Var bounds for the distribution of the loss portfolio: exchangeable Bernoulli models

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Abstract

This thesis studies the modeling of credit risk in static credit portfolios. In this context one of the most important issues is to understand the dependence between defaults, that is one of the measures that helps shaping the probability distribution of the total loss of a credit portfolio. In many cases this dependence is difficult to define, and therefore it is hard to find directly the related risk measures (such as the VaR). For this reason, we investigate the sharp bounds of the VaR of the total loss distribution defined in [1] in two cases : when we do not know correlation between defaults and when we consider an estimation ρ of it. In particular in [1] they consider the indicators of default to be exchangeable and that the marginal distribution of each default is a Bernoulli with probability p.

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Introduction

Credit Risk Management has become crucial among financial institutions. Not only does credit risk management generate stability and reassure stakeholders, but it can also lead to higher returns. However, in many circumstances risk managers do not have enough pieces of information to properly define risk measures. For instance, a variable that is difficult to estimate is correlation between defaults, which plays a paramount role in the determination of the total loss distribution of credit portfolios. Default correlation is present as obligors tend to have strong borrower-lender relationships (if these are financial institutions) and because they are subject to the same macroeconomic factors.

This thesis is based on the results of [1]. Here the authors model credit risk of a portfolio of obligors when defaults are modeled with an exchangeable Bernoulli vector in two contexts: with an estimation of correlation between defaults and with no information about it. In particular, they find sharp bounds of the Value-at-Risk (VaR) of the distribution of the loss for each context. In this work we compare these bounds with other bounds defined thanks to [2] and by setting the hypothesis as paper [1]. The latter bounds, unlike the ones defined in [1], are not sharp. Several numerical examples will then be provided.

The rest of this thesis is organized as follows. In Chapter 1 we give an overview on Risk Management, by introducing the most common financial risks, the approaches and a brief history. Chapter 2 focuses on Credit Risk Management, by showing the importance of dependence modeling and some of the most famous credit risk models, such as Merton model and the mixture models. In particular, mixture models will be defined after setting the mathematical framework with De Finetti's representation theorem. In Chapter 3 the mathematical framework is set (with no mention on correlation), and a definition of the two couples of bounds for the Value-at-Risk is given from [1] and [2]. In Chapter 4 we will compare the bounds, and two numeric examples will be presented. In Chapter 5 we will propose two new couple of bounds for the VaR from [1] and [2] considering an estimation of the correlation between defaults. These bounds will be then compared thanks to two numerical examples.

1 Risk Management

The concept of risk is usually seen in a negative way; it is usually connected to hazard or to bad consequences. The general idea of risk is usually connected only to its downside and less frequently as a possible advantage, i.e. the potential for a gain. We can see this aspect in the concept of *volatility*. A volatile financial asset is considered to be dangerous as its value can easily oscillate and therefore reach low values. At the same, though, when an asset is volatile is means it is more likely for it to reach higher values than another asset with the same expected value but lower volatility.
In this chapter, we will try to present the concept risk from different point of views. In Section 1.1 we will show what are the types of risk that financial institutions need to cope with; in Section 1.2 the focus will be on the reasons why risk management tools should be implemented within a firm; Section 1.3 will discuss the evolution of the approaches adopted in Risk Management; Section 1.4 will talk about the Financial crisis in 2007-2008 and Section 1.5 will discuss about the evolution of the Basel accords, which were implemented to provide more stability and soundness to financial institutions.

1.1 Different types of risks

A financial institution faces all the time different kinds of risks. Here are the most common, as presented from [3]:

- *market risk*: in broad terms, this is referred to a swinging of the price of an asset, given maybe by an asymmetry between supply and demand, or by other factors. Market risk can thus be furtherly divided into *interest rate risk* (risk that the interest rate of a bond will change in the market, with the risk that the bond will lose a lot of its value; this can be measured with *duration*), *equity risk* and *exchange rate risk* (the risk of a sudden change in exchange rates).

- *credit risk*: when we talk about credit risks we don't just talk about the risk that an obligor won't manage to pay back its debt, but also how much of the outstanding debt is at risk. Connected with credit risk we can also find *collateral risk*. When money is lent from a financial institution, the creditor usually requires an asset as collateral, granting him the possibility to keep that asset if the obligor is unable to repay its debt. Since the possibility of default of the obligor exists, the collateral must not be set randomly, but following specific procedures: its market and credit risk will be examined thoroughly. An additional study may be done on the *haircut* coming from that asset. The intuitive idea of an haircut is that if an amount of 100 is given as a loan, the asset's value given as collateral will be 100 plus a percentage given by the haircut (which will be big for less solid creditors). We will discuss later furtherly about credit risks.

- *operational risks*: these are referred to external factors of in general factors that are not directly tied with financial risks, such as the *information security risk* or *reputational risk*.

The boundaries of the aforementioned risk categories are not always clearly defined. We can thus include two new risk categories which surface in nearly all the three categories already presented:

- *model risk*: this is related to the misuse of a risk model, for instance in calibrating the wrong parameters. It can be argued that in every model there will always be some degree of model risk.

- *liquidity risk*: as of [13], it's when a financial asset is not marketable, for instance when it can't be purchased or sold fast enough to avoid or minimize a loss. It is typically explained by unusually big bid-ask spreads. This is the case, for example, if an asset manager absolutely needed to buy an asset when many other managers are rushing to buy it: if most of it has already been sold, the asset manager will end up paying it a lot more.

1.2 Why managing financial risks

It is important to understand the reasons why a company should decide to invest in Risk Management tools. It is important to consider all the points of view and interests of all the stakeholders. Some of the stakeholders are the shareholders, the management board or customers of the financial institution. A good risk management framework needs to be put in place in other to satisfy the interests of all the parties involved. Two different views from [3] will now be presented, one showing the society's interest in investing in Risk Management, and the other the shareholders'.

A societal view

In our society, the stability and soundness of the banking and insurance systems are capital to all of us, and therefore it's just in our interests that the correct risk management frameworks are put in place. It was the scare of systemic risk by the population that lead to the Basel II accords. A systemic risk can be seen as the fear that the troubles characterizing a financial institution may characterize also other entities and therefore disrupt the usual financing systems. So we can say that society looks at risk management in a positive way as it helps preventing systemic situations to take place. Moreover society approves the work of regulators as it sees in them those capable of creating the framework that will safeguard its interests.

The shareholder's view

It's widely believed that the right financial risk management tools can enhance the value of a corporation and therefore the value owned by the shareholders. There are several reasons for which shareholders benefit from good risk management (RM) frameworks:

1. RM can reduce *tax costs*. Under the usual tax regulations the tax amount to be paid to corporations is a convex function of its income. If the firm's cash flow's variability is reduced, then RM can guarantee a higher expected after-tax profit;

2. As RM makes the probability of bankruptcy less likely, RM can enhance the firm value when *bankupcy costs* need to be paid;

3. RM can reduce extrernal financing costs for the corporation, as it facilitates the achievement of optimal investment.

1.3 Approaches to Risk Management

We can enumerate from [3] four different approaches measuring the risk contained in a financial position.

- *Notional-amount approach*: This is the traditional way of managing risk, where the risk of a portfolio is calculated as the sum of the notional values (referred to the value of an underlying asset in a derivatives trade) of all of the single securities in the portfolio, and a weight is associated to each position representing the amount of riskiness embodied. The advantage of this approach is that it's quite simple, but on the other hand it does not take into account the positive aspects of diversification on the overall risk of the portfolio. In fact, this approach assesses in the same way a portfolio consisting of loans to different firms that go bankrupt independently with a portfolio made of one loan to a single company. Moreover, it does not distinguish a long from a short position and can be far from exact in assessing derivatives, as the underlying notional amount and the value of the derivative could be widely different.

- *Factor sensitivity measures*: this risk measure shows the change in portfolio value given the change of an underlying variable. Two examples are the *duration* for bonds (it shows a relationship between a change in interest rates with the resulting change in the value of the bond) and Greeks for derivatives (for instance, *delta* measure shows how the price of a derivative changes when the underlying asset changes of one unit).
  The drawback of this approach stands in the fact that it can be difficult to aggregate these measures on a single portfolio, as it is hard to make a comparison of different measures for different asset classes.

- *Risk measures based on loss distributions*: More recent models use the loss distribution and measures deriving from it to perform risk management analysis. Thanks to this approach the common risk measure of the *Value-at-Risk* (VaR) was born. Broadly speaking, the VaR can be defined as the maximum possible loss after we exclude all worse outcomes. In particular, in this thesis we will also focus on two measures deriving from the VaR, such as the *Left-Tail Value-at-Risk* and the *TVaR*. These measures will be presented more deeply in the following chapters.

- *Scenario-based risk measures*: In this approach, extreme scenarios (like a sudden rise in exchange rates by a central bank, or some event that provokes the exchange rate between two countries to rise) are considered, and the associated maximum loss is measured.

1.4 Financial crisis of 2007-2008

To understand better the importance of risk management, it is worth analyzing one of the most important financial crisis ever happended in the world: the 2007 Global Financial Crisis (GFC). This section is taken from [4], [11], [12].
Although it is impossible to list all the causes that let this crisis happen, the 2007 crisis started first as a crisis connected with the housing market in the United

States, and in particular to the ease at which mortgages were granted to Americans.
Gaining the concession for a mortgage became so easy for Americans since, following the 9/11 attacks, the burst of the dot-com bubble and a series of corporate accounting scandals, the Federal Reserve lowered the Federal funds rate from 6.5 % in May 2001 to a historical low of 1% in June 2003, having as a consequence a huge injection of cheap money in the economy.
As a consequence, the so-called *subprime mortgages* became popular. The idea was that basically everyone (even those with low financial means) was given the possibility to underwrite a mortgage with banks, despite the higher interest rate, and somehow it became convenient from all the parties involved to grant a house to people that were not actually able to afford it. Thanks to less strict restrictions for subprime mortgages, more and more people were now able to buy a house, and this increase in demand obviously made prices go up even more. In contrast to what we would think right now, banks did not worry about the credit-worthiness of their obligors, but they actually enjoying the surpluses in their loans given the fact that the prices of the houses kept on rising. Later in time, financial institutions noticed how exposed they were on the housing market, and thus they managed to give away the risk from these mortgages by using Collateralized Debt Obligations (CDOs). These are products made by putting together many loans (such as credit card debts, student loans, mortgage loans or car loans) in a single product that can then be traded. More in particular, investors on these bonds get a gain when the underlying payments are made (for instance, when the interest on a mortgage is paid). On the other hand, it is the owner of these instruments that now bears the risk of default. These instruments were very attractive because their expected return was high compared to the underlying risks that were perceived. Another tricky side of these instruments is that investors were not seeing the underlying assets composing these CDOs, and for risk purposes investors could only rely on rating agencies. To understand why this complex mechanism reflected in a crisis regarding many financial institutions, It is worth adding that the most common way to ensure oneself from default risk was to by Credit Default Swaps (CDS) from other financial institutions. The idea behind a CDS is that you swap the potential credit default to a counterparty in exchange of an amount (it simply acts as an insurance). During that period, the institution that sold most CDSs was called AIG, and since this firm had granted protection against those financial instruments that then defaulted, its financial stability plummeted after the crisis.

During 2007, this whole system became to be felt less solid and therefore its prices dropped. Moreover, before 2007 banks had taken huge positions in the derivatives market gaining huge amounts of money, but when the crisis started their exposure was unhedged and this cause huge losses for them. Another problem that started at that time is that banks stopped lending themselves money on a short-term basis, as they were not trusting each other any longer. As this is a fundamental procedure for banks, when this stopped financial markets were torn out and there were huge liquidity problems. One of these, Lehman Brothers, was actually obliged to go bankrupt.

It is clear that more developed risk management tools would have been able to partially avoid this crisis. In fact, financial institutions at that time were not prepared to fight against such extreme events, and moreover they didn't consider the powerful effects of contagion within the financial system.
Moreover, people realized that regulation needed to step in more. This idea was motivated for instance by the fact that nothing was known about what backed the CDOs. More over, the financial models that were in place at that time resulted not to be prompt against extreme events. It is for this reason that the already existing Basel Accords (which will be deepened later) were perfectioned to try to be ready in case of a future similar crisis.

1.5 Basel Accords

We will now present a brief history (taken from [11])of the regulations put in place to try to help financial institution in case of market distress.
We know that financial institutions need to have enough capital to help them survive in case of severe losses. It was the explosion of the Latin American debt crisis that lead the Basel Committee on Bank Supervision (BCBS) to establish the Basel Accords in 1988. The Committee firmly believed that a multinational accord to strengthen the stability of the international banking system was needed, together with a way to remove a source of competitive inequality arising from different national capital requirements. It then proposed, under the name of Basel I accords, a minimum ratio of capital to risk-weighted assets of 8% to be put in place by the end of 1992. More in particular, these are guidelines for central banks and governments to make sure that financial institutions in those countries respect these orders. These are not actually laws, but it's the countries taking part to the BCBS that decided themselves to adopt them.

Unfortunately, Basel I accords were not as resolving as expected. Indeed, in 1999 a new framework was demanded to substitute the previous accords. The idea was both to improve the already existing framework on capital requirements and to use effectively disclosure of financial information as a way to increase market discipline and promote sound banking practices.

Even before Lehman Brothers collapsed in September 2008, there was the need to improve the Basel II framework. The crisis proved at first that a new framework of capital requirements needed to be put in place for banks, and therefore higher global minimum capital standards for commercial banks was announced in September 2010. This was anticipated by an agreement reached in July on the overall design of the capital and liquidity reform package, called "Basel III". Other measures were then put in place in the following years:

- tighter requirements for the quantity and quality of regulatory capital, reinforcing in particular the central role of *common equity* (the amount invested in a company by all common shareholders, as of [9]).
- an additional layer of common equity called *capital conservation buffer* that, when not respected, limits payouts helpful to meet the minimum common equity requirement
- the definition of a minimum liquidity ratio, the Liquidity Coverage Ratio (LCR), to make sure to have enough cash to cover funding needs over a stress period of 30 days and of a longer-term ratio, the Net Stable Funding Ratio (NSFR), intended to address maturity mismatches over the whole balance sheet.

2 Credit Risk Management

Credit Risk is the risk that the value of a portfolio changes due to unpredictable changes in the credit quality of the obligors or trading partners.
Credit risk is present everywhere in financial corporations. A typical situation involving credit risk is when a financial institution lends money to an individual or to another institution. On the contrary, a situation in which the presence of credit risk is less crystal clear is when there are certain OTC (over-the-counter, so when there is not an established stock exchange to monitor a deal) derivative operations, such as a swap. In case the counterparty defaults, the other member of the deal won't benefit from any pay-off.
Moreover, as already mentioned, there is not focus on risk management from a financial point of view but also from a regulatory point of view, as the three different Basel Capital Accord testify.
In Section 2.1, a distinction of the different credit risk models will be made, together with a list of the difficulties encountered by credit risk managers; in Section 2.2, we will present two models used in Risk Management, the Merton model and the Bernoulli mixture model.

2.1 Introduction to credit risk modeling

In the upcoming section we give an overview of the different kinds of models employed in credit risk. Then, we will focus on some challenges that are in place in credit risk management.

2.1.1 Credit Risk models

The Basel accords, together with the birth of credit derivatives generated a lot of interest around quantitative credit risk models.
Two are the main areas of application for quantitative credit risk models: credit risk management and analysis of credit-risky securities. For the former category, models of credit risk management are used to represent the distribution of the loss of a bond or loan portfolio over a specified time period. We can then see that, as

the period is fixed, stochastic processes are not present in this framework, but the attention will be on the distribution of the loss. For the analysis of credit-risky securities, instead, dynamics models (that will use stochastic processes usually) are needed, as the pay-off of many financial instruments depends on the exact moment of default.

More in detail, credit risk models can be divided into *firm-value* and *structural models* on the one hand and *reduced-form models* on the other. Behind every firm-value model there is the Merton model, which directly connects the default of a company with its financial stability, in particular with the value in time of its assets and liabilities. To be more precise, the default in a firm-value model takes place when an asset value goes below some a threshold representing liabilities. In this thesis we will focus on the general simple Merton model used to model the default of a single company.
Reduced-form models instead do not have a specified rule for which defaults take place. Here the default time of a firm is modeled as a non-negative random variable whose probability distribution depends on economic covariables. In this thesis we will present some *mixture models*, that can be intended as static portfolio versions of reduced-form models. More in detail, a mixture model assumes conditional independence of defaults given common underlying economic factors.

2.1.2 Challenges in Credit Risk Management

Credit risk management has got a decent amount of challenges that are not present in market risk for what concerns quantitative modeling. Here are some of them.

- *Lack of available data and information*. It's usually difficult to find information publicly available on the credit quality of companies. This is obviously an issue for the lenders, that do not and cannot know about the financial and economic soundness of a firm better than the management board of the same firm. This unavailability of credit data also hinders the employment of statistical methods in credit risk.

- *Skewed loss distribution*. Usually, the probability distributions of credit losses are strongly skewed with a big upper tail, and this is due to the fact that a credit portfolio will generate either frequent small profits or unusual big losses. For these reasons, a big amount of capital is needed to support a portfolio like this. In fact, the amount of capital required for a loan portfolio usually corresponds to the 99.97 quantile of the distribution of the loss.

- *The importance of modelling the dependence of defaults*. One of the main issues of the owner of a credit portfolio is the one concerning the simulta-

neous default of many counterparties. In this context we usually talk about *contagion risk*, which is the risk that one failure of a company can generate successive failures of other companies. This is a cause of the globalization on the financial markes, which made financial institutions more exposed in many different markets around the world. For example, when prices in the U.S. housing market dropped there have been strong consequences in the entire world.
This dependence structure contributes massively to the shape of the upper tail of a portfolio made of many obligors. For a large number of obligors, even when the correlation of two variables is very weak, we know that the tail of the total loss distribution gets way bigger than the case in which dependence is not considered.
Two are the reasons for which there is a strong default dependence between the firms. First is that all the companies are subject to the same macroeconomic factors. Secondly, it's because there are direct economic links between firms, like a strong borrower-lender relationship.

2.2 The basis of all structural models of defaults: the Merton model

A firm-value or structural model is a model that tries to explain the mechanism by which default takes place. The most famous structural model is the Merton model, that will be presented in the following section.
In the next chapters we will refer to stochastic processes in continuous time by (X_t), whereas the value of the process at time $t \geqslant 0$ is given by the random variable X_t.

2.2.1 Merton model

In 1974, the American sociologist and economist Robert Merton introduced a new method to model the credit risk of companies. At the time the so-called Merton model introduced a new approach to studying credit risk, and this model was so versatile that it was then adopted in other financial areas, such as the Black-Scholes model.
Behind this model there is the idea to have a look at the three main parts of the Balance Sheet : assets, debt and equity. First of all, the Merton model makes two assumptions. First, that the total debt of a company can be expressed as a single zero-coupon bond with face value $\bar{D}$ and maturity T. Moreover, for simplicity it is assumed that companies do not finance themselves with new debt or pay dividends to its shareholders. Let's consider the total assets of a company, and let's

assume the amount of assets follows a stochastich process V_t, $0 \leqslant t \leqslant T$. Let us also define D_t and E_t respectively as the debt and equity of the company at time t. If we also assume that there aren't any transaction costs or taxes we have that the total assets of the company is given by the sum of its equity and debt, i.e. $V_t = D_t + E_t$, for $0 \leqslant t \leqslant T$. In this context, the company will default if, at time T, the sum of its total assets won't be enough to repay its debt, so if $V_t < \bar{D}$. We can notice that, for simplicity, the company can only default at time T and not before. Moreover, the outstanding debt $\bar{D}$ is fixed over time.

So, at time T, how will the financial conditions of the debt owner and of the company look like ? If the company does not default, it will give the debt owner the whole amount $\bar{D}$, and the shareholders will have $V_t - \bar{D}$. Instead, if the company does go bankrupt, the debt issuer will receive the maximum possible amount V_t and the shareholders are left with nothing. We can notice that we can describe these profits for debt issuers and shareholders using the payoffs of put and call options.

$$D_T = \min(V_T, \bar{D}) = \bar{D} - (\bar{D} - V_T)^+$$
$$E_T = \max(V_T, \bar{D}) = (V_T - \bar{D})^+$$

The Merton model considers the process of the asset value to be a *geometric Brownian motion* (which is the same hypothesis made about the stock price processes in the Black-Scholes model). So, V_T has the following dynamics:

$$dV_t = rV_t dt + \sigma_V V_t dW_t \tag{2.1}$$

where $r \geqslant 0$, $\sigma_V > 0$, and W_t is a standard Brownian motion. The parameter σ_V stands for the volatility of the assets, and can either be estimated from historical data or extracted from stock prices. By setting $t = T$ and solving eq. (2.1) for V_t , by using Ito's Lemma we get that $V_t = V_0 * exp\big((r - \frac{\sigma_V^2}{2})T + \sigma_V W_T\big)$. Moreover, we can notice that W_T is a standard Brownian motion and therefore W_T follows a Normal distribution with average 0 and variance T. If we divide this process to its standard deviation, the process becomes standard normal, i.e. $W_T/\sqrt{T} \sim N(0,1)$. To make notation more understandable and clearer we introduce the random variable $Z \sim N(0,1)$, so that $W_T = \sqrt{T}Z$. So, thanks to this new notation, and remembering that the company defaults when $V_T < \bar{D}$, we can express the probability of default as:

$$
\begin{aligned}
\mathbb{P}(V_T \leqslant \bar{D}) &= \mathbb{P}\left(\ln(V_T) < \ln(\bar{D}) \right) \\
&= \mathbb{P}\left(\ln(V_0) + \left(r - \frac{\sigma_V^2}{2} \right) T + \sigma_V \sqrt{T} Z < \ln(\bar{D}) \right) \\
&= \mathbb{P}\left(\sigma_V \sqrt{T} Z < \ln\left(\frac{D}{V_0} \right) - \left(r - \frac{\sigma_V^2}{2} \right) T \right) \\
&= \mathbb{P}\left(Z < \frac{ln(\frac{D}{V_0} - (r - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}} \right) \\
&= \phi\left(\frac{ln(\frac{D}{V_0} - (r - \frac{\sigma_V^2}{2})T}{\sigma_V \sqrt{T}} \right)
\end{aligned}
$$

2.3 Mathemathical background and definition of mixture models

In this section another important class of credit risk models will be presented: the Bernoulli mixture models. Before deepening this topic, we will go through the mathematical theorem that has a strong connection with them, namely De Finetti's representation theorem.

2.3.1 De Finetti's Representation model

Before defining de Finetti's theorem, it is worth introducing the definition of exchangeability for an infinite sequence:

Definition 2.1 (Exchangeable sequence)**.** Let us consider the sequence $X_1, X_2, \dots$. The sequence is exchangeable if it holds that

$$\mathbb{P}(X_1 = x_1, ..., X_n = x_n) = \mathbb{P}(X_1 = x_{\sigma(1)}, ..., X_n = x_{\sigma(n)})$$

for all $n \in \mathbb{N}$ and all permutations σ of $\{1, ..., n\}$.

Let us then introduce π_p as the (Bernoulli) probability measure on {0,1} given by $\pi_p(1) = p$ and $\pi_p(0) = 1 - p$. Here is now the de Finetti's theorem, as defined in [6]:

Theorem 2.1 (de Finetti's representation theorem)**.** *Let $\{X_i\}_{i\geqslant 1}$ an infinite sequence of $\{0,1\}$-valued exchangeable random variables. Then there exists a probability measure μ on $[0,1]$ such that for any* N *and any sequence $(x_1,...,x_N) \in \{0,1\}^N$*

$$\mathbb{P}(X_1 = x_1,...,X_N = x_N) = \int_0^1 \prod_{i=1}^{N} \pi_p(x_i)d\mu(p)$$

In other words, the theorem states (as in [6]) that an exchangeable sequence with values in {0,1} is a mixture of independent sequences with respect to a measure μ on [0,1].

An issue regarding de Finetti's theorem is that it needs an infinite sequence. We can easily prove the theorem does not stand when we start with a finite dimension exchangeable vector.
As from [7], let us define the random vector (X_1,X_2) so that:

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 0, X_2 = 1) = \frac{1}{2}$$
$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 1, X_2 = 1) = 0$$

We have constructed the vector (X_1,X_2) to be exchangeable. Nevertheless, there is no probability measure μ satisfying de Finetti's theorem. If fact, if there were a probability measure μ such that

$$0 = \mathbb{P}(X_1 = 1, X_2 = 1) = \int_0^1 p^2 d\mu(p)$$

then necessarely μ will have probability density of 1 in 0, so we wouldn't have:

$$0 = \mathbb{P}(X_1 = 0, X_2 = 0) = \int_0^1 (1-p)^2 d\mu(p)$$

2.3.2 Mixture models

De Finetti's theorem can have numerous applications in finance, in particular in modeling the default distribution of credit portfolios. We see a close connection between this theorem and Bernoulli mixture models. "Financially" speaking, in a mixture model the risk of default of an obligor is assumed to be dependent on a set of economic factors, usually macroeconomic variables, which are also modeled stochastically. Given a realization of the factors, we consider the defaults of individual firms to be independent.

We now provide a general definition of the Bernoulli mixture models, taken from [3].

Definition 2.2 (Bernoulli mixture models)**.** Given some $p < m$ and a p-dimensional random vector $\boldsymbol{\Psi} = (\Psi_1, ..., \Psi_p)'$, the random vector $\boldsymbol{Y} = (Y_1, ..., Y_m)'$ follows a Bernoulli mixture model with factor vector $\boldsymbol{\psi}$ if there are functions $p_i : \mathbb{R}^p \rightarrow [0,1], 1 \leqslant i \leqslant m$, such that conditional on $\boldsymbol{\Psi}$ the components of Y are indipendent Bernoulli random variables satisfying $\mathbb{P}(Y_i = 1 \mid \boldsymbol{\Psi} = \psi) = p_i(\psi)$.

For $\boldsymbol{y} = (y_1, ..., y_m)' \in \{0,1\}^m$ we have that:

$$\mathbb{P}(\boldsymbol{Y} = y \mid \boldsymbol{\Psi} = \psi) = \prod_{i=1}^{m} p_i(\psi)^{y_i} (1 - p_i(\psi))^{1-y_i}$$

We consider the case when Ψ is unidimensional. In this case, the definition of Bernoulli mixture models is just a reformulation of the Definetti's theorem, as the single economic variable stands for the mixing variable μ. Therefore we are assuming that Y satisfies de Finetti's assumption, so that $(Y_1, ..., Y_m)$ are any m elements of an exchangeable sequence. In particular, they are themselves exchangeable.

In the following chapters we consider the framework in [1], where the authors slightly weaken the assumption of de Finetti and require $(Y_1, ..., Y_m)$ exchangeable but not necessarily part of an exchangeable sequence.

3 Problem Description

We will now present the context of our analysis.
Given a credit risk portfolio P of d obligors, let the random vector $\mathbf{X} = (X_1, ..., X_d)$ be the default indicator for the portfolio P. For our purposes, we will just consider the case where each variable X_i is a Bernoulli random variable with the same mean:

$$X_i = \begin{cases} 1 & \text{prob. } p, \\ 0 & \text{prob. } q = 1 - p \end{cases}$$

By taking the same mean for all of the default distributions, we are assuming the obligors belong to the same credit rating.
To model the loss of a credit risk portfolio P of d obligors we consider the weighted sum of the individual losses

$$L = \sum_{i=1}^{d} \omega_i X_i$$

where $\omega_i \in (0, 1]$ and $\sum_{i=1}^{d} \omega_i = 1$. ω_i stands for the weight of the credit granted to obligor i in the portfolio. In this thesis we will consider the case $\omega_i = \frac{1}{d}$, $i \in \{1, ..., d\}$. For equal weights, $L = \frac{S_d}{d}$, where

$$S_d = \sum_{i=1}^{d} X_i$$

In this thesis we will then focus on the random variable S_d, which stands for the sum of the defaults. In particular, we want to study the Value-at-Risk of this distribution. Given $u \in (0, 1)$ and a random variable X, we define its VaR as:

$$\mathrm{VaR}(X) = \inf\{x \in X : (\mathbb{P}(X) \leq x) \geq u\}$$

The VaR can be defined informally as the maximum possible loss after we exclude all worse outcomes (as of [8]). Another useful measure for our purpose is the VaR_u^+, defined as:

$$\mathrm{VaR}_u^+(X) = \sup\{x \in X : (\mathbb{P}(X \leq x) \leq u\}$$

As already mentioned, the correlation between the different obligors plays a vital role in defining the distribution of the total loss. Yet, this correlation is difficult to estimate, and therefore it is hard to define properly the sum of the total loss, and so also its VaR. What we will try to do in this chapter will be to find, thanks to a theorem from [2], bounds for the VaR of S_d when we take a portfolio of $d = 100$ obligors and when we have no information on correlation between defaults. Then we will compare these bounds with others defined in [1].

3.1 Portfolio Var bounds with fixed marginal distributions

This section recalls the similar-VaR bounds measures for given marginal distributions provided in [2]. These are the TVaR and the Left Tail Value-at-Risk LTVaR. Formally, for $\alpha \in (0,1)$, we denote by $\mathrm{TVaR}_\alpha(X)$ the TVaR at level α,

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 Var_u(X)\,du$$

and by $\mathrm{LTVaR}_\alpha(X)$ the LTVaR,

$$\mathrm{LTVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha Var_u(X)\,du$$

So, TVaR_α is the average of all upper Vars from level α onwards. Similarly, LTVaR_α is the average of all lowers VaRs.
We will first show from [2] some already existing bounds for a general case, and we will then apply this case with our underlying hypothesis.

Theorem 3.1 (Unconstrained bounds)**.** *Let $\alpha \in (0,1)$, $X_i \sim F_i$ $(i = 1,2,...,n)$, $S = \sum_{i=1}^n X_i$. Then,*

$$A := \sum_{i=1}^n LTVaR_\alpha(X_i) \leqslant VaR_\alpha(S) \leqslant VaR_\alpha^+(S) \leqslant B := \sum_{i=1}^n TVaR_\alpha(X_i) \qquad (3.1)$$

We can thus see that, for the general case where the single variables X_i don't necessarily have the same distribution, the VaR_α of the sum of the n distributions is bounded by the sum of the LTVaR_α and the sum of the TVaR_α of the single distributions. Therefore, to find the bounds for our case it's enough to calculate the TVaR_α and the LTVaR_α for a Bernoulli distribution with average p.

3.2 Explicit calculation of the bounds

First of all, let's write explicitly the cumulative distribution for a Bernoulli random variable $X \sim B(p)$ (where $q = 1 - p$). We know that:

$$F_X = \begin{cases} 0 & x < 0, \\ q & x \in [0,1], \\ 1 & x \geqslant 1 \end{cases}$$

In order to calculate the TVaR_α and the LTVaR_α for the sum of Bernoulli random variables, we need first to evaluate the $VaR_u^+(X)$ and $VaR_u(X)$ for one Bernoulli distribution, depending on the values of q and u.
We will start with the calculation of the $\text{VaR}_u^+(X)$, in order then to calculate the TVaR:

1. If $u < q$: $VaR_u^+(X) = \sup\{x \in \mathbb{R} : (P(X) \leqslant x) \leqslant u\} = 0$
2. If $u = q$: $VaR_u^+(X) = \sup\{x \in \mathbb{R} : (P(X) \leqslant x) \leqslant u\} = 0$
3. If $u > q$: $VaR_u^+(X) = \sup\{x \in \mathbb{R} : (P(X) \leqslant x) \leqslant u\} = 1$

This can be rewritten as:

$$VaR_u^+(X) = \begin{cases} 0 & u \leqslant q, \\ 1 & u > q \end{cases}$$

Now we are able to calculate explicitly the TVaR for a Bernoulli distribution:

1. if $\alpha \leqslant q$:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 Var_u^+(X)\,du = \frac{1}{1-\alpha}\int_q^1 du = \frac{1-q}{1-\alpha}$$

2. if $\alpha > q$:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 Var_u^+(X)\,du = \frac{1}{1-\alpha}\int_\alpha^1 du = 1$$

To sum up:

$$TVaR_\alpha(X) = \begin{cases} \frac{1-q}{1-\alpha} & \alpha \leqslant q, \\ 1 & \alpha > q \end{cases} \tag{3.2}$$

In a similar way I can calculate the LTVaR of a Bernoulli distribution with average p:

1. If $u < q$: $VaR_u(X) = \inf\{x \in \mathbb{R} : (P(X) \leqslant x) \geqslant u\} = 0$
2. If $u = q$: $VaR_u(X) = \inf\{x \in \mathbb{R} : (P(X) \leqslant x) \geqslant u\} = 0$
3. If $u > q$: $VaR_u(X) = \inf\{x \in \mathbb{R} : (P(X) \leqslant x) \geqslant u\} = 1$

This can be rewritten as:

$$VaR_u(X) = \begin{cases} 0 & u \leqslant q, \\ 1 & u > q \end{cases}$$

We can notice that in the case of a Bernoulli distribution the $\mathrm{VaR}_u^+(X)$ and $\mathrm{VaR}_u(X)$ coincide. Now we are able to calculate explicitly the LTVaR_α for a Bernoulli distribution:

1. if $\alpha \leqslant q$:

$$LTVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha Var_u(X)\,du = \frac{1}{\alpha}\int_0^\alpha 0du = 0$$

2. if $\alpha > q$:

$$LTVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha Var_u(X)\,du = \frac{1}{\alpha}\int_q^\alpha 1du = \frac{\alpha - q}{\alpha}$$

To sum up:

$$LTVaR_\alpha(X) = \begin{cases} 0 & \alpha \leqslant q, \\ \frac{\alpha-q}{\alpha} & \alpha > q \end{cases} \tag{3.3}$$

From eq. (3.1),eq. (3.2) and eq. (3.3), we can define the bounds for the Value-at-risk of the sum of defaults:

Proposition 3.1. Let the random vector $\mathbf{X} = (X_1, ..., X_d)$ be the default indicator for a portfolio P, where $X_i \sim B(p)$ $(q = 1 - p)$. Let $S_d = \sum_1^d X_i$, and let $\alpha \in (0, 1)$. Then:

$$\begin{cases} 0 & \alpha \leqslant q, \\ \frac{(\alpha-q)d}{\alpha} & \alpha > q \end{cases} \leqslant VaR_\alpha(S_d) \leqslant \begin{cases} \frac{d(1-q)}{1-\alpha} & \alpha \leqslant q, \\ 1 & \alpha > q \end{cases} \tag{3.4}$$

3.3 Portfolio bounds for exchangeable Bernoulli defaults

This section introduces analytical bounds for the VaR of the sum of d exchangeable Bernoulli distributions with average p. The results from this sections are taken from [1].

Definition 3.1 (Exchangeable vector)**.** A random vector $\mathbf{X} = (X_1,...,X_d)$ with joint cumulative distribution F is exchangeable if

$$F(x_0,x_1,...,x_d) = F(x_{\pi(0)},x_{\pi(1)},...,x_{\pi(d)})$$

for any permutation of $\{0,1,...,d\}$ and for any $(x_0,x_1,...,x_d) \in \mathbb{R}^d$.

We can see that the authors slightly weaken the assumption of de Finetti and require $(Y_1,...,Y_m)$ exchangeable but not necessarily part of an exchangeable sequence.

Let us then define $\mathcal{S}_d(p)$ as the class of distributions p_S on $\{0,...,d\}$ such that $S_d = \sum_{i=1}^{d} X_i$, with the vector $\mathbf{X}$ belonging to the class of d-dimensional exchangeable Bernoulli distributions with the same Bernoulli marginal distribution $B(p)$. The idea behind the calculation of these bounds is to find the generators of $\mathcal{S}_d(p)$, which can be proved that are all the vectors $p_S = (p_0,...,p_d)$ satisfying this equation

$$\sum_{i=1}^{d}(j-pd)p_j = 0 \tag{3.5}$$

Then, to define the bounds of the VaR, I have to provide the definition of extremal rays.

Definition 3.2 (Extremal Ray)**.** Given the convex cone

$$C_p = \left\{ z \in \mathbb{R}^{d+1} : \sum_{j=0}^{d}(j-pd)z_j = 0,\ Iz \geqslant 0 \right\}$$

which represents the set of positive solutions of eq. (3.5). A solution $\mathbf{r}$ of eq. (3.5) is an extremal ray of C_p iff $I^*z = 0$ for a submatrix $n_{I^*} \times (d+1)$, I^* of I and

$$\text{rank} \begin{bmatrix} A \\ I^* \end{bmatrix} = d.$$

It can be then proved that the extremal rays of the convex cone C_p have at most two non-zero components.

We will now define $R_{(j_1,j_2)}$ and R_{pd} as the random variables whose probability mass functions are defined respectively on the two non-zero values j_1 and j_2 and on the value pd.
Thanks to the following proposition we can find out sharp bounds for the Var of S_d.

Proposition 3.2. Let $S_d \in \mathcal{S}_d(p)$ and let $VaR_u(S_d)$ be its value at risk. Then:

$$\min_R \mathrm{VaR}_u(R) \leqslant \mathrm{VaR}_u(S_d) \leqslant \max_R \mathrm{VaR}_u(R) \tag{3.6}$$

where R are the ray densities of $\mathcal{S}_d(p)$

Then, thanks to proposition 3.3, I'm able to define specifically which are the upper and lower bounds of the VaR:

Proposition 3.3. Let us consider the class $\mathcal{S}_d(p)$ and let $j_1^p = \frac{(p-(1-u))d}{u}$

1. If $j_1^p < 0$, $\min VaR_u(R_{(j_1,j_2)}) = 0$ and $\max VaR_u(R_{(j_1,j_2)}) = j_2^*$, where j_2^* is the larger integer smaller than $\frac{pd}{1-u}$.
2. If $0 \leqslant j_1^p \leqslant j_1^M$, $\min VaR_u(R_{(j_1,j_2)}) = j_1^*$, where j_1^* is the smallest integer greater or equal to j_1^p and $\max VaR_u(R_{(j_1,j_2)}) = d$.
3. If $j_1^p \geqslant j_1^M$, $\min VaR_u(R_{(j_1,j_2)}) = j_2^m = j_1^M + 1$ and $\max VaR_u(R_{(j_1,j_2)}) = d$. In this case, if pd is integer $j_1^M + 1 = pd$

4 Comparison of the bounds

So far, we have developed two different bounds for the VaR of the sum of defaults when no information is provided on correlation between them. Now we will compare the two bounds, according to the different conditions on α and q. To make this comparison possible, we will consider the hypothesis of exchangeability for both the couple of bounds.
In Section 4.1 we will introduce a theorem that compares the two bounds; in Section 4.2 we will show two numeric examples to prove the results found in the theorem.

4.1 Comparison

The following theorem allows us to confront the two couples of bounds defined in chapter 3.

Theorem 4.1. *Let us consider the class $\mathcal{S}_d(p)$. Let $j_1^p = \frac{(p-(1-\alpha))d}{\alpha}$ and let j_1^M be the largest integer smaller than pd. The bounds calculated with Proposition 3.3 are always tighter or equal than those in Proposition 3.1.*

Proof. First of all, we can notice that

$$j_1^p < 0 \iff \frac{(p-(1-\alpha))d}{\alpha} < 0 \iff \frac{(\alpha-q)d}{\alpha} < 0 \iff \alpha < q \tag{4.1}$$

and that

$$0 \leqslant j_1^p \leqslant j_1^M \iff 0 \leqslant \frac{(\alpha-q)d}{\alpha} \leqslant j_1^M \iff 0 \leqslant (\alpha-q) \leqslant \frac{(j_1^M)\alpha}{d} \tag{4.2}$$

We see therefore a close connection between the conditions defining the two different bounds. So, we can easily confront them:

1. if $j_1^p < 0$ $(\alpha < q)$:

   (a) Thanks to equation (4.1), to calculate the upper bound I have to confront $\max_R VaR_u(R) = j_2^*$ from proposition 3.3 with the upper bound

$\frac{(1-q)d}{1-\alpha}$ defined in proposition 3.1. Since by definition j_2^* is the largest integer smaller than $\frac{pd}{1-\alpha} = \frac{(1-q)d}{1-\alpha}$, then j_2^* is the best bound.

   (b) Using the same criteria used for point (a), we easily see that in both cases the lower bound is 0

2. $0 \leqslant j_1^p \leqslant j_1^M \left(0 \leqslant (\alpha - q) \leqslant \frac{(j_1^M)\alpha}{d}\right)$:

   (a) For the upper bound, in both cases the bound is d

   (b) For the lower bound, given proposition 3.3 I have to confront $\max_R VaR_u(R)$ = j_1^* with the other upper bound $\frac{(\alpha-q)d}{\alpha}$. Since by definition j_1^* is the smallest integer greater than $j_1^p = \frac{(\alpha-q)d}{\alpha}$, then j_1^* is the best bound.

3. $j_1^p > j_1^M \left(\alpha - q \geqslant \frac{\alpha(j_1^M)}{d} \geqslant 0\right)$

   (a) For the upper bound, in both cases the bound is d

   (b) We need to prove that $\frac{(\alpha-q)d}{\alpha}(= j_1^p) \leqslant j_1^M + 1$. First of all, since $0 < \alpha < 1$, we can easily see that

$$\frac{(\alpha - q)d}{\alpha} = \frac{(p - (1-\alpha)d)}{\alpha} = \frac{pd}{\alpha} - \frac{(1-\alpha)}{\alpha} < pd$$

   Since by definition $pd \leqslant j_1^M + 1$, then we have proved that the bounds given by Proposition 3.3 are better than those given by Proposition 3.1.

□

Remark 1. We can notice that, when the bounds calculated with Proposition 3.1 are not integers, then we can always "tighten" the bounds to the closest integer forward (for the lower bound) and to the closest integer backwards (for the upper bound). This is because we are considering the distribution of S_d, which is defined on the set of integer values $\{0, 1, ..., d\}$. To make an example, it would make no sense to say that the VaR is "less than 2,9", but we can say it is less than 2. For this reason, having a better look at the two different bounds for all the three conditions 1., 2. and 3. from the previous theorem, we can see that only if $j_1^p < 0$ and if $\frac{pd}{1-d}$ is an integer, then there is an actual advantage of taking the bounds given by the rays, otherwise the upper and lower bounds are always the same. Overall, since the bounds presented in the second methods are sharp, they cannot be improved (this is way the first method never manages to outperform the second one) and moreover it provides explicitly the distribution (the extremal rays) in which they are reached.

The following numeric examples will better clarify what was just stated.

4.2 Numeric examples

We will now perform a couple of numeric examples to show the results proved in the previous theorem and what I highlighted in the previous observation.
Let us suppose we have a credit portfolio P with 100 obligors. Let the random vector $\mathbf{X} = (X_1, ..., X_{100})$ collect the default indicators for the portfolio P and assume the vector $\mathbf{X}$ belongs to the class of d-dimensional exchangeable Bernoulli distributions with the same Bernoulli marginal distribution $B(p)$. The variable S_d represents the number of defaults and the distribution of S represents the distribution of the loss. We will analytically find bounds of VaR_α, for $\alpha = 0.90$, $\alpha = 0.95$ and $\alpha = 0.99$ using the two different methods.

4.2.1 Scenario 1: p=0,11%

Let us assume $p = 0,11\%$, corresponding to S&P's default rate in 2019 for a BBB rating company (as of [5]).

1. $\alpha = 0,90$
   Since $j_1^p = \frac{(0,0011-0,1)100}{0,90} < 0$, the lower and upper bound given by the extremal rays are respectively 0 and the largest integer less than $\frac{(0,0011)100}{0,10}$, so 1.
   With the new method, I get that the lower bound is 0, while the upper bound is 1,1. As pointed in observation 1, we can say the upper bound is still 1.

2. $\alpha = 0,95$
   Since $j_1^p < 0$, the lower and upper bounds given by the extremal rays are respectively 0 and the largest integer less than $\frac{(0,0011)100}{0,05}$, so 2.
   With the new method, I get that the lower bound is 0, while the upper bound is 2,2. For the Observation, we can say the upper bound is 2.

3. $\alpha = 0,99$
   In this specific case, we can see that the method given by the extremal rays finds a better upper bound. As $j_1^p < 0$, the lower and upper bounds given by the extremal rays are respectively 0 and the largest integer less than $\frac{(0,0011)100}{0,05} = 11$, so 10. With the new method, instead, the upper bound would be 11

Previous results can be summarized by this table:

α	left bound 1	left bound 2	right bound 1	right bound 2
0,90	0	0	1	1
0,95	0	0	2	2
0,99	0	0	11	10

Table 4.1: two different bounds of VaR_α of the number of defaults with $p = 0,11\%$.

4.2.2 Scenario 2: p=1,49%

Let us assume $p = 1,49\%$, corresponding to S&P's default rate in 2019 for a B rating company (as of [5]). Here we will also consider the case when $\alpha = 0,999$.

1. $\alpha = 0,90$
   Since $j_1^p = \frac{(0,0149-0,1)100}{0,90} < 0$, the lower and upper bound given by the extremal rays are respectively 0 and 14.
   With the new method, I get that the lower bound is 0, while the upper bound is 14,9. As pointed before, we can say the upper bound is still 14.

2. $\alpha = 0,95$
   Since $j_1^p < 0$, the lower and upper bounds given by the extremal rays are respectively 0 and 35.
   With the new method, I get that the lower bound is 0, while the upper bound is 35,8, which can be rounded to 35.

3. $\alpha = 0,99$
   In this case, $j_1^p = \frac{(0,0149-0,01)100}{0,99} = 0,4949 > 0$. As $j_1^M = 1 > j_1^p$, the lower and upper bounds given by the extremal rays are respectively 1 and 100.
   With the new method, the upper bound is still d whereas the lower one is 0,4949, which can be rounded to 1.

4. $\alpha = 0,999$
   In this case, $j_1^p = \frac{(0,0149-0,001)100}{0,999} = 1,3913 > j_1^M = 1$. So, the lower and upper bounds given by the extremal rays are respectively 2 and 100.
   With the new method, the upper bound is still 100 whereas the lower one is 1,3913, which can be rounded to 2.

Previous results can be summarized by the following table

α	left bound 1	left bound 2	right bound 1	right bound 2
0,90	0	0	14	14
0,95	0	0	35	35
0,99	1	1	100	100
0,999	2	2	100	100

Table 4.2: two different bounds of VaR_α of the number of defaults with $p = 1,49\%$.

5 Problem description with given correlation ρ

So far, we have just considered bounds for the Value-at-Risk without knowing the default correlation between the obligors. In this chapter, we consider an estimation of default correlation to be at level ρ. As we will see later, when correlation is known, we can easily find the variance of the distribution of the total loss S_d.
The chapter is organized as follows. In Section 5.1 we will introduce the new class of ditributions $\mathcal{S}_d(p,\rho)$, and we will define sharp bounds (from [1]) for the Value-at-risk of the loss $S_d \in \mathcal{S}_d(p,\rho)$. In Section 5.2, we will first define new bounds from [2], calculated with a given variance constraint. Then we will show how to calculate the variance of the loss distribution S_d given correlation between defaults. In Section 5.3 we will perform numeric examples to confront the two couples of bounds.

5.1 Portfolio bounds for exchangeable Bernoulli defaults and correlation ρ

Let us first define $\mathcal{S}_d(p,\rho)$ as the class of distributions p_S on $\{0,...,d\}$ such that $S_d = \sum_{i=1}^{d} X_i$, with the vector $\mathbf{X}$ belonging to the class of d-dimensional exchangeable Bernoulli distributions with the same Bernoulli marginal distribution $B(p)$, and given correlation ρ between the Bernoulli distributions.
Sharp bounds for the Var of $S_d \in \mathcal{S}_d(p,\rho)$ are defined thanks to the following proposition from [1]. It is worth noticing that the bounds are constructed in the same way as from space $\mathcal{S}_d(p)$.

Proposition 5.1. Let $S_d \in \mathcal{S}_d(p,\rho)$ and let $VaR_u(S_d)$ be its value at risk. Then:

$$\min_R \mathrm{VaR}_u(R) \leqslant \mathrm{VaR}_u(S_d) \leqslant \max_R \mathrm{VaR}_u(R) \tag{5.1}$$

where R are the ray densities of $\mathcal{S}_d(p,\rho)$

As from space $\mathcal{S}_d(p)$, to define VaR bounds we have to understand the distribution of the extremal rays. We can do so thanks to the following proposition from [1].

Proposition 5.2. Let $\alpha_j = j - pd$ and $\beta_j = j^2 - (pd + d(d-1)\mu_2)$ and let $A_{ij} = \det\begin{bmatrix}\alpha_i & \alpha_j \\ \beta_i & \beta_j\end{bmatrix} = d$.

If

$$\begin{cases} A_{jk} \geqslant 0 \\ A_{ik} \geqslant 0 \\ A_{ij} \geqslant 0 \end{cases}$$

the extremal rays of 5.1 are $\boldsymbol{r}_\rho = (p_0, ..., p_d)$, where $p_l = 0, l \neq i, j, k$,

$$\begin{aligned} p_i &= \frac{jk - (j+k-1)dp + d(d-1)\mu_2}{(k-i)(j-i)} \\ p_j &= -\frac{ik - (i+k-1)dp + d(d-1)\mu_2}{(k-j)(j-i)} \\ p_k &= \frac{ij - (i+j-1)dp + d(d-1)\mu_2}{(k-j)(k-i)} \end{aligned}$$

with $i < j < k$.

5.2 Portfolio Var bounds with fixed marginal distributions and variance constraint

In Section 5.1 we assumed that between obligors there exists a certain default correlation ρ. From basic probability theory, when we know correlation between variables X_i (as well as the probability distributions of the single variables), we are able to calculate the variance of the sum of these distributions. Therefore, in our context, assuming a certain default correlation between the obligors is the same as assuming the respective level of variance for the sum of the defaults.
In this Section, we will first define from [2] new VaR bounds for S_d given a constraint on its variance. Then we will show how to calculate the variance of the sum of defaults given correlation ρ, in order then to be able to compare the two couple of bounds defined in this chapter.

Let us now define the general theorem from [2] with which we are able to define bounds of VaR(S_d) given a variance constraint.

Theorem 5.1 (constrained bounds)**.** *Let $\alpha \in (0,1)$, $X_i \sim F_i$ $(i = 1,2,...,n)$, $S = \sum_{i=1}^n X_i$ satisfy $var(S) \leqslant s^2$. Let $\mu = \mathbb{E}\Big(\sum_{i=1}^n X_i\Big)$.*

$$a := max\left(\mu - \sqrt{\frac{1-\alpha}{\alpha}}, A\right) \leqslant m \leqslant VaR_\alpha(S)$$

$$\leqslant VaR^+_\alpha(S) \leqslant M \leqslant b \leqslant b := min\left(\mu + s\sqrt{\frac{\alpha}{1-\alpha}}, B\right)$$

where A and B are the bounds defined without variance constraint in Theorem 3.1.
In particular, if $s^2 \geqslant \alpha(A-\mu)^2 + (1-\alpha)(B-\mu)^2$, then $a = A$ and $b = B$ (the unconstrained bounds are not improved by the presence of the constraint on variance).

This theorem allows us to find non-sharp bounds for the distribution of the number of defaults. We can notice that this theorem works also when we define ex-ante a fixed amount of variance.

In order to be able to compare the two bounds, let's now go through the connection between correlation and variance.
Consider a random vector $\boldsymbol{X} = (X_1, ..., X_d)$ with $Corr(X_i, X_j) = \rho$, $\forall$ i,j. Let $var(X_i) = \sigma^2$, $\forall i \leqslant d$. We have that:

$$var\left(\sum_{i=1}^{d} X_i\right) = var(X_1 + ... + X_d) = dvar(X_1) + d(d-1)\rho/\sigma^2$$

So, when considering the sum of exchangeable Bernoulli distributions with average p with given correlation ρ, we know the variance of the sum of the defaults S_d will be:

$$var(S_d) = dp(1-p)(1+\rho d - \rho) \tag{5.2}$$

Therefore, it now sounds natural to confront the bounds given by Proposition 5.1 with the ones given by Theorem 5.1 setting the corresponding level of variance. In particular, in this section we do not confront "theorically" the two couples of bounds as in chapter 4, but we provide significant numerical examples.

5.3 Comparison of the bounds with correlation ρ

Some numerical examples will now be performed to compare the two previous theorems. A default correlation ρ will be set ex-ante.

The sharp bounds proposed in the following scenarios are taken from [1]. The bounds from theorem 5.1 will be calculated based on the set ρ, and the corresponding variance s^2 (that will act as our "variance constrain" of theorem 5.1) will be calculated ex-post thanks to eq. (5.2).

Let us suppose we have a credit portfolio P with 100 obligors. Let the random vector $\mathbf{X} = (X_1, ..., X_{100})$ collect the default indicators for the portfolio P and assume the vector $\mathbf{X}$ belongs to the class of d-dimensional exchangeable Bernoulli distributions with the same Bernoulli marginal distribution $B(p)$ and correlation ρ. The variable S_d represents the number of defaults. We will analytically find bounds of VaR_α, for $\alpha = 0.90$, $\alpha = 0.95$ and $\alpha = 0.99$ using the two methods defined in this chapter.
We remind that the bounds A and B of the sum of defaults (that we need to calculate the bounds in theorem 5.1) set in Proposition 3.1 are:

$$A := \begin{cases} 0 & \alpha \leqslant q, \\ \frac{(\alpha-q)d}{\alpha} & \alpha > q \end{cases} \leqslant VaR_\alpha(S_d) \leqslant B := \begin{cases} \frac{d(1-q)}{1-\alpha} & \alpha \leqslant q, \\ 1 & \alpha > q \end{cases} \tag{5.3}$$

5.3.1 Scenario 1: p=0,17%, ρ= 1/6

Let us assume $p = 0,17\%$ and $\rho = 1/6$.

1. $\alpha = 0,90$
   From [1], the sharp bounds are respectively 0 and 16.
   To calculate the new bounds, let us first calculate s^2 thanks to eq. (5.2).

   $$s^2 = 100 * (0,0017)(1-0,0017)(1+(1/6)*100-1/6) = 29,24$$

   We can easily calculate $A = 0$ and $B = 17$. To see of we can improve these bounds, we confront s^2 with

   $$\begin{aligned}\alpha(A-\mu)^2+(1-\alpha)(B-\mu)^2 =&(0,90)(0-1,7)^2)+ \\ &(1-0,90)(17-1,7)^2 = 26,01\end{aligned} \tag{5.4}$$

   So, we can't improve the bounds A and B.

2. $\alpha = 0,95$
   From [1], the sharp bounds are respectively 1 and 25.
   We can easily calculate that $A = 0$ and $B = 34$. To see of we can improve these bounds, we confront s^2 with

   $$\begin{aligned}\alpha(A-\mu)^2+(1-\alpha)(B-\mu)^2 =&(0,95)(0-1,7)^2)+ \\ &(1-0,95)(34-1,7)^2 = 54,91\end{aligned} \tag{5.5}$$

Since $54,91 \geqslant 26,01$, we can improve the bounds A and B. From Theorem 5.1, we find the new lower and upper bounds are respectively $\mu - (1-\alpha)/\alpha)^{1/2} = 0,459$ and $\mu + s(\alpha/(1-\alpha) = 25,27$. Following the same reasoning done in the section 4, we can round the bounds to 1 and 25.

3. $\alpha = 0,99$
From [1], the sharp bounds are respectively 2 and 55.
We can easily calculate that $A = 0,707$ and $B = 100$. To see of we can improve these bounds, we confront s^2 with

$$\begin{aligned}\alpha(A-\mu)^2 + (1-\alpha)(B-\mu)^2 =& (0,99)(0,707-1,7)^2) + \\ & (1-0,99)(100-1,7)^2 = 97,60\end{aligned} \tag{5.6}$$

Since $97,6 \geqslant 26,01$, we can improve the bounds A and B. From Theorem 5.1, we find the new lower and upper bounds are respectively $\mu - (1 - \alpha)/\alpha)^{1/2} = 1,1564$ and $\mu + s(\alpha/(1-\alpha) = 55,5$. Following the same reasoning done in the section 4, we can round the bounds to 2 and 55.

Previous results can be summarized by the following table:

α	left bound 1	left bound 2	right bound 1	right bound 2
0,90	0	0	16	17
0,95	1	1	25	25
0,99	2	2	55	55

Table 5.1: two different bounds of VaR_α of the number of defaults with $p = 1,7\%$ and $\rho = 1/6$.

5.3.2 Scenario 2: p=0,17%, ρ= 1/2

Let us assume $p = 0,17\%$ and $\rho = 1/2$.

1. $\alpha = 0,90$
From [1], the sharp bounds are respectively 0 and 9.
To calculate the new bounds, let us first calculate s^2 thanks to eq. (5.2).

$$s^2 = 100 * (0,0017)(1-0,0017)(1+(1/2)*100-1/2) = 84,39$$

We can easily calculate $A = 0$ and $B = 17$. To see of we can improve these bounds, we confront s^2 with

$$\begin{aligned}\alpha(A-\mu)^2 + (1-\alpha)(B-\mu)^2 =& (0,90)(0-1,7)^2) + \\ & (1-0,90)(17-1,7)^2 = 26,01\end{aligned}$$

Since $26,01 \leqslant 84,39$ we can't improve the bounds A and B.

2. $\alpha = 0,95$
   From [1], the sharp bounds are respectively 0 and 25.
   We can easily calculate $A = 0$ and $B = 34$. To see if we can improve these bounds, we confront s^2 with:

$$\begin{aligned}\alpha(A-\mu)^2+(1-\alpha)(B-\mu)^2 =&(0,95)(0-1,7)^2)+\\ &(1-0,95)(34-1,7)^2 = 54,91\end{aligned}$$

   Since $54,91 \leqslant 84,39$ we can't improve the bounds A and B.

3. $\alpha = 0,99$
   From [1], the sharp bounds are respectively 1 and 93.
   We can easily calculate that $A = 0,707$ and $B = 100$. To see of we can improve these bounds, we confront s^2 with

$$\begin{aligned}\alpha(A-\mu)^2+(1-\alpha)(B-\mu)^2 =&(0,99)(0,707-1,7)^2)+\\ &(1-0,99)(100-1,7)^2 = 97,60\end{aligned} \tag{5.7}$$

   Since $97,6 \geqslant 84,39$, we can improve the bounds A and B. From Theorem 5.1, we find the new lower and upper bounds are respectively $\mu-(1-\alpha)/\alpha)^{1/2} = 0,78$ and $\mu+s(\alpha/(1-\alpha) = 93,1$. Following the same reasoning done in the section 4, we can round the bounds to 1 and 93.

Previous results can be summarized by the following table:

α	left bound 1	left bound 2	right bound 1	right bound 2
0,90	0	0	9	17
0,95	0	0	25	34
0,99	1	1	93	93

Table 5.2: two different bounds of VaR_α of the number of defaults with $p = 1,7\%$ and $\rho = 1/2$.

Overall, what we notice from these examples is that when we manage to improve the bounds A and B, then the bounds given by Theorem 5.1 assume the same values as the sharp bounds found in paper [1].

6 Conclusions

The purpose of this thesis was to compare two different couples of bounds of the Value-at-Risk of the total loss of a portfolio, with some underlying hypothesis. While in a first scenario we didn't take into account the correlation between defaults, in the second case we assumed there was between them a correlation ρ. Although it seemed the bounds given by the extremal rays were much finer, we could then realize that just in some cases they can improve the boundaries set by [2] (in particular, as we saw in chapter 4, bounds given by [1] are rarely better than the ones taken from [2] when no information on correlation is provided). The effectiveness of the bounds from [2] can be seen by two aspects. First, it's that they do not require the further hypothesis that the default vector is exchangeable. Secondly, they are more intuitive, and the calculation done in this specific case can be repeated in other cases, with other distributions of the variables X_i. On the other hand, it was worth remembering that only the bounds proposed in [1] are sharp, as they provide the exact maximum and minimum possible values of the Value-at-Risk.

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