$$

$$\frac{1}{\mu(h)}\int_{\Sigma_i}\overline{t}^{\phi}(\overline{x},\overline{n})d\sigma^{\phi} \to -(\hat{k}\cdot\hat{k})\overline{t}^{\phi}(\overline{x}_0,\hat{k}) = -(\hat{k},\overline{e}_i)\overline{t}^{\phi}(\overline{x}_0,\overline{e}_i)\,, \tag{2.19}$$

where there is no sum on the index i in the last expression. Also from Theorem 1 we have

$$\frac{1}{\mu(h)}\int_{\partial P(h)}\overline{t}^{\phi}(\overline{x},\overline{n})d\sigma^{\phi} \to 0\,, \tag{2.20}$$

and this yields

$$\overline{t}^{\phi}(\overline{x}_0,\hat{k}) = \sum_i(\hat{k}\cdot\overline{e}_i)\overline{t}^{\phi}(\overline{x}_0,\overline{e}_i) = (\overline{t}^{\phi}(\overline{x}_0,\overline{e}_i)\otimes\overline{e}_i)\hat{k}\,, \tag{2.21}$$

which is the first point of the theorem, if one defines $\sigma^{\phi} = \overline{t}^{\phi}(\overline{x}_0,\overline{e}_i)\otimes\overline{e}_i$.

Now if one uses the *Stress principle of Euler and Cauchy*, we have

$$\int_{V^{\phi}}\overline{f}^{\phi}dV^{\phi} + \int_{\partial V^{\phi}}\sigma^{\phi}\overline{n}d\sigma^{\phi} = 0\,, \tag{2.22}$$

and, by using the divergence theorem, one has

$$\int_{V^{\phi}}(\overline{f}^{\phi} + div^{\phi}\sigma^{\phi})d\sigma^{\phi} = 0\,. \tag{2.23}$$

Since this is true for every $V^{\phi}\subset\overline{\Omega}^{\phi}$ this means that

$$-div^{\phi}\sigma^{\phi} = \overline{f}^{\phi}\,, \tag{2.24}$$

which is the second point of the theorem.

The third point of the theorem in the same way comes from the *Stress principle of Euler and Cauchy* together with the Theorem 2. The last property is a direct consequence of the definition of σ^{ϕ} and the definition of $\overline{t}^{\phi}$. $\square$

The tensor σ^{ϕ} is called the *Cauchy stress tensor*. It is useful to underline that σ^{ϕ}_{ij} represents the i-th component of the *Cauchy vector stress* along the j-th direction.

Thanks to this Theorem one can define the equilibrium equations in the deformed configuration:

Problem 1. *Given $\Omega^{\phi}\subset\mathbb{R}^3$, $\overline{f}^{\phi}$ and $\overline{g}^{\phi}$ functions defined respectively on $\overline{\Omega}^{\phi}$ and $\Gamma^{\phi}\subset\partial\Omega^{\phi}$, find the tensor field σ^{ϕ} such that*

$$-div^{\phi}(\sigma^{\phi}) = \overline{f}^{\phi} \quad in \quad \Omega^{\phi}\,, \tag{2.25}$$

$$\sigma^{\phi} = (\sigma^{\phi})^T \quad in \quad \Omega^{\phi}\,, \tag{2.26}$$

$$\sigma^{\phi}[\overline{n}] = \overline{g}^{\phi} \quad in \quad \Gamma^{\phi}\,. \tag{2.27}$$

This formulation of the problem is not easy to solve, since we usually do not know the domain Ω^ϕ, which is actually what we want to know. A more handy, but equivalent, formulation of the problem to solve is in the reference configuration. This formulation that we will give is actually more convenient, since it allows us to formulate a weak form of the problem[5] that allows us to use numerical methods. This weak form is what actually our numerical scheme uses to solve.

We need to find a way to formulate the problem in the reference configuration, where the geometry is known. The transformation that allows us to go in the reference configuration is the *Piola transformation*, and it is defined as the following one.

Definition 1. *Let $\overline{\phi}$ be a deformation of Ω that is injective on $\overline{\Omega}$, so that the matrix $\nabla\overline{\phi}$ is invertible at all points of the reference configuration. Let $T^\phi(\overline{x}^\phi)$ be a tensor field for $\forall \overline{x}^\phi \in \overline{\Omega}^\phi$. We associate with $T^\phi(\overline{x}^\phi)$ a tensor field $T(x)$ defined as*

$$T(x) = det(\nabla\phi)T^\phi(\overline{x}^\phi)(\nabla\phi)^{-T} = T^\phi(\overline{x}^\phi)Cof(\overline{\phi}), \quad \forall \overline{x}^\phi = \phi(x)\,. \tag{2.28}$$

Remarks: The new tensor defined through this transformation has one index in the reference configuration one in the deformed configuration. Also it has the following property[6]

$$div(T(x)) = (det(\nabla\overline{\phi}(x))div^\phi T^\phi(\overline{x}^\phi) \quad \forall \overline{x}^\phi = \overline{\phi}(\overline{x}), \overline{x} \in \overline{\Omega}\,. \tag{2.29}$$

Using now the Piola transformation to *Cauchy stress tensor* σ^ϕ one obtains a new stress tensor called *First Piola-Kirchhoff stress tensor*. The advantage of this tensor is the relation with the divergences of the two tensors. This will allow to write the equilibrium equations in a similar "divergence structure". As we already pointed out, the first Piola-Kirchhoff stress tensor has one index in the reference and one index in the deformed configuration, which makes it meaningless to ask if the tensor is symmetric. Instead, if we denote the first Piola-Kirchhoff tensor with Σ the following holds

1. $$div(\Sigma(x)) = (det(\nabla\overline{\phi}(x))div^\phi(\sigma^\phi(\overline{x}^\phi))\,,$$

2. $$\Sigma^T = \nabla\overline{\phi}(\overline{x})^{-1}\Sigma\nabla\overline{\phi}(\overline{x})^{-T}\,.$$

[5] Also in this form we can give a weak formulation but it is of no use, since our integrals are over Ω^ϕ that we usually do not know.

[6] See [11] for a proof.

The constitutive relation that we will introduce later, will assume a simpler form if we have a symmetric stress tensor also in the reference configuration. In order to have a symmetric tensor, we have to transform the index of Σ that is still in the deformed configuration. We define the new tensor, called *second Piola-Kirchhoff stress tensor* as

$$\sigma(\overline{x}) = \nabla\overline{\phi}(\overline{x})^{-1}\Sigma(\overline{x}) = (det(\nabla\overline{\phi}(\overline{x}))\nabla\overline{\phi}(\overline{x})^{-1}\sigma^{\phi}(\overline{x}^{\phi})\nabla\overline{\phi}(\overline{x})^{-T} \quad \forall\overline{x}^{\phi} = \overline{\phi}(\overline{x})\,. \tag{2.30}$$

The tensor defined in this way is indeed symmetric.

The last question we need to address before formulating our problem in the deformed configuration, is to see how the force density fields transform when we go into the reference configuration. Given a force density per unit volume field in the deformed configuration, one defines a force density per unit volume field in the reference configuration requiring that the force acting on a volume element is the same, i.e.

$$\overline{f}dV = \overline{f}^{\phi}dV^{\phi} \quad \forall\overline{x}^{\phi} = \overline{\phi}(\overline{x})\,, \tag{2.31}$$

which yields $\overline{f}(\overline{x}) = det(\nabla\overline{\phi})\overline{f}^{\phi}(\overline{x}^{\phi})$. Analogously for the force density per unit area field, should be related in such a way that

$$\overline{g}(\overline{x})d\sigma = \overline{g}^{\phi}d\sigma^{\phi} \quad \forall\overline{x}^{\phi} = \overline{\phi}(\overline{x}) \in \Gamma^{\phi}\,, \tag{2.32}$$

which gives us the following relation $\overline{g}(\overline{x}) = det(\nabla\overline{\phi}) \mid \nabla\overline{\phi}(\overline{x})^{-1}\hat{n} \mid \overline{g}^{\phi}(\overline{x}^{\phi})$.

We are ready now to formulate our problem in the reference configuration.

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded and sufficiently regular subset, and let $\overline{\phi}(\overline{x})$ be a deformation filed for Ω. Then the second Piola-Kirchhoff stress tensor $\sigma(x)$ satisfies the following equations in the reference configuration Ω:*

1. $-div(\nabla\overline{\phi}(\overline{x})\sigma(\overline{x})) = \overline{f}(\overline{x}) \quad \forall\overline{x} \in \Omega,$
2. $\sigma(\overline{x}) = \sigma(\overline{x})^T \quad \forall\overline{x} \in \Omega,$
3. $\nabla\overline{\phi}(\overline{x})\sigma(\overline{x})\hat{n} = \overline{g}(\overline{x}) \quad \overline{x} \in \Gamma.$

These are the equilibrium equations in the reference configuration. They can be put also in a weak form that is useful for the finite element method techniques, but we will not go into that formulation.

The next quantity we need to introduce is the strain tensor. Let Ω and Ω^{ϕ} be the body in the reference and deformed configuration. Let P and Q be two neighbouring points in space with local coordinate x^i and $x^i + dx^i$ respectively. The distance between the two points is

$$ds^2 = dx^i dx^i = g_{ij}dx^i dx^j\,, \tag{2.33}$$

where one defines the metric tensor g_{ij}. After the deformation, the points will be mapped in P' and Q' with global coordinates given by the following relation,$\phi^i(x^1, x^2, x^3)$. The distance between the P' and Q' is given by

$$ds'^2 = \phi^k_{,i}\phi^k_{,j}dx^i dx^j = \tilde{g}_{ij}dx^i dx^j \ , \tag{2.34}$$

the main point here is that one assumes that the local coordinate x^i remains the same, i.e $d\tilde{v}^i = dv^i$, this is what characterizes the Langrangian description. The difference between this two distances is given by

$$ds'^2 - ds^2 = (\tilde{g}_{ij} - g_{ij})dx^i dx^j = 2\epsilon_{ij}dx^i dx^j \ , \tag{2.35}$$

where one defines the strain tensor as

$$\epsilon_{ij} = \frac{1}{2}\left(\tilde{g}_{ij} - g_{ij}\right) \ . \tag{2.36}$$

As already stated, the deformed and undeformed configurations are connected by ϕ^i, which are related to the deformation vector through $\overline{\phi} = \overline{id} + \overline{u}$. In this way, one obtains that the strain tensor, as a function of u^i has the following form

$$2\epsilon_{ij} = u^i_{,j} + u^j_{,i} + u^k_{,i}u^k_{,j} \ , \tag{2.37}$$

where ϵ_{ij} are the components of the strain tensor along the x^i expressed as a function of the local frame of reference v^i. Clearly the strain tensor is symmetric, and covariant, so it transforms in the following way

$$\epsilon^*_{ij} = h^k_{,i}h^p_{,j}\epsilon_{pk} \tag{2.38}$$

with h^i being the mapping between two global frame of references.

We introduce two more principles needed to define the constitutive relation. We take the same approach as in [15]. The stress tensor[7] characterizes the contact forces acting on each part $V \subset \Omega$; its definition at a certain point depends on the contact forces acting on an infinitesimal volume containing that point. Also the causality principle implies that the configuration at a certain time is determined only by its past history. In the development of the theory, we assume that the stress tensor is sufficient to describe all mechanical interactions and we discard all non mechanical quantities. With these observations one introduces the following two principles

1. The stress in a body is determined by the history of the motion of the body.

[7]From now one, when we refer to stress tensor we mean the second Piola-Kirchhoff stress tensor, if not said otherwise.

2. In determining the stress of a given particle $X \in \Omega$, one can discard the motion outside a neighbourhood of this particle.

This two principles translate into saying that there exists a functional $\mathcal{F}_t$ with the following properties:

a) In every possible kinematic process the stress $\overline{\overline{\sigma}}$, at time t is related to the motion $\overline{u}(t)$ of the body Ω by

$$\overline{\overline{\sigma}} = \mathcal{F}_t(\overline{u})\,, \tag{2.39}$$

b) For any two motions $\overline{u}$ and $\overline{u}^*$ that coincide in some neighbourhood $U(\overline{x})$ for $\forall\tau \leq t$, the values of $\mathcal{F}_t$ is the same. Formally

$$\mathcal{F}_t(\overline{u}) = \mathcal{F}_t(\overline{u}^*)\,. \tag{2.40}$$

One must observe that the functional $\mathcal{F}$ is constrained by the Galilean covariance. So, all the functionals defined by (a), with the restriction (b) and Galilean covariance, define the general constitutive equation for purely mechanical theories of continuum media. With these requests, one obtains the most general form for the constitutive equation. The functional $\mathcal{F}$ is called response functional at the particle X.

Different materials have different kind of behaviour, which are linked to different constitutive equations. We will deal only with Hookean elasticity, i.e. materials whose response functional is linear and time independent, but may in principle be space dependent. In the Hookean elasticity the response functional is a linear relation between the stress and strain tensor. From a mathematical point of view this means that the response functional is an isomorphism on the space of symmetric rank 2 covariant tensors, this means that it can be represented as a rank 4 tensor, i.e. A_{ijkl}. Thanks to representation theory such automorphism on $\mathbb{R}^3 \otimes \mathbb{R}^3$, this tensor can be expressed as [8]

$$A^{ijkl} = sym(\overline{e}^i \otimes \overline{e}^j)A[sym(\overline{e}^k \otimes \overline{e}^l)]\,, \tag{2.41}$$

and this tensor is clearly symmetric in in the i,j and k,l exchange, i.e

$$A^{ijkl} = A^{jikl} = A^{ijlk}\,, \tag{2.42}$$

which brings down the unknowns from 81 to 36, and the general constitutive relation in Hookean elasticity becomes

$$\sigma^{ij} = A^{ijkl}\epsilon_{lk}\,, \tag{2.43}$$

where the isomorphism is called elasticity tensor and its components are called elasticities.

[8] See [15].

We are interested in isotropic materials. A material is called isotropic if and only if its symmetry group is the orthogonal group, $O(3,\mathbb{R})$. We are not going to show how to obtain the form of A^{ijkl}, which is present in any book of representation theory or elasticity, and the form we will report is a necessary and sufficient condition for the body to be isotropic.

A material is isotropic at point $\overline{x}$ if and only if its elasticity tensor have the following form

$$A^{ijkl} = \mu\left(\delta^{ik}\delta^{jl} + \delta^{jk}\delta^{il}\right) + \lambda\delta^{ij}\delta^{kl}\,, \tag{2.44}$$

I follow the approach in [16], which I find most clear. Thanks to this the equilibrium problem, in terms of the displacement field, becomes

Problem 2. *Let Ω be a domain in $\mathbb{R}^3$ with boundary $\partial\Omega = \Sigma_0 \cup \Sigma_1$ with $\Sigma_0 \cap \Sigma_1 = \varnothing$. Let $f^i \in L^2(\Omega)$ and $h^i \in L^2(\Sigma_1)$ for $i = 1, 2, 3$, be two functions that are respectively the force density per unit volume and unit surface applied on the volume Ω.*
Find $\overline{u} : \Omega \to \mathbb{R}^3$, $\overline{u} \in W(\Omega) = \{\overline{v} \in W^{1,4}(\Omega)|\overline{v} = 0 on \Sigma_0\rangle$, where $\overline{u}$ is the displacement vector in Cartesian coordinates, it satisfies the following displacement-traction problem:

1. $-\partial_j(\sigma^{jk} + \sigma^{ji}\partial_i u^k) = f^k$ *in* Ω,
2. $u_i = 0$ *on* Σ_0,
3. $(\sigma^{ij} + \sigma^{jk}\partial_k u^i)\hat{n}_j = h^i$ *on* Σ_1.

where

$$\sigma^{ij} = A^{ijkl}\epsilon_{lk}\,, \tag{2.45}$$

$$2\epsilon_{ij} = \partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k\,. \tag{2.46}$$

This problem can be put in an equivalent weak form, where one considers that $\forall\overline{v} \in W(\Omega)$ the following equation holds

$$\int_\Omega (\sigma^{ik} + \sigma^{ji}\partial_i u^k)\partial_j v_k dV = \int_\Omega f^k v_k dV - \int_{\Sigma_1} h^k v_k d\sigma\,, \tag{2.47}$$

or equivalently as a variational problem as

$$\hat{E}(\overline{u}) = \inf_{\overline{v}\in W(\Omega)} \hat{E}(\overline{v})\,, \tag{2.48}$$

where $\hat{E}(\overline{v}) : W(\Omega) \to \mathbb{R}$ is the three dimensional energy in Cartesian coordinates defined as

$$\hat{E}(\overline{v}) = \frac{1}{2}\int_\Omega \{\lambda(\epsilon_{kk})^2 + 2\mu\epsilon_{ij}\epsilon_{ji}\}dV - \{\int_\Omega f^k v_k dV + \int_{\Sigma_1} h^k v_k d\sigma\}\,. \tag{2.49}$$

In order to be able to introduce the thin shell theory, in which the various quantities are expressed in terms of the curvilinear coordinates, we need to express the previous equations in terms of three dimensional curvilinear coordinates. This means that having a domain $\Omega_1 \subset \mathbb{R}^3$ and a smooth enough injective mapping $\overline{\theta} : \Omega \to \mathbb{R}^3$ such that $\overline{\theta}(\overline{\Omega}_1) = \overline{\Omega}$ and the vectors $\overline{g}_i = \partial_i \overline{\theta}$ are linearly independent at each point of the domain.
So we want to express the previous quantities in terms of curvilinear coordinates $\overline{x} = \overline{\theta}(w^1, w^2, w^3)$, and this transformation will be actually done on the variational formulation of our problem and not on the PDEs, this choice is due to the fact that we already know how volumes and surface integrals transform under such change of variables.

Theorem 5. *Let Ω be a domain in $\mathbb{R}^3$ such that $\partial\Omega = \Sigma_0 \cup \Sigma_1$ with $\Sigma_0 \cap \Sigma_1 = \emptyset$. Let $\hat{f}^i \in L^2(\Omega)$ and $\hat{h}^i \in L^2(\Sigma_1$ be two functions that represents respectively the force density per unit volume and surface acting on our body Ω. Let $\hat{\overline{u}} \in W(\Omega)$ be the minimizer of the energy functional $E(\overline{v})$ over the space $W(\Omega)$.*

Let now Ω_1 be a domain in $\mathbb{R}^3$ and let there exist $\overline{\theta}$ a C^2-diffeomorphism of $\overline{\Omega}_1$ into $\overline{\Omega} = \overline{\theta}(\overline{\Omega}_1)$, such that the three vectors $\overline{g}_i = \partial_i \overline{\theta}$ are linearly independent over all the domain.
Let define the vectors $\overline{g}_i$ such that $\overline{g}^i \cdot \overline{g}_j = \delta^i_j$, $g(x) = \det(\overline{g}_i \cdot \overline{g}_j)$ and $g^{ij}(\overline{w}) = \overline{g}^i \cdot \overline{g}^j$.
Then the vector field $\overline{u} : \Omega_1 \to \mathbb{R}^3$ defined by

$$\hat{u}_i \hat{e}^i := u_i g^i \quad \forall \overline{x} = \overline{\theta}(\overline{w}) \;\; \overline{w} \in \Omega_1 \,, \tag{2.50}$$

satisfies the following minimisation problem over $W(\Omega_1) := \{\overline{v} \in W^{1,4}(\Omega_1) | \overline{v} = 0 \text{ on } \Sigma_0^\theta\}$,

$$E(\overline{u}) = \inf_{\overline{v} \in W(\Omega_1)} E(\overline{v}) \,, \tag{2.51}$$

where

$$E(\overline{v}) := \frac{1}{2} \int_{\Omega_1} A^{ijkl} \tilde{\epsilon}_{kl}(\overline{v}) \tilde{\epsilon}_{ij}(\overline{v}) \sqrt{g} dV^\theta - \{ \int_{\Omega_1} f^i v_i \sqrt{g} dV^\theta + \int_{\Sigma_1^\theta} h^i v_i \sqrt{g} d\sigma^\theta \} \,, \tag{2.52}$$

where $\Sigma_0^\theta \cap \Sigma_1^\theta = \overline{\theta}^{-1}(\Sigma_0) \cap \overline{\theta}^{-1}(\Sigma_1) = \emptyset$, and the functions $f^i \in L^2(\Omega_1)$ and $h^i \in L^2(\Sigma_1^\theta)$ are defined by

$$\hat{f}^i(\overline{x}) \hat{e}_i dV = \sqrt{g(\overline{w})} f^i(\overline{w}) \overline{g}_i(\overline{w}), \;\; \overline{x} = \overline{\theta}(\overline{w}), \;\; \overline{w} \in \Omega_1 \,, \tag{2.53}$$

$$\hat{h}^i \hat{e}_i d\sigma = \sqrt{g(\overline{w})} h^i(\overline{w}) \overline{g}_i(\overline{w}) d\sigma^\theta, \;\; \overline{x} = \overline{\theta}(\overline{w}), \;\; \overline{w} \in \Sigma_1^\theta \,, \tag{2.54}$$

the components of the elasticity tensor $A^{ijkl} = A^{jikl} = A^{ijlk} \in C^1(\overline{\Omega}_1)$ are defined by

$$A^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}) \,, \tag{2.55}$$

and finally, the strain tensor in terms of the curvilinear coordinates and the transformed filed assumes the following form

$$\tilde{\epsilon}_{ij}(\overline{v}) = \frac{1}{2}\left(D_j v_i + D_i v_j + g^{mn} D_i v_m D_j v_n\right) \tag{2.56}$$

where

$$D_j v_i = \partial_j v_i - \Gamma^p_{ij} v_p, \;\; \textit{with } \Gamma^p_{ij} = \frac{1}{2} g^{pl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) = \Gamma^p_{ji} \in C^0(\overline{\Omega_1}) \tag{2.57}$$

is the covariant derivative.

Recall that we defined the strain tensor as the change of metric tensor. It can be shown that the same definition still holds, in fact the components of the strain tensor are sometimes called *covariant components of the change of metric tensor.*

In order to obtain the equations of equilibrium one can show that $E(\overline{v})$ is weakly differentiable, and determine the Gateaux derivative, which determines the weak formulation of the problem, from which one obtains at the end the equilibrium equations.

Chapter 3

Thin shell theory

In the following chapter, we will develop the elasticity theory for thin elastic shells. There are many different approaches to derive a thin shell theory. All of these theories are based on the concept of dimensional reduction, which allows us to go from three dimensional to two dimensional theory. The author of this work prefers the approach taken by Niordson in [17]. This approach has the advantage that is variational by constriction and allows to increase or decrease the complexity of the formulation as needed. Moreover it allows to know exactly the error committed in the approximation in a given theory, as we shall see when recovering the Love's linear theory of cylindrical shells.

When a body is delimited by two outer surfaces Σ^+ and Σ^-, and there exists a surface Σ, called *middle surface*, such that its normal intersects the two outer surfaces $\Sigma^\pm$ at $\pm\frac{h}{2}$, then the body is called a shell of thickness h (in principle non constant).

The dimensional reduction we were talking about before, starts from the three dimensional stress expressed in curvilinear coordinate (u_1, u_2, z), where u^α with $\alpha = 1, 2$ are the intrinsic coordinates of the middle surface with parametric relations $x^i = g^i(u^1, u^2)$. These quantities are reduce to statically equivalent forces and moments acting at the middle surface, for which one obtains two dimensional equilibrium equations.

Let us consider now a connected set $\Gamma \subset \Sigma$, of the middle surface, with smooth boundary $\partial\Gamma$. The boundary $\partial\Gamma$ is characterized by two dimensional vector $u^\alpha(s)$, which are the curvilinear coordinates of the surface and s is the arclength of $\partial\Gamma$. We recall that we can define three mutually orthogonal vector, t^α,n^α and X^α, where

$$t^\alpha = \frac{du^\alpha}{ds}\,, \tag{3.1}$$

is the tangent to the curve,

$$n^\alpha = a^{1/2} e^{\alpha\beta} t_\beta\,, \tag{3.2}$$

where $e^{\alpha\beta}$ is the alternating tensor in two dimension, and finally

$$X^{\alpha} = a^{-1/2} e_{\alpha\beta\gamma} n^{\beta} t^{\gamma}\,, \tag{3.3}$$

where e_{ijk} is the Levi-Civita symbol or the alternating symbol in three dimensions. Now regarding the the statically equivalent formulation of the stress. We consider $\sigma^{ij}(u^1, u^2, z)$ in normal coordinates, and let $n^i(z)$ be the normal to the surface element $ds_z dz$. Let us consider a unit parallel vector field A^i, i.e. $D_j A^i = 0$, that defines a given direction. As it is well known fact, $\Gamma^3_{3k} = 0$ in normal coordinates, and one of the equations, $j = i = 3$, that define the parallel vector field is

$$\partial_3 A^3 = 0\,, \tag{3.4}$$

this means that the third component of this field must be z independent. It's therefore customary to consider two separate parallel field, the first one having $A^3 = 0$ and the second cone having $A^1 = A^2 = 0$. In the first case, one can easily determine the z dependence of A^{α}, which is the one we need in order to integrate over the thickness. one of the equations that define the filed is $D_3 A^{\alpha} = 0$ which equivalently

$$\partial_3 A^{\alpha} = \Gamma^{\alpha}_{3k} A^k = \Gamma^{\alpha}_{3\gamma} A^{\gamma}\,, \tag{3.5}$$

and one can easily show that in normal coordinates and for a point which is not too far from the surface,$g_{\alpha\beta} = a^{\gamma\delta}(a\alpha\gamma - d_{\alpha\gamma} z)(a_{\beta\delta} - d_{\beta\delta} z)$, and $g_{\alpha\delta}\Gamma^3_{3k} = d^{\epsilon}_{\gamma}(a_{\delta\epsilon} - d_{\delta\epsilon})$, where $a_{\alpha\beta}$ is the surface metric tensor and $d_{\alpha\beta}$ is the curvature tensor of the surface. So thanks to this relations, one obtains

$$(a_{\delta\epsilon} - d_{\delta\epsilon} z)(a^{\epsilon}_{\alpha} - d^{\epsilon}_{\alpha} z)\partial_3 A^{\alpha} = d^{\epsilon}_{\gamma}(a_{\delta\epsilon} - d_{\delta\epsilon} z)A^{\gamma}\,. \tag{3.6}$$

From the basic geometric theories of surfaces, we know that $\det(a_{\delta\epsilon} - d_{\delta\epsilon} z) = (ag)^{1/2} \neq 0$, we have the following equation

$$(a^{\epsilon}_{\alpha} - d^{\epsilon}_{\alpha} z)\partial A^{\alpha} = d^{\epsilon}_{\gamma} A^{\gamma}\,, \tag{3.7}$$

which is equivalent to

$$\partial_3((a^{\epsilon}_{\alpha} - d^{\epsilon}_{\alpha} z)A^{\alpha}) = 0\,, \tag{3.8}$$

which trivially yields the solution we were looking for

$$A_{\alpha} = (a^{\beta}_{\alpha} - d^{\beta}_{\alpha} z)\tilde{A}_{\beta}\,, \tag{3.9}$$

with $\tilde{A}_{\beta} := \tilde{A}_{\beta}(u^1, u^2, 0)$. In the second case, we don't have to determine the dependence of z of the parallel field since there is no such dependence.

In the first case, the normal to a similar boundary curve $\partial\gamma$, at a small distance z from the middle surface is given by $n^{\alpha} = g^{1/2} e_{\alpha\beta} \frac{du^{\beta}}{ds_z}$, and the resultant,$R_A$, of the force acting on a arclength ds of the boundary $\partial\Gamma$, in the direction A^i is given by

$$R_A ds = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta} g^{1/2} e_{\alpha\delta} du^\delta (a_\beta^\gamma - d_\beta^\gamma z) A_\gamma dz \,. \tag{3.10}$$

The resultant stress tensor is denoted usually by $N^{\alpha\beta}$ and is defined in such a way that $R_A ds = n_\alpha A_\gamma N^{\alpha\gamma} ds$, is the resultant force component acting on the element ds in the direction A_γ. The normal to the element ds has been defined previously and thus the tangential components of the stress tensor have the following form,

$$N^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\gamma} (\delta_\gamma^\beta - d_\gamma^\beta z)(1 - 2Hz + Kz^2) dz \,, \tag{3.11}$$

where we have used $(g/a)^{\frac{1}{2}} = 1 - 2Hz + Kz^2$ with H and K being respectively the mean and Gaussian curvature.

Analogously, in the case of the normal parallel field A^i having only the $i = 3$ different from zero. One has directly that the stress component, denoted by Q^α where one implies that the other index is equal to 3, is equal to

$$Q^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{3}} \sigma^{\alpha 3} (1 - 2Hz + Kz^2) dz \,. \tag{3.12}$$

One needs to observe is that $N^{\alpha\beta}$ is non symmetric, and this means that we have in principle $4 + 2$ unknown statically equivalent stresses to find. But as we'll see, thanks to the virtual displacement principle, these quantities will not be independent from each other.

The next quantity we need to define is the statically equivalent moment what acts on the middle surface. The statically equivalent moment that lies in the tangential plane to the middle surface is decomposable in twisting moment along the normal to the boundary and in bending moment along the negative of the tangential direction to the boundary. In completely analogous way one can define the moment tensor $M^{\alpha\beta}$:

$$M^{\alpha\gamma} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta} (\delta_\beta^\gamma - d_\beta^\gamma z)(1 - 2Hz + Kz^2) z dz \,, \tag{3.13}$$

and from this the bending and twisting moments are naturally defined as

$$M_B = M^{\alpha\beta} n_\alpha n_\beta \,, \quad M_T = M^{\alpha\beta} n_\alpha t_\beta \,. \tag{3.14}$$

The total momentum that lies in the tangential plane to the middle surface is

$$M_\gamma = M_T n_\gamma - M_B t_\gamma = e_{\gamma\beta} M^{\alpha\beta} n_\alpha \,. \tag{3.15}$$

The loads that are applied to the shell, are formulated in an equivalent way to statically equivalent loads applied to the middle surface. The in-plane statically equivalent loads are

$$F^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} f^\delta(\delta_\delta^\alpha - d\alpha_\delta z)(1-2Hz+Kz^2)dz + \sum_{\{\Sigma^+,\Sigma^-\}} (f^\alpha)_{\pm\frac{h}{2}}(1\mp Hh+\frac{1}{4}Kh^2)\,, \tag{3.16}$$

the normal statically equivalent load[1] is

$$p = \int_{-\frac{h}{2}}^{\frac{h}{2}} f^3(1-2Hz+Kz^2)dz + \sum_{\{\Sigma^+,\Sigma^-\}} (f^3)_{\pm\frac{h}{2}}(1\mp Hh+\frac{1}{4}Kh^2)\,. \tag{3.17}$$

The forces contribute also to the momentum equilibrium which is $\partial_\gamma g^i e^{\gamma\beta} m_\beta$ where

$$m^\alpha = -\int_{-\frac{h}{2}}^{\frac{h}{2}} f^\gamma(\delta_\gamma^\alpha - d_\gamma^\alpha z)(1-2Hz+Kz^2)zdz - \sum_{\{\Sigma^+,\Sigma^-\}} (F^\alpha)_{\pm\frac{h}{2}}(1\mp Hh+\frac{1}{4}Kh^2)(\pm\frac{h}{2})\,. \tag{3.18}$$

3.1 Equilibrium equations

In order to have equilibrium one requires that for any subdomain $\Gamma \subset \Sigma$ the resultant force vanishes on the boundary, i.e.

$$\int_\Gamma (\partial_\alpha g^i F^\alpha + X^i p)dA + \oint_{\partial\Gamma} (\partial_\beta g^i N^{\alpha\beta} n_\alpha + X^i Q^\alpha n_\alpha)ds = 0\,, \tag{3.19}$$

using the divergence theorem and the fact that this equation must hold for $\forall\Gamma \subset \Sigma$ we have the following equilibrium equations

$$D_\alpha\left[\partial_\beta g^i N^{\alpha\beta} + X^i Q^\alpha\right] + \partial_\alpha g^i F^\alpha + X^i p = 0\,, \tag{3.20}$$

projecting this equations on the local frame of reference one obtains

$$\begin{cases} D_\alpha N^{\alpha\beta} - d_\alpha^\beta Q^\alpha + F^\beta = 0, & (3.21) \\ D_\alpha Q^\alpha + d_{\alpha\beta} N^{\alpha\beta} + p = 0, & (3.22) \end{cases}$$

these are 3 equations in the 4+2 unknown stresses. In completely analogous way, writing the moment equilibrium first in the cartesian frame of reference and then projecting on the local frame of reference one gets the following equations

$$\begin{cases} e_{\alpha\beta}(N^{\alpha\beta} - d_\gamma^\beta M^{\gamma\alpha}) = 0, & (3.23) \\ D_\alpha M^{\alpha\beta} + Q^\beta + m^\beta = 0. & (3.24) \end{cases}$$

[1]We denote it as p for pressure.

one can use the last equation to eliminate Q^α from the previous equations. With this last equations we get 6 equations in the 10 unknown quantities $N^{\alpha\beta}$,Q^α and $M^{\alpha\beta}$. Moreover, statically equivalent stresses and momentum are not symmetric so we cannot relate them through one-to-one correspondence with the strain and curvature tensors, i.e. $\epsilon_{\alpha\beta}$ and $k_{\alpha\beta}$.
Starting from the equilibrium equations for the three dimensional problem in the deformed configuration one can derive an equation for the virtual work principle. This equation can be expressed in terms of the reduced quantities, from which one can define obtain effective symmetric stress and moment. We will not derive this quantities, but their expressions are given by

$$\begin{cases} \tilde{N}^{\alpha\beta} = N^{\alpha\beta} - d^\beta_\gamma(M^{\gamma\alpha} + e^{\gamma\alpha}\Phi), & (3.25) \\ \tilde{M}^{\alpha\beta} = M^{\alpha\beta} - e^{\alpha\beta}\Phi, & (3.26) \end{cases}$$

where Φ is a scalar function and $e^{\alpha\beta}$ the alternating tensor in two dimensions. Assuming that the membrane as a loading $T^i = \partial_\alpha g^i T^\alpha + X^i T$ on the boundary, and rewriting the equilibrium equations in terms of the effective quantities, they assume the following form with the given boundary conditions:

$$\begin{cases} D_\alpha \tilde{N}^{\alpha\beta} + 2d^\beta_\gamma D_\alpha \tilde{M}^{\gamma\alpha} + \tilde{M}^{\gamma\alpha} D_\alpha d^\beta_\gamma + F^\beta + d^\beta_\alpha m^\alpha = 0 \quad \text{in } \Sigma, & (3.27) \\ D_\alpha D_\beta \tilde{M}^{\alpha\beta} - d_{\alpha\gamma} d^\gamma_\beta \tilde{M}^{\alpha\beta} - d_{\alpha\beta}\tilde{N}^{\alpha\beta} + D_\alpha m^\alpha - p = 0 \quad \text{in } \Sigma, & (3.28) \\ \tilde{M}^{\alpha\beta} n_\alpha n_\beta = M_B \quad \text{on } \partial\Sigma, & (3.29) \\ T - \dfrac{\partial M_T}{\partial s} = -(D_\alpha M^{\alpha\beta}) n_\beta - \dfrac{\partial}{\partial s}(M^{\alpha\beta} n_\alpha t_\beta) \quad \text{on } \partial\Sigma, & (3.30) \\ \left[N^{\alpha\beta} + (2d^\alpha_\gamma - d^\alpha_\rho n^\rho n_\gamma) M^{\beta\gamma}\right] n_\beta = T^\alpha \quad \text{on } \partial\Sigma. & (3.31) \\ & (3.32) \end{cases}$$

Where s is the curvilinear coordinate of the $\partial\Sigma$.

Observation: One needs to observe that these equations are in the deformed configuration.

3.2 Energy and stress-strain relation

For a material that obeys Hook's law, we know that the problem can be formulated in a variational way where one minimises an energy functional which is quadratic in the strain.

The stress-strain energy density per unit area is given by

$$W = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{ij}\epsilon_{ij}(g/a)^{\frac{1}{2}} dz\,. \tag{3.33}$$

Now we want to formulate our elasticity problem for thin shells in analogous way. The state of the middle surface is completely determined by the strain tensor $\epsilon_{\alpha\beta}$ and the bending tensor $\kappa_{\alpha\beta}$. The most general dimensionless expression in terms on $\epsilon_{\alpha\beta}$,$\kappa_{\alpha\beta}$, ν,H,K and E is

$$\frac{W}{Eh} = C^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + D^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\kappa_{\gamma\delta} + F^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta} + Q(\epsilon_{\alpha\beta}, \kappa_{\alpha\beta}, \nu, Hh, Kh^2)\,, \tag{3.34}$$

where the last term Q contain the covariant derives of $\epsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. The energy must be invariant with respect to any coordinate transformation of any given transformation. This means that each of the four order tensor, and the other terms, have to be invariant, and be functions of h,ν,K,H,$a_{\alpha\beta}$ and $d_{\alpha\beta}$, since this quantities characterise the middle surface.

The strain and bending tensor are symmetric quantities, this means that $C^{\alpha\beta\gamma\delta}$,$D^{\alpha\beta\gamma\delta}$ and $F^{\alpha\beta\gamma\delta}$ are symmetric. The most general expression for the first term is of the following form

$$C^{\alpha\beta\gamma\delta} = C_1 a^{\alpha\beta}a^{\gamma\delta} + C_2 a^{\alpha\gamma}a^{\beta\delta} + C_3 h a^{\alpha\beta}d^{\gamma\delta} + C_4 h a^{\alpha\gamma}d^{\beta\delta} + C_6 h^2 d^{\alpha\beta}d^{\gamma\delta} + C_6 h^2 d^{\alpha\gamma}d^{\beta\delta}\,, \tag{3.35}$$

where $C_1, \ldots$ are constant dependent on the following dimensionless quantities ν,Hh and Kh^2, and the factor h has been introduced in order to make the constants dimensionless.

If the normal to the surface changes, the terms containing and even number of times the curvature tensor, will change sign. The only quantity that changes sign is Hh and in order to avoid the sign changing all the constants can be defined as follows

$$\begin{cases} C_1(\nu, Hh, Kh^2) = \overline{C}_1(\nu, (Hh)^2, Kh^2), & (3.36)\\ C_2(\nu, Hh, Kh^2) = \overline{C}_2(\nu, (Hh)^2, Kh^2), & (3.37)\\ C_3(\nu, Hh, Kh^2) = Hh\overline{C}_3(\nu, (Hh)^2, Kh^2), & (3.38)\\ C_4(\nu, Hh, Kh^2) = Hh\overline{C}_4(\nu, (Hh)^2, Kh^2), & (3.39)\\ C_5(\nu, Hh, Kh^2) = \overline{C}_5(\nu, (Hh)^2, Kh^2), & (3.40)\\ C_6(\nu, Hh, Kh^2) = \overline{C}_6(\nu, (Hh)^2, Kh^2). & (3.41) \end{cases}$$

One can assume that these constants are smooth enough in the constants, and one can expand in terms of h/R, where R is the smallest radius of curvature at a given point. We rain only terms up to $\frac{h}{R}$ and discard hither order terms. The first two terms, can be expressed as $C_{1,0}(e_1 + e_2)^2 + C_{2,0}(e_1^2 + e_2^2)$ where e_1 and e_2 are the principal strains such that $|e_1| \geq |e_2|$, and the constants are the zero order expansions.

Let us show that the third and forth term are of order $(h/2)^2$. Using that $|d^{\gamma\delta}\epsilon_{\gamma\delta}| \leq 2\frac{|e_1|}{R}$

$$|C_3 h a^{\alpha\beta}d^{\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}| \leq |C_{3,0}Hh^2(2\frac{|e_1|^2}{R})^2| \sim \left(\frac{h}{R}\right)^2\,, \tag{3.42}$$

using the same property we have that also the fifth and sixth term are of order $\left(\frac{h}{2}\right)^2$. So the first tensor, retaining only the terms up to first order in h/R becomes

$$C^{\alpha\beta\gamma\delta} \simeq C_{1,0}a^{\alpha\beta}a^{\gamma\delta} + C_{2,0}a^{\alpha\gamma}a^{\beta\delta}\,. \tag{3.43}$$

The third tensor is similar to the first. The particular one is the second tensor. Since the tensor $\epsilon_{\alpha\beta}\kappa_{\gamma\delta}$ changes with the direction of the normal vector and thus also $D^{\alpha\beta\gamma\delta}$ must change sign. So we can write is as

$$D^{\alpha\beta\gamma\delta} = D_{1,0}Hh^2a^{\alpha\beta}a^{\gamma\delta} + D_{2,0}Hh^2a^{\alpha\gamma}a^{\beta\delta} + D_{3,0}h^2a^{\alpha\beta}d^{\gamma\delta} + D_{4,0}h^2a^{\gamma\delta}d^{\alpha\beta} + D_{5,0}h^2a^{\alpha\gamma}d^{\beta\delta}\,,$$

where we neglected the higher order terms. Multiplying by $\epsilon_{\alpha\beta}$, $\kappa_{\gamma\delta}$ one obtains

$$\begin{aligned} D_{1,0}Hh^2\,(e_1+e_2)\,(k_1+k_2) + D_{2,0}Hh^2\epsilon^\alpha_\beta\kappa^\beta_\alpha + D_{3,0}h^2(e_1+e_2)d^\alpha_\beta\kappa^\beta_\alpha + \\ + D_{4,0}h^2(k_1+k_2)d^\alpha_\beta\epsilon^\beta_\alpha + D_{5,0}h^2d^\alpha_\beta\epsilon^\beta_\gamma\kappa^\gamma_\alpha\,, \end{aligned} \tag{3.44}$$

we compare the terms of this sum one by one, in order to do so we need the following inequality valid for any a, x and y,

$$|2axy| \le \frac{|a|}{\sqrt{bc}}(bx^2+cy^2) \qquad b>0 \qquad c>0\,.$$

The first tern is

$$|D_{1,0}Hh(e_1+e_2)(k_1+k_2)| \le \frac{|D_{1,0}|}{2\sqrt{C_{1,0}F_{1,0}}}(C_{1,0}(e_1+e_2)^2 + h^2F_{1,0}(k_1+k_2)^2)|Hh|\,,$$

which shows that the first tern is of order $\frac{h}{R}$, for the other terms

$$\begin{cases} |D_{2,0}Hh^2\epsilon^\alpha_\beta\kappa^\beta_\alpha| \le \dfrac{|D_{2,0}|}{2\sqrt{C_{2,0}F_{2,0}}}(C_{2,0}e_1^2 + h^2F_{2,0}k_1^2)|Hh| & (3.45) \\ |D_{3,0}h^2(e_1+e_2)d^\beta_\alpha\kappa^\alpha_\beta| \le \dfrac{|D_{3,0}|}{\sqrt{C_{1,0}F_{2,0}}}(C_{1,0}(e_1+e_2)^2 + h^2F_{2,0}k_1^2)|h/R| & (3.46) \\ |D_{4,0}h^2(k_1+k_2)d^\alpha_\beta\epsilon^\beta_\alpha| \le \dfrac{|D_{4,0}|}{\sqrt{C_{2,0}F_{1,0}}}(C_{2,0}e_1^2 + h^2F_{1,0}(k_1+k_2)^2)(h/R) & (3.47) \\ |D_{5,0}h^2d^\alpha_\beta\epsilon^\beta_\gamma\kappa^\gamma_\alpha| \le \dfrac{2|D_{5,0}|}{\sqrt{C_{2,0}F_{2,0}}}(C_{2,0}e_1^2 + h^2F_{2,0}k_1^2)(h/R) & (3.48) \end{cases}$$

Indeed, this shows that the term $D^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\kappa_{\gamma\delta}$ is at most or order h/R.

One can do the exactly same procedure and obtain that the third tensor become

$$F^{\alpha\beta\gamma\delta} \simeq F_{1,0}a^{\alpha\beta}a^{\gamma\delta} + F_{2,0}a^{\alpha\gamma}a^{\beta\delta}\,.$$

Covariant derivatives are of the order $(h/L)^2$ where L is the deformation pattern and we neglect them too. The energy density with this approximation becomes

$$\begin{aligned} W = Eh[&\left(C_1 a^{\alpha\beta} a^{\gamma\delta} + C_2 a^{\alpha\gamma} a^{\beta\delta}\right) \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + D^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\kappa_{\gamma\delta} + \\ &+ \left(F_1 a^{\alpha\beta} a^{\gamma\delta} + F_2 a^{\alpha\gamma} a^{\beta\delta}\right) \kappa_{\alpha\beta}\kappa_{\gamma\delta}] + O\left(\left(\frac{h}{R}\right)^2 + \left(\frac{h}{L}\right)^2\right) . \end{aligned} \quad (3.49)$$

This is the most general for under this approximation. The constants are usually determined comparing the energy of known solution of the three dimensional problem with this solution. The stress-strain and moment-bending relation are given by the functional derivative of the energy, i.e.

$$\delta W = \frac{\partial W}{\partial \epsilon_{\alpha\beta}}\delta\epsilon_{\alpha\beta} + \frac{\partial W}{\partial \kappa_{\alpha\beta}}\delta\kappa_{\alpha\beta} = N^{\alpha\beta}\delta\epsilon_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} . \quad (3.50)$$

One needs to observe that if we neglect also the term $\epsilon_{\alpha\beta}\kappa_{\gamma\delta}$, the constitutive relations are decoupled. The mixed term adds a coupling between stress and bending as well as between moment and strain.

Just for the sake of completeness we will derive the constitutive relations starting from the Equation (3.50), but in the following we will use the ones obtained neglecting the mixed term. Let us derive now the relations.

$$N^{\alpha\beta} = \frac{\partial W}{\partial \epsilon_{\alpha\beta}} ,$$

the derivative can be split in two pieces, the quadratic in strain and mixed term. Let us now evaluate the first term

$$\frac{\partial}{\partial \epsilon_{\alpha\beta}}\left(C_1\epsilon^\gamma_\gamma\epsilon^\sigma_\sigma + C_2\epsilon^\gamma_\sigma\epsilon^\sigma_\gamma\right) = 2C_1\frac{\partial \epsilon^\gamma_\gamma}{\partial \epsilon_{\alpha\beta}}\epsilon^\sigma_\sigma + 2C_2\frac{\partial \epsilon^\gamma_\sigma}{\partial \epsilon_{\alpha\beta}}\epsilon^\sigma_\gamma = 2C_1 a^{\alpha\beta}\epsilon^\gamma_\gamma + 2C_2\epsilon^{\alpha\beta} ,$$

in the same way the second term yields

$$\frac{\partial(D^{\gamma\delta\sigma\xi}\epsilon_{\gamma\delta}\kappa_{\sigma\xi})}{\partial \epsilon_{\alpha\beta}} = D^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} ,$$

and putting everything together the relation becomes

$$N^{\alpha\beta} = Eh\left(2C_1 a^{\alpha\beta}\epsilon^\gamma_\gamma + 2C_2\epsilon^{\alpha\beta}\right) + EhD^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} , \quad (3.51)$$

This is the constitutive relation that links stress tensor with the strain and curvature tensor. In analogous way performing the functional derivative with respect to $\kappa_{\alpha\beta}$ we obtain the following relation

$$M^{\alpha\beta} = EhD^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta} + Eh^3\left(F_1 a^{\alpha\beta}\kappa^\gamma_\gamma + F_2\kappa^{\alpha\beta}\right) . \quad (3.52)$$

We now consider the case where one neglects the mixed term, and the dimensionless energy becomes

$$W = Eh[\left(C_1 a^{\alpha\beta}a^{\gamma\delta} + C_2 a^{\alpha\gamma}a^{\beta\delta}\right)\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}+ \\ + \left(F_1 a^{\alpha\beta}a^{\gamma\delta} + F_2 a^{\alpha\gamma}a^{\beta\delta}\right)\kappa_{\alpha\beta}\kappa_{\gamma\delta}] + O\left(\left(\frac{h}{R}\right) + \left(\frac{h}{L}\right)^2\right) , \quad (3.53)$$

where we introduce an error of the order $\frac{h}{R}$. In this case the constitutive relations become

$$\begin{cases} N^{\alpha\beta} = Eh\left(2C_1 a^{\alpha\beta}\epsilon^\gamma_\gamma + 2C_2\epsilon^{\alpha\beta}\right) , & (3.54) \\ M^{\alpha\beta} = Eh^3\left(F_1 a^{\alpha\beta}\kappa^\gamma_\gamma + F_2\kappa^{\alpha\beta}\right) , & (3.55) \end{cases}$$

in this case it's easy to determine the coefficients. We compare the energy of the three dimensional rectangular body problem with the obtained energy.

The strain and bending tensor, defined respectively as the change in metric and curvature tensor, are given by

$$\epsilon_{\alpha\beta} = \frac{1}{2}\left(D_\alpha v_\beta + D_\beta v_\alpha\right) - d_{\alpha\beta}w + \frac{1}{2}(D_\alpha v^\gamma D_\beta v^\gamma - d_{\beta\gamma}wD_\alpha v^\gamma + \\ - d_{\alpha\gamma}wD_\beta v^\gamma + d_{\alpha\gamma}d^\gamma_\beta w^2 + d_{\alpha\beta}d_{\beta\delta}v^\gamma v^\delta + d_{\alpha\gamma}v^\gamma w_{,\alpha} + w_{,\alpha}w_{,\beta}) , \quad (3.56)$$

instead the exact expression of the bending tensor is too cumbersome and we will not report it here, it can be found in [17], instead the linearised approximation is

$$\kappa_{\alpha\beta} \simeq D_\alpha D_\beta w + d_{\alpha\gamma}D_\beta v^\gamma + d_{\beta\gamma}D_\alpha v^\gamma + v^\gamma D_\beta d_{\gamma\alpha} - d_{\beta\gamma}d^\gamma_\alpha w \quad (3.57)$$

Pure tension: $v^1 = \epsilon u^1$, $v^2 = -\nu\epsilon u^2$, $w^3 = -\nu\epsilon z$

In this case strain and stress are

$$\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & -\nu\epsilon & 0 \\ 0 & 0 & -\nu\epsilon \end{pmatrix}$$

$$\sigma = \begin{pmatrix} E\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And the three dimensional energy is given by

$$W_{3D} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{ij}\epsilon_{ji}dz = \frac{1}{2}E\epsilon^2 h\,.$$

In the two dimensional case, the covariant derivatives become ordinary derivatives and the energy is

$$W_{2D} = \left(C_1(1-\nu)^2 + C_2(1+\nu)^2\right) Eh\epsilon^2 \, .$$

so we get the first equation that we need in order to determine the coefficients.

Pure shear: $v^1 = \gamma u^2$, $v^2 = \gamma u^1$, $v^3 = 0$
In this case the strain and stress are

$$\epsilon = \begin{pmatrix} \gamma & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma = \frac{E\nu}{1+\nu} \begin{pmatrix} 1-\nu & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And the three dimensional energy becomes

$$W_{3D} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{ij}\epsilon_{ji} dz = \frac{1}{1-\nu}\gamma^2 \, ,$$

and similarly the two dimensional energy is

$$W_{2D} = 2C_2\gamma^2 \, ,$$

we gent the second equation needed in order to determine the constants.

We can now determine the first two constants which are given by

$$\begin{cases} C_1 = \dfrac{1}{2}\dfrac{\nu}{1-\nu}, & (3.58) \\ C_2 = \dfrac{1}{2}\dfrac{1}{1-\nu}. & (3.59) \end{cases}$$

In order to determine the constants F_1 and F_2 we consider pure bending and twist, in this cases the strain tensor of the two dimensional problem vanishes, and the bending tensor is non zero.

Pure bending: $v^1 = -2ku^1 z$, $v^2 = 0$, $v^3 = k(u^1)^2$
In this case the three dimensional stress and strain are

$$\epsilon = \begin{pmatrix} -2kz & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.60}$$

$$\sigma = -2Ekz \begin{pmatrix} \frac{1-\nu}{1+\nu} & 0 & 0 \\ 0 & -\frac{\nu}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & -\frac{\nu}{(1+\nu)(1-2\nu)} \end{pmatrix} \tag{3.61}$$

and the three dimensional energy is

$$W_{3D} = \frac{1}{6}\frac{Eh^3}{1-\nu^2}k^2\,,$$

instead the two dimensional energy is given by

$$W_{2D} = 4Eh^3k^2(F_1+F_2)\,,$$

which gives the fist equation we need:

$$F_1 + F_2 = \frac{1}{24}\frac{1}{1-\nu^2}\,. \tag{3.62}$$

Pure twist: $v^1 = \phi u^2 z$, $v^2 = \phi u^i z$, $v^3 = -\phi u^1 u^2$
In this case the three dimensional strain and stress are given by

$$\epsilon = \begin{pmatrix} 0 & \phi z & 0 \\ \phi z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma = \frac{E}{1+\nu}\begin{pmatrix} 0 & \phi z & 0 \\ \phi z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with three dimensional energy

$$W = \frac{1}{12}\frac{h^3}{1+\nu}Eh^3\phi^2\,,$$

instead the two dimensional energy is

$$W_{2D} = 2Eh^3F_2\phi^2\,.$$

We are able now to determine the coefficients F_1 and F_2, they are given by

$$\begin{cases} F_1 = \dfrac{1}{24}\dfrac{\nu}{1-\nu^2}, & (3.63) \\ F_2 = \dfrac{1}{24}\dfrac{1}{1-\nu^2}. & (3.64) \end{cases}$$

So the energy density per unit area of the membrane with the introduced error and the constitutive relations become

$$
\begin{cases}
W = \dfrac{Eh}{2(1-\nu)}\left(\nu\epsilon_\alpha^\alpha\epsilon_\beta^\beta + \epsilon_\beta^\alpha\epsilon_\alpha^\beta\right) + \dfrac{Eh^3}{24(1-\nu^2)}\left(\nu\kappa_\alpha^\alpha\kappa_\beta^\beta + \kappa_\beta^\alpha\kappa_\alpha^\beta\right), & (3.65)\\
N^{\alpha\beta} = \dfrac{Eh}{1-\nu}\left((1-\nu)\epsilon^{\alpha\beta} + \nu a^{\alpha\beta}\epsilon_\gamma^\gamma\right), & (3.66)\\
M^{\alpha\beta} = \dfrac{Eh^3}{12(1-\nu^2)}\left((1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa_\gamma^\gamma\right). & (3.67)
\end{cases}
$$

We arrived at the end of the derivation of the theory. We are able now to formulate a well-posed mathematical problem. using the expression of the strain and bending tensor in terms of the displacements v^α and w, one can obtain three differential equations with the boundary conditions.

Even with the various simplifications we introduced in the formulation of the theory, it is nonlinear and analytical solutions are not available except for particularly simple cases. We will formulate our problem in the case of thin shells of revolution.

3.3 Cylindrical thin shell

Let us now derive the theory of cylindrical thin elastic shells. There is a lot of well known works done for this geometry. A cylindrical shell is described in cylindrical coordinates and the azimuthal and angular coordinates coincide with the intrinsic coordinates of the surface, thus the parametrisation of the surface is

$$
\begin{cases}
x^1 = x, & (3.68)\\
x^2 = R\sin\left(\dfrac{\phi}{R}\right), & (3.69)\\
x^3 = R\cos\left(\dfrac{\phi}{R}\right), & (3.70)
\end{cases}
$$

with metric and curvature tensor given by

$$
a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.71}
$$

$$
d = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}, \tag{3.72}
$$

with Christoffel symbol vanishes and the covariant derivatives become ordinary derivatives.

3.3.1 Love's linear equations of cylindrical shell

Love equations' for elastic cylindrical linear shell were derived from the Kirchhoff hypothesis, see [18]. We will go with different way. We start from the quadratic energy density where we retain the mixed term, but written in the following way

$$(1-\nu^2)D^{\alpha\beta\gamma\delta} = d_1\frac{h^2}{R}a^{\alpha\beta}a^{\gamma\delta} + d_2\frac{h^2}{R}a^{\alpha\gamma}a^{\beta\gamma} + d_3h^2a^{\alpha\beta}d^{\gamma\delta} + \\ + d_4h^2(a^{\beta\delta}d^{\alpha\gamma} + a^{\alpha\delta}d^{\beta\gamma}) + d_5h^2a^{\gamma\delta}d^{\alpha\beta}\,, \quad (3.73)$$

where the $d_1, \ldots$ are dimensionless functions of ν.

The equations of equilibrium are given by

$$\begin{cases} N^{11}_{,1} + N^{21}_{,1} + F^1 = 0, & (3.74)\\ N^{12}_{,1} + N^{22}_{,2} - \dfrac{2}{R}\left(M^{21}_{,1} + M^{22}_{,2}\right) + F^2 = 0, & (3.75)\\ M^{11}_{,11} + 2M^{12}_{,12} + M^{22}_{,22} - \dfrac{1}{R^2}M^{22} + \dfrac{1}{R}N^{22} - p = 0. & (3.76)\end{cases}$$

The constitutive relations are given by

$$\begin{cases} N^{\alpha\beta} = \dfrac{Eh}{1-\nu}\left((1-\nu)\epsilon^{\alpha\beta} + \nu a^{\alpha\beta}\epsilon^{\gamma}_{\gamma}\right) + \dfrac{Eh}{1-\nu^2}D^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}, & (3.77)\\ M^{\alpha\beta} = \dfrac{Eh^3}{1-\nu^2}\left((1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^{\gamma}_{\gamma}\right) + \dfrac{Eh}{1-\nu^2}D^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}, & (3.78)\end{cases}$$

with linearised strain and bending tensor which are given by

$$\begin{cases} \epsilon_{\alpha\beta} = \dfrac{1}{2}\left(D_\alpha v_\beta + D_\beta v_\alpha\right) - d_{\alpha\beta}w, & (3.79)\\ \kappa_{\alpha\beta} = D_\alpha D_\beta w + d_{\alpha\gamma}D_\beta v^\gamma + d_{\beta\gamma}D_\alpha v^\gamma - d_{\beta\gamma}d^\gamma_\alpha w, & (3.80)\end{cases}$$

where expressing in matrix form and using the convention $v^1 = u$ and $v^2 = v$ we get

$$\epsilon = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial \phi} + \frac{w}{R} \end{pmatrix} \quad (3.81)$$

$$\kappa = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x\partial \phi} - \frac{1}{R}\frac{\partial v}{\partial x} \\ \frac{\partial^2 w}{\partial x\partial \phi} - \frac{1}{R}\frac{\partial v}{\partial x} & \frac{\partial^2 w}{\partial \phi^2} - \frac{2}{R}\frac{\partial v}{\partial \phi} - \frac{w}{R^2} \end{pmatrix} \quad (3.82)$$

If one substitutes the expressions of the stress and moment as functions of the displacement field, one gets the following equations

$$(d_1+d_2)\frac{h^2}{R}\left(\frac{\partial^3 w}{\partial x\partial\phi^2} + \frac{\partial^3 w}{\partial x^3}\right) + \frac{\partial^2 u}{\partial^2 x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \phi^2} + \\ + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial\phi} + \frac{\nu}{R}\frac{\partial w}{\partial x} + \frac{1-\nu^2}{Eh}F_x = 0, \quad (3.83)$$

$$\left(d_1+d_2-d_5-\frac{1}{6}\right)\frac{h^2}{R}\left(\frac{\partial^3 w}{\partial x^2\partial\phi}+\frac{\partial^3 w}{\partial\phi^3}\right)+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial^2\phi^2}+ \\ +\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial\phi}+\frac{1}{R}\frac{\partial w}{\partial\phi}+\frac{1-\nu^2}{Eh}F_\phi=0, \quad (3.84)$$

$$\left(d_1+d_2-d_5-\frac{1}{6}\right)\frac{h^2}{R}\left(\frac{\partial^3 v}{\partial\phi^3}+\frac{\partial^3 v}{\partial x^2\partial\phi}\right)+(d_1+d_2)\frac{h^2}{R}\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x\partial\phi^2}\right)+ \\ +\left(2d_1-2d_5-\frac{1}{6}\nu\right)\frac{h^2}{R^2}\frac{\partial^2 w}{\partial x^2}+\left(2d_1+2d_2-2d_5-\frac{1}{6}\right)\frac{h^2}{R^2}\frac{\partial^2 w}{\partial\phi^2}+ \\ +\frac{\nu}{R}\frac{\partial u}{\partial x}+\frac{1}{R}\frac{\partial v}{\partial\phi}+\frac{w}{R^2}+\frac{h^2}{12}\Delta^2 w-\frac{1-\nu^2}{Eh}p=0, \quad (3.85)$$

where one has set $d_3=d_4=0$ since they do not contribute to any simplification. Here we have also omitted terms that appear twice but with higher order in h/R, e.g. the term $\frac{\partial w}{\partial x}$ in the first equation turns to be $\nu/R+d_1h^2/R^2$, and we omit the term of order h^2/R^2. One can set d_1, d_2 and d_5 in such a way that simplifies the differential equations. Clearly setting $d_1=-d_2$ and $d_5=-\frac{1}{6}$ the third order derivatives vanish. Further simplification is obtained if one sets the coefficients in front of the terms $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial\phi^2}$ equal to each other one bets that $d_2=\frac{1}{12}(1-\nu)$. With this simplifications the equations become

$$\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}(1-\nu)\frac{\partial^2 u}{\partial\phi^2}+\frac{1}{2}(1-\nu)\frac{\partial^2 v}{\partial x\partial\phi}+\frac{\nu}{R}\frac{\partial w}{\partial x}+\frac{1-\nu^2}{Eh}F_x = 0, \quad (3.86)$$

$$\frac{1}{2}(1+\nu)\frac{\partial^2 u}{\partial x\partial\phi}+\frac{1}{2}(1-\nu)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial\phi^2}+\frac{1}{R}\frac{\partial w}{\partial\phi}+\frac{1-\nu^2}{Eh}F_\phi = 0, \quad (3.87)$$

$$\frac{\nu}{R}\frac{\partial u}{\partial x}+\frac{1}{R}\frac{\partial v}{\partial x}+\frac{w}{R}+\frac{h^2}{12R^2}\left(R\Delta+\frac{1}{R}\right)^2 w-\frac{1-\nu^2}{Eh}p = 0. \quad (3.88)$$

These are the Morley-Koiter equations originally derived by Love in 1888. Koiler showed that these equations are the simplest possible within a first-approximation theory, for a cylindrical thin shell. From the original system one can obtain the boundary conditions to apply to the system.

3.3.2 Twistless linear theory

From the previous equation we can derive a set of ordinary differential equations for the case in which the cylindrical shell is of revolution also in its reference state. More specifically we consider axisymmetric, twist-less deformation. By assumption the displacement has zero azimuthal component, i.e.

$v^2 = v = 0$, and the radial, axial and forces does not depend on ϕ. The strain-displacement relation becomes

$$\begin{cases} \epsilon_{11} = \dfrac{du}{dx} + \dfrac{1}{2}\left(\dfrac{dx}{dx}\dfrac{du}{dx} + \dfrac{dw}{dx}\dfrac{dw}{dx}\right), & (3.89) \\ \epsilon_{12} = 0, & (3.90) \\ \epsilon_{22} = \dfrac{w}{R} + \dfrac{w^2}{R^2}. & (3.91) \end{cases}$$

The variational free energy we write it as

$$\delta W = \int_{l_1}^{l_2} \left(N^{\alpha\beta}\delta\epsilon_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right) 2\pi R dz + \\ + \int_{l_1}^{l_2} (F_z\delta u + p\delta w)\, 2\pi R dz + \sum_{i=1}^{2} 2\pi R\, (p(l_i)\delta w + F_z(l_i)\delta u) \ , \quad (3.92)$$

where we included the potential energy of the external load.

Depending on the approximation that one wants to use, we can derive a linear, quasi linear and non-linear theory for twistless shells. The linear theory, can be obtained from the previous equations, where one puts the $d_1, \ldots$ coefficients to zero or from the Morley-Koiter equations, where one uses the hypothesis of $v = 0$ and displacement being independent of ϕ. The two ways produce similar equations with a slight change in the constant coefficients in the ODEs.

We now want to consider a quasi-linear theory. This consist of retaining second order terms in w wherever first order are not present, but neglecting second order terms in u. This means that the strain tensor becomes in this approximation

$$\epsilon = \begin{pmatrix} \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 & 0 \\ 0 & \frac{w}{R} \end{pmatrix} \quad (3.93)$$

Regarding the bending tensor, it is trickier to evaluate, since the linear approximation we have, Equation (3.57), does not contain all the needed terms. One can do some trivial but long algebra using the expression given in [17], and obtain that the diagonal terms of the tensor are given by

$$\kappa_{11} = \sqrt{\frac{1}{(2\epsilon_{11}+1)(2\epsilon_{22}+1)}}\left(\frac{d^2w}{dx^2} + \frac{du}{dx}\frac{d^2w}{dx^2} + \frac{w}{R}\frac{d^2w}{dx^2} - \frac{dw}{dx}\frac{d^2u}{dx^2}\right) , \quad (3.94)$$

$$\kappa_{22} = \sqrt{\frac{1}{(2\epsilon_{11}+1)(2\epsilon_{22}+1)}}\left(-\frac{1}{R} - \frac{1}{R}\frac{du}{dx} - \frac{2w}{R} - \frac{w}{R^2}\frac{du}{dx} - \frac{w^2}{R^3}\right) + \frac{1}{R}, \quad (3.95)$$

and in the quasi-linear approximation the tensor becomes

$$\kappa = \begin{pmatrix} \frac{d^2w}{dx^2} & 0 \\ 0 & -\frac{1}{R}\frac{du}{dx} - \frac{2u}{R^2} \end{pmatrix}, \tag{3.96}$$

one can observe that these expressions are equal to the ones in [7] but with the reverse sign for the bending tensor due to how one defines the local coordinate system.

In order to derive the equations of equilibrium we need to express the variation $\delta\epsilon_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ as variation of δu and δw. This is quite easy since we know the expression of the strain and bending in term of the displacement filed.

$$\begin{cases} \delta\epsilon_{11} = \dfrac{d\delta u}{dx} + \dfrac{dw}{dx}\dfrac{d\delta w}{dx}, & (3.97) \\ \delta\epsilon_{22} = \dfrac{\delta w}{R}, & (3.98) \\ \delta\kappa_{11} = \dfrac{d^2\delta w}{dx^2}, & (3.99) \\ \delta\kappa_{22} = -\dfrac{1}{R}\dfrac{d\delta u}{dx} - \dfrac{2\delta u}{R^2}, & (3.100) \end{cases}$$

as it clearly appears we weed to to some derivation by parts where we have the total derivative of the displacement variation. Let us now isolate the terms in two parts, the ones that will end up multiplying δu and the ones δw.

1.Variation for δu:

$$\int_{l_1}^{l_2}\left(N^{11}\frac{d}{dx}\delta u - \frac{M^{22}}{R}\frac{d}{dx}\delta u + F_z\delta u\right)2\pi R dz + \sum_{i=1}^{2} 2\pi R F_z(l_i)\delta u = 0\,, \tag{3.101}$$

which integrated by parts becomes

$$\int_{l_1}^{l_2}\left(-\frac{dN^{11}}{dx} + \frac{1}{R}\frac{dM^{22}}{dx} + F_x\right)2\pi R dx + \left[N^{11}(l_2) - \frac{M^{22}(l_2)}{R} + F_x(l_2)\right]2\pi R\delta u(l_2) + \\ + \left[-N^{11}(1_1) + \frac{M^{22}(l_1)}{R} + F_x(l_1)\right]2\pi R\delta u(l_1) = 0\,, \tag{3.102}$$

which must be true for any variation δu, so we have the first equilibrium equation with the boundary conditions.

$$\begin{cases} \dfrac{dN^{11}}{dx} - \dfrac{1}{R}\dfrac{dM^{22}}{dx} = F_x, & (3.103) \\ N^{11}(l_2) - \dfrac{M^{22}(l_2)}{R} + F_x(l_2) = 0, & (3.104) \\ N^{11}(l_1) - \dfrac{M^{22}(l_1)}{R} - F_x(l_1) = 0. & (3.105) \end{cases}$$

We have implicitly assume that the boundary conditions are linearly independent. This helps when one implements a shooting method for solving the final system.

2.Variation for δw:

$$\int_{l_1}^{l_2}\left(N^{11}\frac{dw}{dx}\frac{d}{dx}\delta w+\frac{N^{22}}{R}+M^{11}\frac{d^2}{dx^2}\delta w\right)2\pi R dx+\sum_{i=1}^{2}2\pi r p(l_i)=0\,, \tag{3.106}$$

integrating by parts few times, we get

$$\begin{aligned}\int_{l_1}^{l_2}\left(-\frac{d}{dx}\left(N^{11}\frac{dw}{dx}\right)+\frac{N^{22}}{R}+\frac{d^2M^{11}}{dx^2}+p\right)\delta w 2\pi R dx+\\
+\left[N^{11}(l_2)\frac{dw(l_2)}{dx}-\frac{dM^{11}(l_2)}{dx}+p(l_2)\right]2\pi R\delta w(l_2)+\\
+\left[-N^{11}(l_1)\frac{dw(k_1)}{dx}+\frac{dM^{11}(l_1)}{dx}+p(l_1)\right]2\pi R\delta w(l_1)+\\
+\left[M^{11}2\pi R\frac{d}{dx}\delta w\right]_{l_1}^{l_2}=0,\end{aligned} \tag{3.107}$$

which gives the following set of ODE with boundary conditions

$$\begin{cases}\dfrac{d^2M^{11}}{dx^2}+\dfrac{N^{22}}{R}-\dfrac{d}{dx}\left(N^{11}\dfrac{dw}{dx}\right)+p=0, & (3.108)\\
N^{11}(l_2)\dfrac{dw(l_2)}{dx}-\dfrac{dM^{11}(l_2)}{dx}+p(l_2)=0, & (3.109)\\
-N^{11}(l_1)\dfrac{dw(k_1)}{dx}+\dfrac{dM^{11}(l_1)}{dx}+p(l_1)=0, & (3.110)\\
M^{11}(l_1)=0, & (3.111)\\
M^{11}(l_2)=0. & (3.112)\end{cases}$$

We see that in the twistless case the partial differential equations become ordinary differential equations, that we can express in term of the displacement field thanks to the constitutive relations. Now we can either use the uncoupled constitutive relation and the coupled constitutive relations. Let us proceed in a classical point of view, where we use the uncoupled constitutive relations, Equations (3.66) and (3.67). Substituting everything, one obtains the following differential equations

$$\frac{d^2u}{dx^2}\left(\frac{Eh}{1-\nu^2}+\frac{Eh^3}{R^2(1-\nu^2)}\right)+\frac{dw}{dx}\frac{d^2w}{dx^2}\frac{Eh}{1-\nu^2}+$$
$$+\frac{dw}{dx}\frac{Eh\nu}{R(1-\nu^2)}-\frac{d^3w}{dx^3}\frac{Eh^3\nu}{R(1-\nu^2)}+$$
$$+\frac{du}{dx}\frac{2Eh^3}{R^3(1-\nu^2)}=F_x \quad (3.113)$$

$$\frac{d^4w}{dx^4}\frac{Eh^3}{12(1-\nu^2)}-\frac{du}{dx}\frac{d^2w}{dx^2}\frac{Eh}{1-\nu^2}-\frac{3}{2}\frac{Eh}{1-\nu^2}\left(\frac{dw}{dx}\right)^2\frac{d^2w}{dx^2}-\frac{d^2u}{dx^2}\frac{dw}{dx}\frac{Eh}{1-\nu^2}+$$
$$-\frac{d^3u}{dx^3}\frac{Eh^3\nu}{12R(1-\nu^3)}+\frac{du}{dx}\frac{Eh\nu}{R(1-\nu^2)}-\left(\frac{du}{dx}\right)^2\frac{Eh\nu}{2R(1-\nu^2)}-w\frac{d^2w}{dx^2}\frac{Eh\nu}{R(1-\nu^2)}+$$
$$+\frac{w}{R^2}\frac{Eh}{1-\nu^2}-\frac{d^2u}{dx^2}\frac{2Eh^3\nu}{12R^2(1-\nu^2)}+p=0\,, \quad (3.114)$$

with the six boundary conditions given by

$$\frac{Eh}{1-\nu^2}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2+\frac{\nu}{R}w\right]_{x=l_1}+$$
$$+\frac{Eh^3}{12R(1-\nu^2)}\left[\frac{1}{R}\frac{du}{dx}+\frac{2u}{R}-\nu\frac{dw^2}{dx^2}\right]_{x=l_1}-F_x(l_1)=0, \quad (3.115)$$

$$\frac{Eh}{1-\nu^2}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2+\frac{\nu w}{R}\right]_{x=l_1}+$$
$$-\frac{Eh^3}{1-\nu^2}\left[\frac{d^3w}{dx^3}-\frac{\nu}{R}\frac{d^2u}{dx^2}-\frac{2\nu}{R^2}\frac{du}{dx}\right]_{x=l_1}-p(l_1)=0, \quad (3.116)$$

$$\left[\frac{d^2w}{dx^2}-\frac{\nu}{R}\frac{du}{dx}-\frac{2\nu}{R^2}u\right]_{x=l_1}=0, \quad (3.117)$$

$$\frac{Eh}{1-\nu^2}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2+\frac{\nu}{R}w\right]_{x=l_2}+$$
$$+\frac{Eh^3}{12R(1-\nu^2)}\left[\frac{1}{R}\frac{du}{dx}+\frac{2u}{R}-\nu\frac{dw^2}{dx^2}\right]_{x=l_2}+F_x(l_2)=0, \quad (3.118)$$

$$\frac{Eh}{1-\nu^2}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2+\frac{\nu w}{R}\right]_{x=l_2}+ \\ -\frac{Eh^3}{1-\nu^2}\left[\frac{d^3w}{dx^3}-\frac{\nu}{R}\frac{d^2u}{dx^2}-\frac{2\nu}{R^2}\frac{du}{dx}\right]_{x=l_2}+p(l_2)=0, \quad (3.119)$$

$$\left[\frac{d^2w}{dx^2}-\frac{\nu}{R}\frac{du}{dx}-\frac{2\nu}{R^2}u\right]_{x=l_2}=0. \quad (3.120)$$

We obtained a system of two nonlinear ordinary differential equations, that is of a forth order in w and second in u with six boundary conditions. The author of this work has not tried to solve this system analytically. The system can be solved computationally using shooting methods in standard libraries that are able to implement these kinds of boundary conditions.

Chapter 4

Constricting ring-like stimulus

4.1 Numerical validation of the Hookean elasticity

As we previously said, the Hookean relation between stress and strain is justified if the components of the strain tensor are small with respect to unity. Since we have a symmetric deformation, we will consider the behavior of the gradient along the longitudinal direction.

4.2 Validation of the algorithm in the linear elastic regime and the necessity of non-linear simulations

First we validate the algorithm through the analysis of the variation of the volume, radius and thickness of the cylindrical shell in the linear regime, due to the presence only of a constant internal pressure. Thanks to the axisymmetri of our model, we can easily compute the various displacements and stresses. The 3D elasticity equations in cylindrical coordinates and with the assumption $\sigma_{rz} = \sigma_{r\theta} = \sigma_{z\theta} = 0$ read

$$\begin{cases} \frac{\partial \sigma_{rr}}{\partial r} - \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0 \\ \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \\ \frac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases} , \tag{4.1}$$

with the initial boundary conditions

$$\sigma_{rr}(r = R_{in}) = -p, \qquad \sigma_{rr}(r = R_{out}) = 0, \qquad \sigma_{zz}(z = \pm L/2) = 0\,, \tag{4.2}$$

where $R_{in} = R - h/2$,$R_{out} = R + h/2$ are the internal and external radii or the cylindrical shell and L is its length. As we said, our system is axisymmetric, thus we can assume that our quantities depends only on r and z. Thus from the second equation of Equation(4.1) we have that the stress $\sigma_{\theta\theta}$ is a function of r and z only.

From the third equation of the of Equation(4.1) and the boundary conditions, we have $\sigma_{zz} = 0$ this means that the strain-stress relations reduces to

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}\left(\epsilon_{rr} + \nu\epsilon_{\theta\theta}\right) \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\epsilon_{\theta\theta} + \nu\epsilon_{rr}\right) \end{cases}, \tag{4.3}$$

where the strain components have the following displacement dependence,

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \epsilon_{zz} = \frac{\partial u_z}{\partial z}. \tag{4.4}$$

Using this relations, and the stress-strain relations, the equilibrium equations reduces to

$$\begin{cases} \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0 \\ \frac{du_z}{dz} = -\frac{\nu}{1-\nu}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) \end{cases}, \tag{4.5}$$

the first equation ha a solution of the form $u_r(r) = ar^{-1} + br$, and the solution of the second equation is $u_z(z) = -\frac{2b\nu}{1-\nu}z + c$. Imposing the boundary conditions and the $z \mapsto -z$ symmetry, we obtain the following solutions

$$\begin{cases} u_r(r) = \frac{P}{E\left(\frac{R_{out}^2}{R_{in}^2} - 1\right)}\left((1+\nu)\frac{R_{out}^2}{r} + (1-\nu)\,r\right) \\ u_z(z) = -\nu\frac{P}{E\left(\frac{R_{out}^2}{R_{in}^2} - 1\right)}z \\ u_\theta = 0 \end{cases}. \tag{4.6}$$

From this expressions we can obtain the relative change in length, radius and thickness, which are given by the following expressions

$$\begin{cases} \frac{\Delta L}{L} = -\nu\frac{P}{E\left(\frac{R_{out}^2}{R_{in}^2} - 1\right)} \underset{\frac{h}{R}\ll 1}{\simeq} -\frac{\nu}{2}\frac{PR}{Eh} \\ \frac{\Delta R}{R} = \frac{u_r(R)}{R} \underset{\frac{h}{R}\ll 1}{\simeq} \frac{PR}{Eh}\left(1 + (\nu - 1)\frac{h}{2R}\right) \\ \frac{\Delta h}{h} = \frac{u_r(R_{out}) - u_r(R_{in})}{h} \underset{\frac{h}{R}\ll 1}{\simeq} -\frac{PR}{Eh}\left(\nu + (1-\nu)\frac{h}{2R}\right) \end{cases}. \tag{4.7}$$

We observe, from Figure(4.1) that in the limit of small values of P/E, smaller than 0.01 the simulations agrees well enough with the linear model. As P/E increases the nonlinear behaviour becomes important and one should take into account the nonlinear terms of the strain tensor. Observing that the expressions of the quantities in Equation(4.7) in the limit $\frac{h}{R} \ll 1$ are linear in $\frac{PR}{Eh}$, plotting the data from the simulations against this quantity we see that the points collapse on a single curve. It is evident that this property extends also to the nonlinear region. In [2] this observation was used to determine an analytic expansion that well describes the numerical simulation also in the nonlinear region.

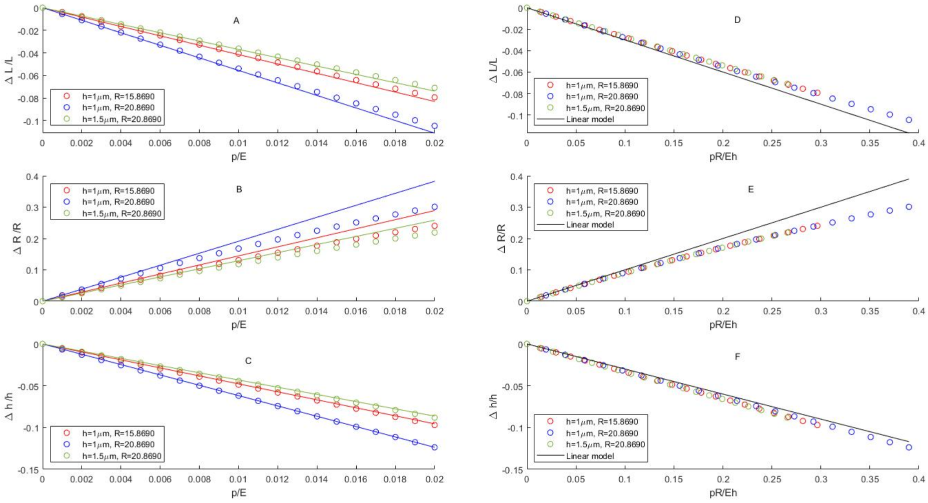

Figure 4.1: Effect of internal pressure on a cylindrical membrane. Relative change in the length L (A), radius R (B) and thickness h (C) of the cylindrical shell. The scattered points are obtained from simulations with the reported values of thickness and radius, the lines are the expressions of the Equation(4.7). The plots (D,E,F) are obtained plotting the same quantities as functions of $\frac{PR}{Eh}$, and the straight lines are the expressions of Equation(4.7) in the $\frac{h}{R} \ll 1$ limit.

In the case of the nematode, Young modulus, internal pressure, radius and thickness are such that $\frac{PR}{Eh} \simeq 0.2$. This value shows us that we actually need a non-linear simulation, rather than a linear one.

4.3 Validation of the numerical scheme using the elastic theory of axisymmetric shells

We are describing the mechanical properties of the nematode as the ones of a thin pressurised elastic cylindrical shell. The only way we have to approach this problem in it is full generality is through numerical simulations, this is due to the geometric nonlinearities which are intrinsic to the problem. We want to go deeper into the validation of the numerical simulations and we will do it using some results of the this shell of revolution, in particular the case of axisymmetric, twistless deformations.
As we developed the equations in 3.3.2, where one assumes that the azimuthal

displacement is zero, and the radial and axial displacement are independent on the azimuthal coordinate. We consider the case of small applied pressure, where we use a quasilinear theory.

We consider a thin cylindrical shell with free edges under uniquely a radial load. Under this assumptions, our system is axially symmetric and the equations we need to solve are given by[1]

$$\begin{cases} \frac{d^2 M_s}{ds^2} - \frac{N_\theta}{\rho} + \frac{d(N_s v_s')}{ds} + f_r(s) = 0 \\ \frac{1}{\rho}\frac{dM_\theta}{ds} + \frac{dN_s}{ds} = 0 \end{cases}, \tag{4.8}$$

with the following boundary conditions

$$\begin{cases} \left.\frac{dM_s}{ds}\right|_{s=s_1} + \left.N_s v_s'\right|_{s=s_i} - F_r^i(s_i) = 0 \\ N_s(s_i) + \frac{M_\theta(s_i)}{\rho} = 0 \\ M_s(s_i) = 0 \end{cases}, \tag{4.9}$$

where in our case $s_1 = -L/2$ and $s_2 = L/2$ with L being the length of the worm. The stresses and moments are linked respectively to the strain and curvature strain through the following relations

$$\begin{cases} N_s = \frac{Eh}{1-\nu^2}\left(\epsilon_s + \nu\epsilon_\theta\right), \qquad N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_\theta + \nu\epsilon_s\right) \\ M_s = D\left(k_s + \nu k_\theta\right), \qquad M_\theta = D\left(k_\theta + \nu k_s\right) \end{cases}, \tag{4.10}$$

where in the quasi-linear approximation, that we're considering here, the strain and the curvature strain in terms of the radial and axial displacement are given by

$$\begin{cases} \epsilon_s = v_z' + \frac{1}{2}\left(\frac{dv_r}{ds}\right)^2, \qquad \epsilon_\theta = \frac{v_r}{\rho} \\ k_s = -v_r'', \qquad k_\theta = \frac{v_z'}{\rho} \end{cases}. \tag{4.11}$$

Inserting the stress and bending expressions in the Equation(4.8), adimensionalization and using the boundary conditions, we obtain the following boundary value problem

$$\begin{cases} \frac{d^4 v}{d\xi^4} - \frac{d^2 v}{d\xi^2}\left(A_1 + A_2\left(\frac{dv}{d\xi}\right)^2 + A_2 v\right) - \frac{dv}{d\xi}\frac{dv}{d\xi} + v = A_4 f_r(B\xi) \\ \frac{dv_z}{ds} = \frac{D\nu\rho(1-\nu^2}{Eh\rho^2+D(1-\nu^2)}\frac{d^2 v_r}{ds^2} - \frac{Eh\rho^2}{Eh\rho^2+D(1-\nu^2)}\left(\frac{1}{2}\left(\frac{dv_r}{ds}\right)^2 + \frac{\nu}{\rho}v_r\right) \\ \left.\frac{d^2 v}{d\xi^2}\right|_{\xi=\xi_i} + \frac{\rho\nu A_2}{A}\left.\left(\frac{dv}{d\xi}\right)^2\right|_{\xi=\xi_i} + \frac{2\rho\nu}{A}v(\xi_i) = 0 \\ \left.\frac{d^3 v}{d\xi^3}\right|_{\xi=\xi_i} - \left.\frac{dv}{d\xi}\right|_{\xi=\xi_i}\left(\frac{A_1}{2} + \frac{A_2}{3}\left(\left.\frac{dv}{d\xi}\right|_{\xi=\xi_i}\right)^2 + A_3 v(\xi_i)\right) = 0 \\ v_z(0) = 0 \end{cases}, \tag{4.12}$$

[1]The following relations change in the case of the radius depending on arc length s. We consider the case of constant radius.

where the non dimensional quantities are defined as

$$\begin{aligned} v_r &= Av, \qquad s = B\xi, \\ A &= \frac{2\rho}{\nu}\frac{Eh\rho^2 + D(1-\nu^2)^2}{EhB^2}, \\ B^4 &= \frac{\rho^2}{Eh}\frac{D(Eh\rho^2 + D(1-\nu^2)^2}{Eh\rho^2 + D} \end{aligned} \tag{4.13}$$

$$\begin{aligned} A_1 &= -\frac{2Eh\nu^2 B^2}{Eh\rho^2 + D(1-\nu^2)^2}, \qquad & A_2 &= \frac{3}{2}\frac{EhA^2}{Eh\rho^2 + D(1-\nu^2)^2}, \\ A_3 &= \frac{\nu}{\rho}\frac{EhB^2 A}{Eh\rho^2 + D(1-\nu^2)^2}, \qquad & A_4 &= \frac{Eh\rho^2 + D(1-\nu^2)}{Eh\rho^2 + D(1-\nu^2)^2}\frac{B^4}{A}, \end{aligned} \tag{4.14}$$

Here we considered the $s \mapsto -s$ symmetry of out system, in the general case, one must consider the second equation being a second order ODE and the whole system will be a proper boundary value problem where the second boundary condition is on the first derivative of w.

We can easily integrate this system with numerical methods, we used the MATLAB function 'bvp5c' for v_r and a numerical integration for v_z. This way we obtain the radial and axial displacement. in particular we considerate a pressurized cylinder with a pressure applied on the region $(-r_{ind}, r_{ind})$ with $r_{ind} = 2\mu m$, where $f_r(s)$ has the following expressions

$$f_r(s) = \begin{cases} p & \text{if} s \leq -r_{ind} \\ p - f & \text{if} - r_{ind} < s < r_{ind} \\ p & \text{if} s \geq r_{ind} \end{cases} . \tag{4.15}$$

We determine the mechanical response of a thin pressurized cylinder, where the pressure f is applied on the external surface and p on the internal one. The numerical methods are solving directly the three dimensional elastic equations. And we confront the simulations with the solutions we obtain solving the one dimensional equation for the this shells of revolution.

We show in Figure 4.2 the result of different mesh size. Clearly the deformation profile is captured by our simulations even with a not quite fine mesh size.

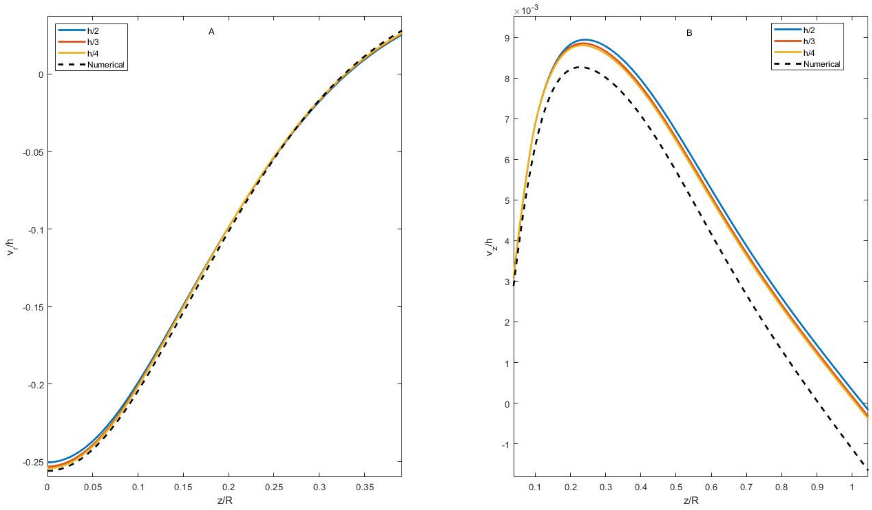

Figure 4.2: Mechanical response of a pressurized thin cylindrical shell to a radial pressure on symmetric region. (**A**) Radial displacement along the axial direction. (**B**) Axial displacement along the axial direction. The numerical solution (black line) is well approximated by the numerical simulation as the mesh becomes finer, where h is the shell thickness and the mesh size is given in terms of h. The parameters of the simulations are $h/R = 5 \cdot 10^{-2}$, $h = 1.0708 \mu m$, $p/E = 10^{-4}$, and $f = 10^{-2} \mu N/\mu m^2$

4.4 Channel's response as function of the filament's orientation

Until now we analysed the validation of the numerical scheme. Now we analyse the mechanical response of the neural channel as function of the filament orientation. Since the opening probability depends on the tangential components of the forces acting on the channel, i.e. Equation(1.9), we want to see now how this components depends on the initial orientation of the filament.

In [2] is shown that the numerical data are described better if the initial orientation of the filament is along the axial direction. We will assume that all the filaments are oriented axially, but before that we see the tangential components of the force acting on the filament at different orientation, in order to see if our results are consistent with the one shown in [2].

The first consideration comes from thin shell theory. In this theory, the terms ϵ_{xz} and ϵ_{yz} are assumed to be small compared to the other components. The direct consequence of this hypothesis, using the relation $d\overline{r}'_1 \cdot d\overline{r}'_2 - d\overline{r}_1 \cdot d\overline{r}_2 = 2\epsilon_{ij} dr_{1,j} dr_{2,j}$, that vectors initially normal to the surface remain normal also after the deformation. This is important since the neural membrane deforms

in the same way as the nematodes surface, so the normal components remains normal also after the deformation. The only available rotation in this case are the ones around the normal, but as we see from Figure(4.3B), the strain is essentially diagonal so also ϵ_{xy} is negligible.
A single channel is subjected to an elastic force $\overline{F}_{elastic} = k\overline{x}$, where $\overline{x} = \Delta\overline{r}^f - \Delta\overline{r}^c$. The tangential components of the force are non negligible if the elastic filament is oriented in the tangential plane to the channel and are negligible if it is in the orthogonal direction. We can observe in Figure(4.3), that the force have different sign if the filament is oriented in the $\overline{w}_1$ or $\overline{w}_2$, but the amplitude is quite different. This is totally consistent due to the fact that the applied stimulus is neither globally or locally invariant by rotation of $\pi/2$ around the normal.

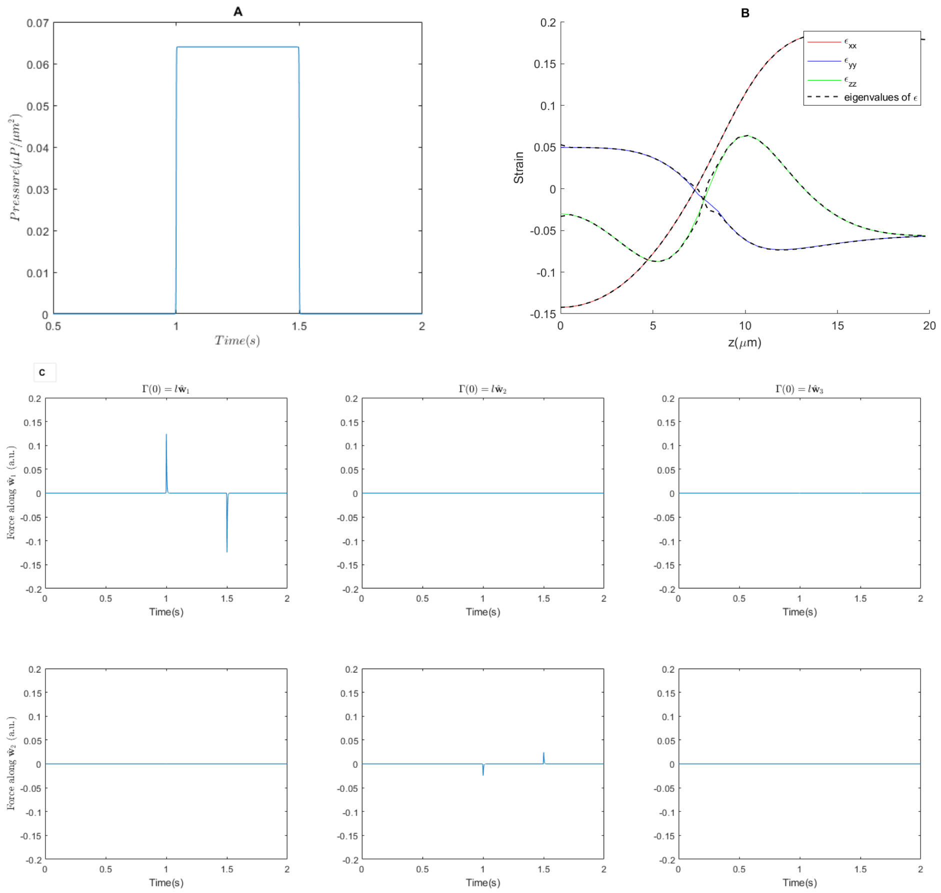

Figure 4.3: Mechanical stimuli of the channel due to a step. (**A**) Stimulus profile due to a step. (**B**)Diagonal components of the strain tensor and its eigenvalues. (**C**) The two tangential components of the force acting on a single channel for the stimulus in panel A, with filament orientation respectively along $\overline{w}_1$,$\overline{w}_2$ and $\overline{w}_3$.
The parameters of the simulations are $P/E = 10^{-2}$, $p_{\text{external}} = 5 \cdot 10^{-2} \mu N/\mu m^2$, cylinders radius $R = 20\mu m$, length of the application region $r = 4\mu m$.

The type of stimulus we are considering gives analogous force components as observed in [2], and since we do not have experimental data to determine the best orientation of the filament, we can orient them parallel to the neural membrane.

4.5 Mechanical and neural response dependencies

As shown in [2], the mechanical and neural response strongly depend on the radius of the indenting beam. Here we will do a similar analysis, where we will look at the length of the application region and the values of the pressure we're applying.
We concentrate in particular on the PVM touch receptor neuron, and apply the pressure at the middle of the worm. In this way the neuron membrane extends roughly the same size in both directions. For now we neglect the neural response to the other neighbouring TRNs, and we will take them into account subsequently.

The radial displacement for different values of the the application region's length, r, are shown in Figures (4.4A,4.5A, 4.6A). As one can notice the maximum displacement and the profile of the deformation strongly depend on the value of r. In Figures (4.4B, 4.5B, 4.6B) are shown the profiles of the mean current for different values of r due to a stimulus profile of the form of Figure4.7A.

The displacement profile for small values of r are similar, indeed one can verify that the displacement at the origin as function of r is a linear one; likewise does the current. One can observe this growth in the Figure(4.7B) for the maximum average current flowing inside the neuron as function of r. For values or $r \leq 11$ one can see a change of behaviour in the mean current, Figure(4.4B,4.5B,4.6B), and the maximum current, Figure(4.7B). This change of behaviour is strongly reflected in the gating probability, Figure(4.4C,4.5C,4.6C).

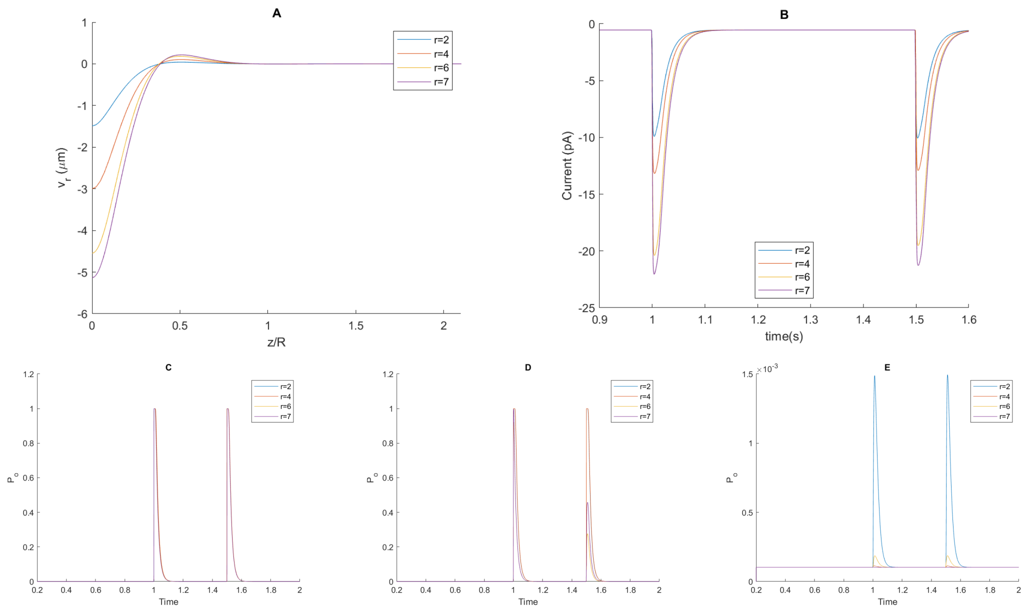

Figure 4.4: Mechanical and neural behaviour for different values of the application region. (**A**)The deformation behaviour depends on the length of the application region, r. (**B**)The mean neural current response to a step. (**C**-**E**)Gating probability for an individual channel for different r. The position of the channel is respectively at $z = 0$,$z = r/2$ and $z = 5r$, where the frame if reference is set in such a way that the application region is $(-r/2, r/2)$.

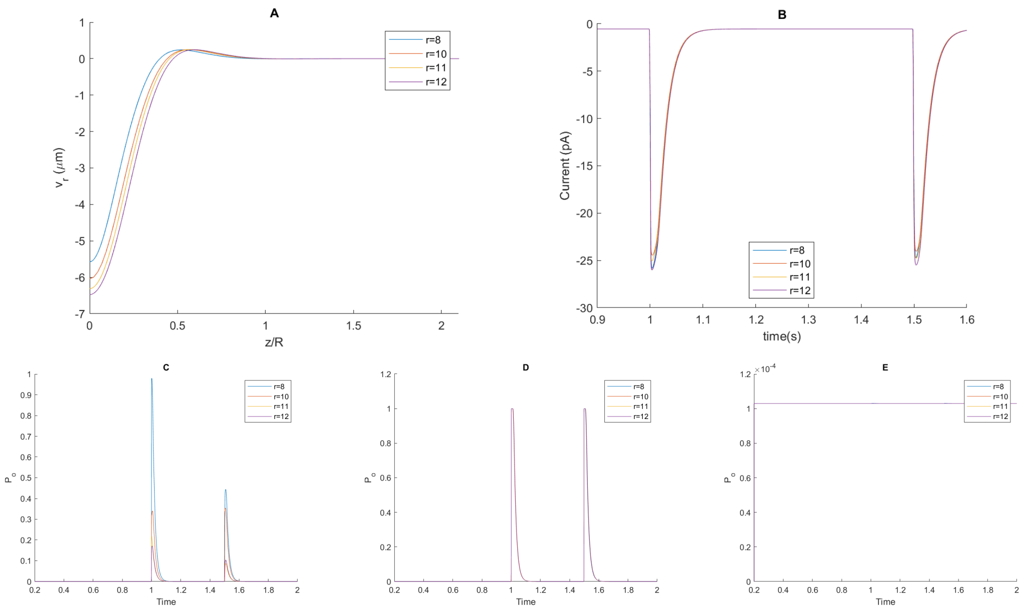

Figure 4.5: Mechanical and neural behaviour for different values of the application region. (**A**)The deformation behaviour depends on the length of the application region, r. (**B**)The mean neural current response to a step. (**C**-**E**)Gating probability for an individual channel for different r. The position of the channel is respectively at $z = 0$,$z = r/2$ and $z = 5r$, where the frame if reference is set in such a way that the application region is $(-r/2, r/2)$.

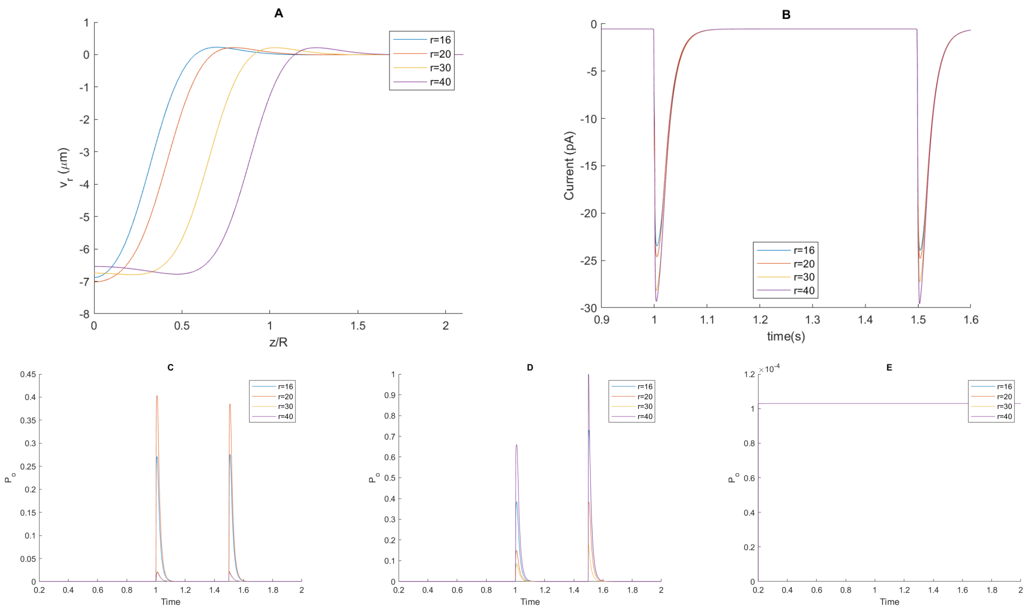

Figure 4.6: Mechanical and neural behaviour for different values of the application region. (**A**)The deformation behaviour depends on the length of the application region, r. (**B**)The mean neural current response to a step. (**C**-**E**)Gating probability for an individual channel for different r. The position of the channel is respectively at $z = 0$,$z = r/2$ and $z = 5r$, where the frame if reference is set in such a way that the application region is $(-r/2, r/2)$.

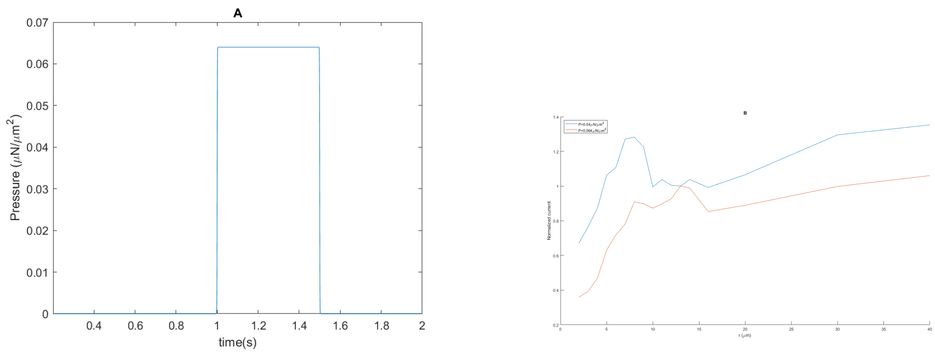

Figure 4.7: (**A**)Application profile for a step stimulus. (**B**) The maximum current for a given r for a step stimulus. The current is normalized by its value for $r = 10$.

As stated in [19], the PVM by itself does not mediate a noticeable touch response, i.e. the worm will not have a forward/backward escape behaviour. From this observation we want to see if the proposed model, captures the inability of the PVM to distinguish between front or back stimulus. We will analyse the current in the PVM as function of the distance between the center of the stimulus and the end of the neuron. We show the results in Figure(4.8) and one can observe that the neural response is statistically the same from left and right, so the PVM cannot distinguish the same stimulus if it's applied on the posterior or anterior part of the nematode.

We present another quantity which is the local current normalized by it is maximum value i_0. We can observe similar behaviour between the various curves for small values of r. For large values of r the normalized local current changes drastically behaviour. This way one can obtain the range over which a stimulus of this kind is felt by a single channel.

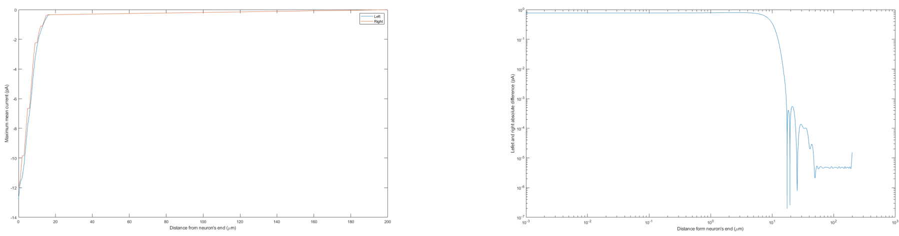

Figure 4.8: (**A**)Maximum current in the PVM against the distance of the stimulus, respectively on the left and right. (**B**) Log-log plot of the absolute difference between the left and right mean current vs the distance from the neuron's end.

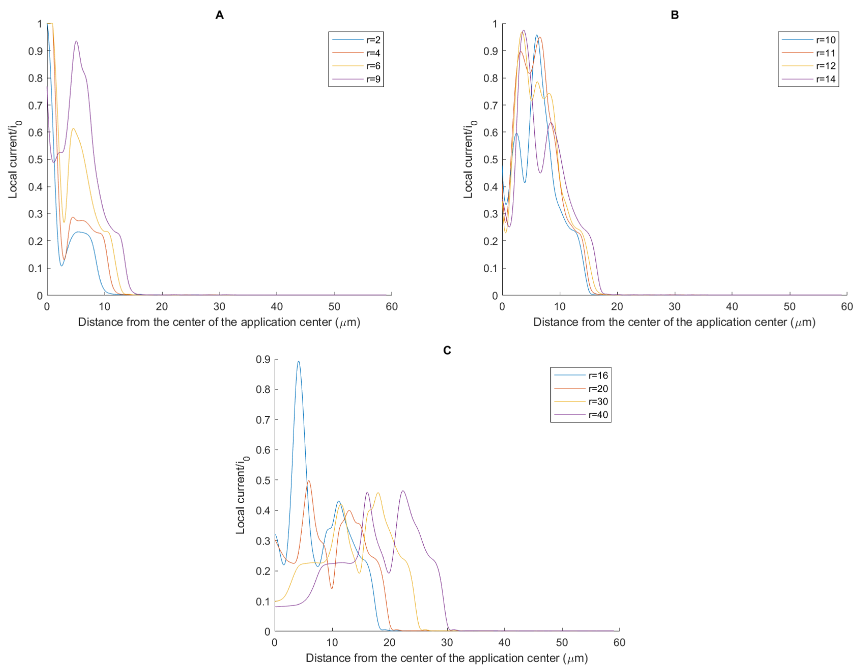

Figure 4.9: Local current behaviour. (**A**-**C**) Normalized local mean current versus the distance from the center or the application region.

4.6 Multiple neuron response

Until now we considered the neural response only of the PVM touch receptor neuron. Now we look at different neurons with a step stimulus of the type in Figure(4.7A), where the stimulus is applied in the anterior part and the posterior part of the nematode. The results for different touch receptor neurons are shown in Figure(4.10) and Figure(4.11).

One can observe that the PVM reacts in the same way for an anterior or posterior stimulus. The ALMs responds to an anterior stimulus and the PLMs responds to a posterior one.

We were not able to analyse the dependence between various neurons. As stated in [19], the ALML and ALMR work independently during in an early larva stage but then the AVM develops, which connects to the ALM cells. This connection between various neurons make the neuronal circuit more complex, and may be important in the *C.elegans* development, since the structure of the single neuron is simple.

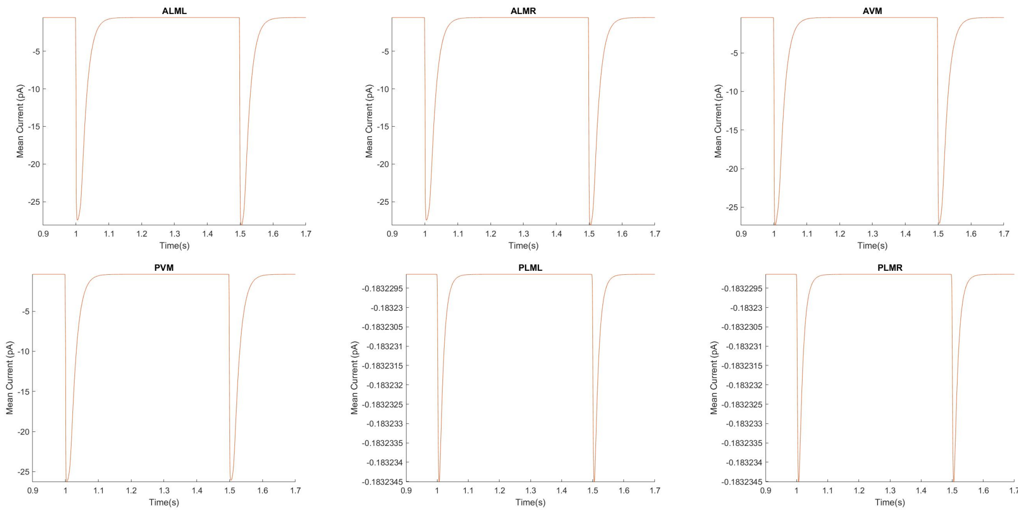

Figure 4.10: Neuron response of different neurons for various values of the application region. The step stimulus is applied on the anterior part of the worm.

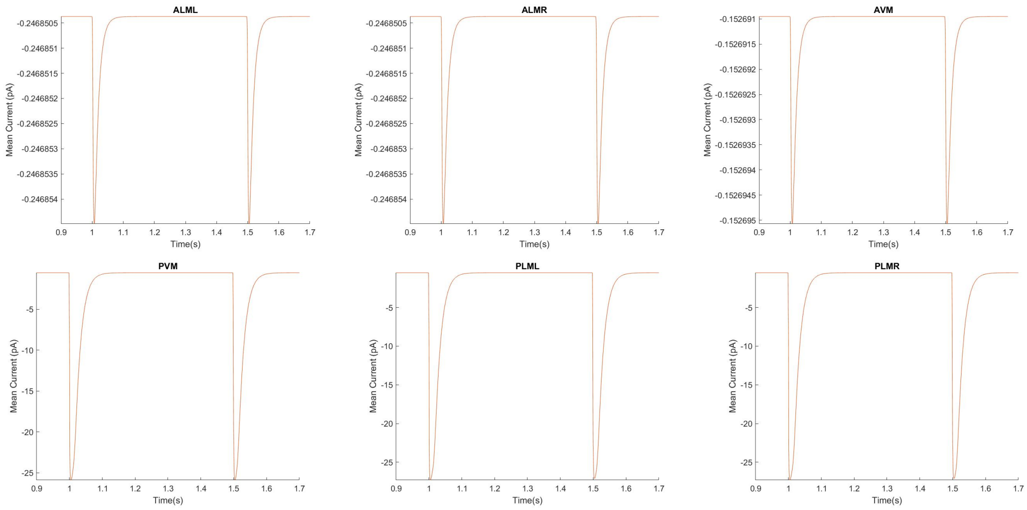

Figure 4.11: Neuron response of different neurons for various values of the application region. The step stimulus is applied on the posterior part of the worm.

Bibliography

[1] A. L. Eastwood, A. Sanzeni, B. C. Petzold, S. Park, M. Vergassola, B. L. Pruitt, and M. B. Goodman. Tissue mechanics govern the rapidly adapting and symmetrical response to touch. *Proceedings of the National Academy of Sciences*, 112(50):E6955–E6963, dec 2015.

[2] A. Sanzeni, S. Katta, B. Petzold, B. L Pruitt, M. B Goodman, and M. Vergassola. Somatosensory neurons integrate the geometry of skin deformation and mechanotransduction channels to shape touch sensing. *eLife*, 8, aug 2019.

[3] M. Goodman. Mechanosensation. *WormBook*, 2006.

[4] Sean. M. Maguire, C. M. Clark, J. Nunnari, J. K. Pirri, and M. J. Alkema. The c. elegans touch response facilitates escape from predacious fungi. *Current Biology*, 21(15):1326–1330, aug 2011.

[5] A. R. Schmidt, H. Dorfelt, and V. Perrichot. Carnivorous fungi from cretaceous amber. *Science*, 318(5857):1743–1743, dec 2007.

[6] Electricité de France. Finite element *code_aster*, analysis of structures and thermomechanics for studies and research,. Open source on www.code-aster.org, 1989–2017.

[7] Yves Pomeau Basile Audoly. *Elasticity and Geometry: From Hair Curls to the Non-Linear Response of Shells*. OXFORD UNIV PR, 2018.

[8] A. L. Brown, Z. Liao, and M. B. Goodman. MEC-2 and MEC-6 in the caenorhabditis elegans sensory mechanotransduction complex: Auxiliary subunits that enable channel activity. *Journal of General Physiology*, 131(6):605–616, may 2008.

[9] J. G. Cueva, A. Mulholland, and M. B. Goodman. Nanoscale organization of the MEC-4 DEG/ENaC sensory mechanotransduction channel in caenorhabditis elegans touch receptor neurons. *Journal of Neuroscience*, 27(51):14089–14098, dec 2007.

[10] Robert O'Hagan, Martin Chalfie, and Miriam B Goodman. The MEC-4 DEG/ENaC channel of caenorhabditis elegans touch receptor neurons transduces mechanical signals. *Nature Neuroscience*, 8(1):43–50, dec 2004.

[11] Philippe Ciarlet. *Mathematical elasticity.* North-Holland Elsevier Science Publishers, B.V, Amsterdam New York Amsterdam, 1993.

[12] A. I. Lurie and Alexander Belyaev. *Theory of Elasticity.* Springer Berlin Heidelberg, 2005.

[13] Walter Noll. Lectures on the foundations of continuum mechanics and thermodynamics. *Archive for Rational Mechanics and Analysis*, 52(1):62–92, mar 1973.

[14] C Truesdell. *Linear theories of elasticity and thermoelasticity ; Linear and nonlinear theories of rods, plates, and shells.* Springer-Verlag, Berlin New York, 1984.

[15] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics.* Springer Berlin Heidelberg, 2004.

[16] Philippe Ciarlet. *Mathematical elasticity, Volume 3.* North-Holland Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co, Amsterdam New York New York, N.Y., U.S.A, 1988.

[17] Frithiof Niordson. *Shell theory.* North-Holland Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co, Amsterdam New York New York, N.Y, 1985.

[18] Augustus Edward Hough Love. XVI. the small free vibrations and deformation of a thin elastic shell. *Philosophical Transactions of the Royal Society of London. (A.)*, 179:491–546, dec 1888.

[19] M Chalfie, JE Sulston, JG White, E Southgate, JN Thomson, and S Brenner. The neural circuit for touch sensitivity in caenorhabditis elegans. *The Journal of Neuroscience*, 5(4):956–964, apr 1985.