POLITECNICO DI TORINO

Corso di Laurea Magistrale in Physics of complex systems

Tesi di Laurea Magistrale

Interaction between energetic particles and axisymmetric modes in magnetically confined fusion plasmas

Thesis supervisor

Prof. Francesco PORCELLI

Thesis candidate

Tommaso BARBERIS

Anno Accademico 2019/2020

Abstract

The most promising path towards controlled thermonuclear fusion as a new source of energy makes use of devices, called tokamaks, for the confinement of high temperature hydrogen plasma by means of intense magnetic fields in toroidal configurations.
The deuterium-tritium reactions in fusion reactors generate α particles with 3.5 MeV energy. These particles are supposed to transfer their energy to the plasma in order to self-sustain the conditions required for thermonuclear burn. Therefore, in order to realize such a fusion burning plasma, α particles in the MeV range must also be magnetically confined.
However, under these circumstances, instabilities related to resonant interactions between the periodic motions of energetic particles and normal mode fluctuations of the thermal plasma may be excited.
Of particular interest in this thesis are toroidally axisymmetric normal modes of the thermal plasma that arise spontaneously in a confinement configuration where the plasma cross-section is non-circular. Indeed, so-called *magnetic divertor* configurations and plasma shaping are common features in modern-day tokamak experiments. The elongation of the plasma cross section is linked to axisymmetric movements of the toroidal plasma column, associated with so-called vertical displacement events (VDE). These are related to normal modes, which conserve toroidal symmetry, i.e. with toroidal wave number n = 0, resulting in a rigid shift of the plasma column. This kind of vertical instability can lead to a termination of the plasma discharge on a characteristic Alfven time scale $\tau_A = R_0/v_A$ (with R_0 the toroidal major radius and v_A the propagation velocity of Alfven waves), which is typically of the order of microseconds, with consequent large electromagnetic stresses on the confinement device. Hence, external feedback currents are used in tokamak experiments to avoid the occurrence of VDEs, which result in characteristic stable plasma oscillations with a frequency close to the characteristic Alfven frequency $2\pi/\tau_A$.
On the other hand, orbits of energetic particles are periodic and exhibit a characteristic frequency called transit frequency $\omega_t \sim (v/R_0)(r/R_0)^{1/2}$ (with v the energetic particle speed and r the minor toroidal plasma radius). When the energy of energetic ions is of the order of 1 MeV, ω_t can become comparable with the oscillation frequency of n=0 axisymmetric vertical displacements, giving rise to a resonant interaction.
This resonance leads to a collisionless transfer of energy between the plasma wave and the particles. Depending on the fast particles distribution function, this can result into either damping of the wave and consequent particle acceleration, or growth of the wave amplitude together with particle decel-

eration. The latter is clearly a dangerous unstable situation, as a growing vertical displacement may reach amplitudes that are no longer controllable by the external feedback system, giving rise to uncontrolled VDEs. This resonant interaction must be studied using a *hybrid kinetic-MHD* (Magneto Hydro Dynamic) model, in which the energetic particles are treated using kinetic plasma theory and the thermal plasma is modelled according to MHD.
The subject of this work is a first study of the resonant interaction between the oscillating n=0 vertical displacement modes and energetic particles at their transit frequency. Thresholds for the resonant excitation of these modes are obtained. As far as we are aware, this is the first theoretical study of this interaction. There is, however, some evidence from the JET (Joint European Torus) tokamak experiment (see, e.g., HJC Oliver et al, Phys. Plasmas 2017, 24, 122505) that these modes can be excited in the presence of energetic deuterons.

Contents

List of Figures

Chapter 1

Introduction

Nuclear fusion reactions, which merge two light nuclei, release great amount of energy due to a small loss of mass during the process. This reactions are the primary energy source of stars. Following the first fusion experiments of the second half of the XX century, research on controlled thermonuclear fusion for the production of electricity has continued to progress. The realization of a fusion reactor represents one of present-day greatest scientific challenges, and it is clear that the availability of controlled thermonuclear fusion as a new energy source could have a huge impact from both social and environmental points of view.

1.1 Fusion power and Lawson criterion

The difficulties related to the control and exploitation of the nuclear fusion are mainly due to the extreme conditions required to achieve fusion. In fact, the fuel atoms need to posses enough energy to overcome the coulomb barrier to approach each other close enough for the triggering of nuclear fusion.

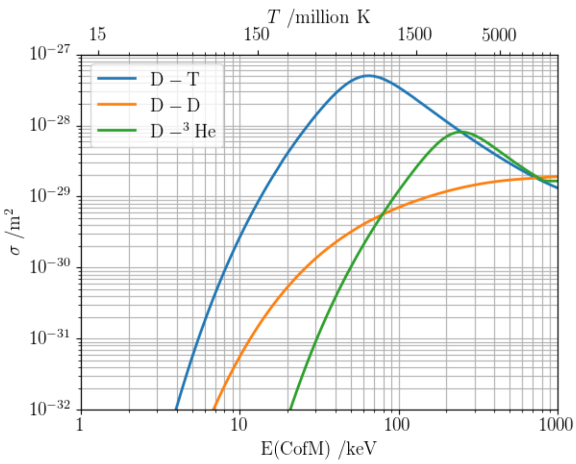

Figure 1.1: Fusion reaction cross sections as a function of the centre of mass energy (temperature)

Figure (1.1) shows the cross section for different nuclear fusion reaction as a function of centre of mass energy. Due to the high energies required for having a non-negligible cross section the fuel will be in the plasma state, i.e. a ionized gas characterized by collective phenomena. Figure (1.1) shows also one of the reasons why now-days fusion research focuses on D-T plasmas: the reaction

$$^2_1D +^3_1 T \rightarrow^4_2 He(3.5MeV) +^1_0 n(14.1MeV) \tag{1.1}$$

presents the highest fusion cross section at the lowest temperature.

In order to exploit fusion energy in the laboratory, a well known criterion, called Lawson criterion, must be fulfilled. Considering the D-T reaction, ^4_2He (or α) particles are supposed to transfer their energy to the plasma in order to self-sustain the high-temperature plasma conditions required for thermonuclear burn and balance energy losses. Consider the power balance

$$\frac{dW}{dt} = P_{aux} + P_\alpha - P_{loss} \tag{1.2}$$

where $W \sim 3nT$ is the energy density of a plasma with temperature T and density $n = n_{electrons} = n_{ions}$. The external power used for the heating of the plasma is represented by P_{aux}, while P_α is the α particles heating power density and P_{loss} takes into account the different power losses caused by conducting, radiative or convective phenomena. Introducing the empirical definition of the energy confinement time τ_E, it is possible to describe the power losses with:

$P_{loss} = W/\tau_E \sim 3nT/\tau_E$. For a D-T plasma with $n_D = n_T = n/2$ it is possible to write P_α as:

$$P_\alpha = n_D n_T \langle \sigma v \rangle \mathcal{E}_\alpha = \frac{1}{4} n^2 \langle \sigma v \rangle \mathcal{E}_\alpha \tag{1.3}$$

where $\mathcal{E}_\alpha = 3.5MeV$ and $\langle \sigma v \rangle$ is the velocity averaged cross section, which is showed in Figure (1.2) for different fusion reactions.

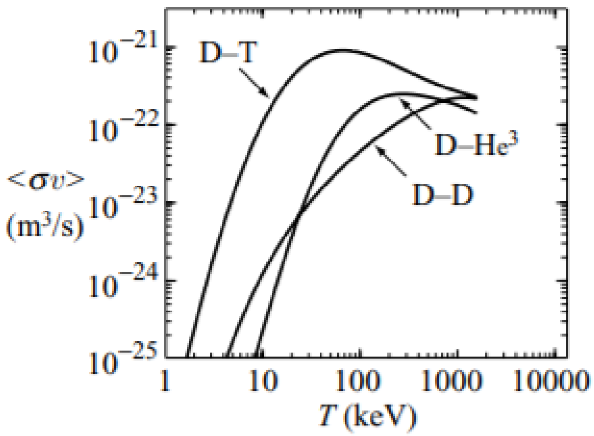

Figure 1.2: Velocity averaged cross section for the D–T, D–He3, and D–D fusion reactions as a function of temperature. [1]

When the α particles heating power density can balance the power losses without auxiliary power the plasma reaches the so-called thermonuclear ignition. Thus, we can define the condition for ignition as $P_\alpha \geq P_{loss}$:

$$\frac{1}{4} n^2 \langle \sigma v \rangle \mathcal{E}_\alpha \geq \frac{3nT}{\tau_E} \tag{1.4}$$

$$n\tau_E \geq \frac{12T}{\langle \sigma v \rangle \mathcal{E}_\alpha} = f(T) \tag{1.5}$$

The function $f(T)$ has an absolute minimum at $T \sim 25keV$, thus replacing $f(T)$ with this minimum value in Eq.(1.5), we obtain the Lawson criterion:

$$n\tau_E > 1.5 \times 10^{20} s \cdot m^{-3} \tag{1.6}$$

Considering instead T between $10keV$ and $20keV$, for D-T plasmas the fusion reaction rate can be approximated by $\langle \sigma v \rangle \simeq c_0 T^2$. Therefore, the criterion of Eq.(1.5) can be rewritten as:

$$n\tau_E T > \frac{12}{\mathcal{E}_\alpha c_0} > 3 \times 10^{21} keV \cdot s \cdot m^{-3} \tag{1.7}$$

where $n\tau_E T$ is typically referred as the "fusion triple product".
In order to satisfy the relation (1.7) mainly two different approaches have been studied: *inertial* confinement and *magnetic* confinement. The first makes use of high power lasers in order to heat and compress a fuel pellet reaching very high densities $n \sim 10^{31} m^{-3}$ for a very short confinement time $\tau_E \sim 10^{-11} s$. On the other hand, the magnetic confinement method makes use of strong magnetic fields in order to confine a low density plasma ($n \sim 10^{20} m^{-3}$) for a confinement time that must be larger than $1s$. The latter represents the most promising and mature method for achieving controlled fusion.

1.2 Magnetic confinement: Tokamak

The magnetic confinement method exploits the fact that the plasma is composed by charged particles and is susceptible to electromagnetic fields. In fact, these particles will move along the magnetic field lines. In magnetic confinement machines the magnetic field is designed to keep plasma particles away from the device's walls and acts as a recipient for the plasma. From the beginning of the research for controlled nuclear fusion several magnetic confinement devices have been designed and studied. Most of present-day fusion experiment are based on the successful design called **Tokamak**.
The tokamak is a toroidal machine designed by the soviets in the late '60s. Since its initial concept the tokamak design has been studied and modified in order to enhance its performance, but the main principles are still the same even in now-days machines. Figure (1.3) shows the principal schematic of a tokamak: outer toroidal field coils induce a toroidal magnetic field while a poloidal field is generated by the plasma current itself resulting in an helical magnetic field. This plasma current is induced as the secondary of a transformer circuit where the internal solenoid represents the primary. Outer poloidal coils are the used for plasma shaping and vertical positioning. The plasma current heats the plasma via ohmic heating, but the efficiency of this heating method decreases with the temperature. Thus, in order to reach the high temperatures required, different heating method have been developed, mainly neutral beam injection (NBI), consisting in shooting high energy neutrals in the plasma, and ion cyclotron resonant heating (ICRH) using high power electromagnetic waves.

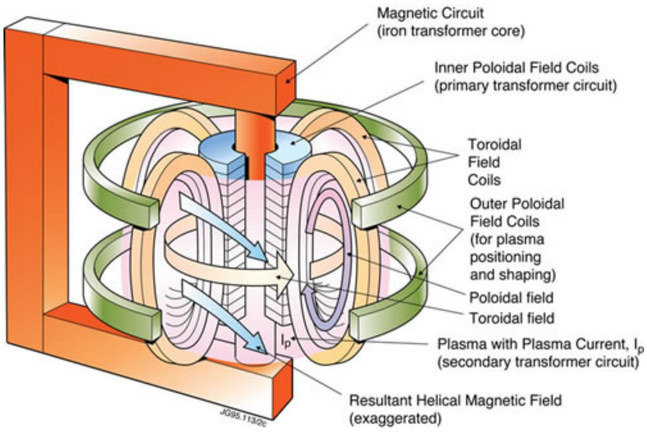

Figure 1.3: Simplified Tokamak scheme. [2]

The most promising path towards controlled thermonuclear fusion as a new source of energy makes use of tokamak devices, and now-days there are many projects under development which can potentially lead to a breakthrough in the fusion research. The leading next-step tokamak experiment is ITER, the International Tokamak Experimental Reactor, at present under construction in southern France under an international effort including all technologically advanced countries in the world. Apart from ITER, and on a national level, DTT, the Divertor Test Tokamak experiment, has been given the green light and will soon start construction at the ENEA laboratories in Frascati. SPARC is a university-scale, compact, high magnetic field device under consideration at MIT. Several tokamak experiments are operating in the world, and among these the JET (Joint European Torus) tokamak experiment is possibly the most advanced and most interesting for the work presented in this thesis.

1.2.1 Equilibrium and stability

The performances of a magnetic confinement device are strictly related to the concepts of equilibrium and stability. The study of equilibrium and stability properties of plasmas is typically done using a fluid model called *Magneto Hydro Dynamics*(MHD), which will be in part treated in Chapter 2 and that is described more in detail in Ref.[1]. As an introduction to the topic, it is possible to summarise that MHD equilibrium and stability are necessary requirements for a fusion reactor. Indeed, instabilities can lead to several consequences ranging from deterioration of the device performances to dangerous termination of the plasma called *disruptions*.
Considering different assumptions and simplifications, it is possible to distin-

guish different versions of MHD: the simplest one is called "Ideal MHD" , which is based on many simplifications but is related to some of the most dangerous plasma instabilities. More complex versions are often addressed as "extended MHD" considering for example finite resistivity or kinetic effects. The main objective of MHD is the study of a magnetic configuration which can sustain a stable equilibrium and that has its parameters, like the plasma pressure and current, large enough to satisfy the criteria of Equations (1.6) and (1.7). One of the important results of the ideal MHD is that the plasma stability can be studied by means of normal mode fluctuations of the plasma and the confining magnetic field.
Among those, Toroidally axisymmetric normal modes of the thermal plasma are of particular interest for this thesis. These are modes that arise spontaneously in a confinement configuration where the plasma cross-section is non-circular. Indeed, modern tokamak experiments often make use of so-called *magnetic divertor* configuration (Figure (1.4)) and plasma shaping in order to enhance their performances.

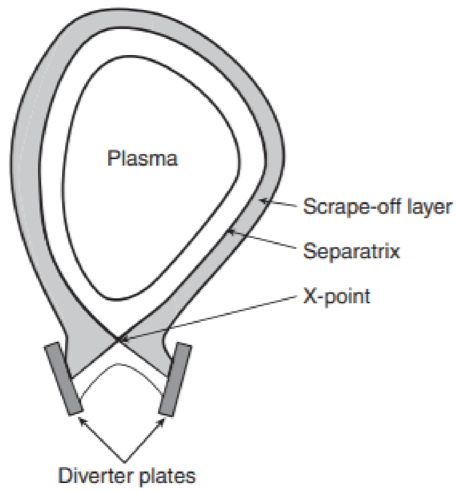

Figure 1.4: Schematic diagram of a divertor (single null configuration),[3]

Magnetic divertors are special configurations where the plasma is bounded by a last-closed-magnetic-surface with one or two magnetic X-points (that are therefore referred to as single or double null divertor configurations). Divertors are thought to be the best way to limit the flux of thermal plasma to the inner metallic wall of the toroidal confinement chamber, therefore mitigating issues of plasma-wall interactions. Plasma shaping, on the other hand, optimise the total current carried by the plasma and therefore enhances confinement performances. However, plasma shaping, and in particular the elongation of the plasma cross section, is linked to potentially dangerous axisymmetric movements of the toroidal plasma column, associated with so-called vertical displacement events (VDE). These are related to normal modes which conserve the toroidal symmetry, i.e. with toroidal wave number n = 0, that result in a rigid shift of the plasma column in the vertical direction (i.e., towards the X-points of the divertor configuration). Vertical plasma displacements are typically studied in an ideal-MHD framework. This kind of vertical instability can lead to a sudden termination of the plasma discharge. For instance, an unstable VDE may cause the multi-Mega-Ampere plasma current to terminate on a characteristic Alfven time scale $\tau_A = R_0/v_A$ (with R_0 the toroidal major radius and v_A the propagation velocity of Alfven waves) , which is typically of the order of micro-seconds, with consequent adverse electromagnetic stresses on the confinement device. Hence, care is used in tokamak experiments to avoid the occurrence of VDEs by external feedback currents, which results in characteristic stable oscillations with a frequency close to the characteristic Alfven frequency $2\pi/\tau_A$.

1.3 Resonant interactions

As described previously, the confinement of energetic particles that result from fusion reactions (fusion α_s) with energies in the MeV range is a crucial aspect for the realization of a fusion reactor. In present-day tokamak energetic particle populations can be produced, using heating techniques, in order to study their effect on the plasma behaviour.
The presence of energetic particles in the plasma can indeed lead to instabilities related to resonant interactions between these particles and normal mode fluctuations of the thermal plasma and the confining magnetic field. The resonance condition involves frequencies related to the periodic motion of the energetic particles within the confinement region matching the frequency of oscillations of thermal plasma collective modes. The relation between resonant interactions and plasma stability is based on a well-known physical effect called "Landau damping" that will be briefly described in the following chapters.
Due to their high energy, energetic particle exhibit several kind of periodic motions, thus each one of their associated frequencies could be resonant with different normal modes. For example the precessional motion around the torus of energetic ions presents frequencies that resonates with the m=1 ideal MHD mode generating so-called "fishbone oscillations" as described in Ref.[6] and [7]. In this work the resonant interaction between energetic particle motion in the poloidal plane and n=0 vertical displacements is investigated.

1.3.1 Experimental evidence

The main experimental evidence that these modes can be excited in presence of energetic particles populations comes from the JET experiment as reported in Ref.[8]. A combination of NBI and ICRH was used in order to produce an energetic population of Deuterium with energies in the range $100keV - 1MeV$. The observed frequency of the axisymmetric mode was $f_{obs} = 325kHz$. Energetic particles can be classified as either passing or trapped with respect to the shape of their periodic motions in the poloidal plane: their orbits can be complete revolutions around the magnetic axis (passing) or so called “banana orbits” (trapped) depending on their velocities.
In particular, considering the plasma parameters for the JET experiments, passing particles present a transit frequency ω_t close to the ω_{obs}. In fact it is possible to show that:

$$\omega_t = \frac{\pi v\sqrt{\epsilon}}{\sqrt{2}R_0 q}\frac{\kappa}{\mathcal{K}(1/\kappa^2)} \quad , \quad \kappa^2 = \frac{1}{2} + \frac{1}{2\epsilon}(1-\Lambda) \tag{1.8}$$

Where $\epsilon = r/R_0$, $\mathcal{K}(x)$ is the complete elliptic integral of the first kind and v is the modulus of the energetic particles velocity. Λ is a parameter defined as $(1+\epsilon cos(\theta))v_\perp^2/v^2$ where θ is the poloidal angle and $v_\perp$ is the velocity of the particle in the direction perpendicular to the magnetic field line. It can be showed that the parameter Λ is a constant of the particle motion, and that it ranges form 0 to $1+\epsilon$. It is related to the shape of the particle orbit in the poloidal plane. In fact, it is easy to show that when $\Lambda < 1-\epsilon$ the projection of the guiding center motion of the particle in the poloidal cross section is a complete orbit around the magnetic axis (passing particle). Instead, when $1-\epsilon \leq \Lambda \leq 1+\epsilon$, the particle will be trapped in a portion of the poloidal plane following a so-called “banana orbit” (trapped particle).

Then, using typical JET parameters: $R_0 \sim 3m \quad , \quad r \sim 0.9m \quad , \quad q \sim 1$ and $800keV$ deuterons corresponding with a velocity $v = \sqrt{2E/m} \sim 8.8 \times 10^6 m/s$. Considering $\Lambda \sim 0.6$, the resulting transit frequency is $f_t = \omega_t/2\pi \sim 325kHz$, which is of the order of f_{obs}. Thus, is reasonable that a resonant interaction between the axisymmetric toroidal mode and passing energetic particles is the mechanism that excited the observed mode during the experiment. This work will provide a first theoretical study of this resonant interaction in order to determine thresholds for the excitation of resonant vertical displacements in the presence of energetic particles.

Chapter 2

Plasma stability

The main objective of the plasma physics research in the framework of magnetic confinement thermonuclear fusion is to devise a magnetic configuration for the containment of the high temperature plasma that satisfies the Lawson criterion (1.6), i.e. able to confine the plasma for a period long enough for fusion reactions to take place and produce a net gain in energy. Given a certain geometric configuration the first step is determine how the magnetic geometry provides forces to confine the plasma that naturally tends to expand, defining the *equilibrium* configuration. Then, the responses of this equilibrium to small perturbations should be investigated in order to determine its *stability.* After introducing a small perturbation in the plasma, if the system tends to return to the previous equilibrium configuration the equilibrium is called *stable*. Instead, if the perturbation will grow in time the equilibrium would be *unstable* and the confined plasma will tend to disassemble. Plasma instabilities represent the main reason that has prevented the control and exploitation of the thermonuclear fusion so far. Thus, the study of the plasma stability is a crucial aspect of the nuclear fusion research field.

2.1 Ideal MHD

A plasma is a complex system made of a huge number of particles related with long range Coulomb interactions making its study very involving. Due to this complexity, a compromise between model simplicity and detail is required. The *kinetic* model keeps an high level of detail making use of distribution functions for the description of position and velocity of the particles. On the other hand this model must be treated numerically, and often for the configurations of interest the computational cost is too high. Analytical investigation of the plasma behaviour can be carried out using fluid models such as the *Magneto*

Hydro Dynamic model. The MHD model treats the plasma as a continuum fluid instead that a collection of a huge number of particles. Doing so, the model is simple enough to be solved exactly in some cases of interest, but due to its very low level of detail the description of some physical phenomena is lost.
The so-called *Ideal MHD* model is the most simple fluid model for a plasma and, due to this simplicity, a large number of physical phenomena of interest cannot be studied using this model. On the other hand, it describes the effect of the magnetic configuration on the equilibrium and stability of the plasma.
The Ideal MHD model is given by the following set of equations (in c.g.s. units):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.1}$$

$$\rho \frac{d\mathbf{u}}{dt} = \frac{1}{c}\mathbf{J} \times \mathbf{B} - \nabla p \tag{2.2}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{2.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.4}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{2.5}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B} = 0 \tag{2.6}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the convective derivative and $\gamma = 5/3$ is the ratio of specific heats.

Equations (2.1), (2.2) and (2.3) are the fluid equations, respectively, for the mass, momentum and energy in terms of the fluid variables mass density ρ, fluid velocity $\mathbf{u}$ and scalar pressure p. The electromagnetic variables magnetic field $\mathbf{B}$, electric field $\mathbf{E}$ and current density $\mathbf{J}$ are described by the low-frequency, pre-Maxwell equations. Since in Ideal MHD low-frequency phenomena are considered, in Eq.(2.5), displacement current has been neglected. Furthermore, as a consequence of Eq.(2.4), we have that $\nabla \cdot \mathbf{B} = 0$. Finally, ideal Ohm's law is considered in Eq.(2.6). This approximation can be justified considering that the typical macroscopic dimension of fusion plasmas is large enough so that characteristic ideal MHD time scale is much smaller than resistive diffusive time. Thus, the plasma is considered to be a perfect conductor and this assumption is what gives the name "ideal" to the model.
The mass equation describes the conservation of the mass in the plasma meaning that phenomena such as the fuel depletion due to nuclear reactions are neglected in the MHD framework. The momentum equation describes the

interaction between three different forces acting on the plasma: the pressure gradient ∇p, the Lorentz force $\mathbf{J} \times \mathbf{B}$ and the inertial force $\rho d\mathbf{v}/dt$. In particular, for static equilibrium it describes how the $\mathbf{J} \times \mathbf{B}$ magnetic force balances the ∇p force that tens to expand the plasma, thus describing how the plasma is confined. The energy equation represents the equation of state used for the closure of the fluid model.
The derivation of the Ideal MHD model starts from the complete mathematical description of the plasma by means of the kinetic Boltzmann equation coupled with Maxewll's equations:

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla f_j + \frac{q_j}{m_j}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = \left(\frac{\partial f_j}{\partial t}\right)_{coll} \tag{2.7}$$

where $f_j(\mathbf{r}, \mathbf{v}, t)$ is the distribution function of the j species (j = electrons, ions, ...) and $(\partial f_j/\partial t)_{coll}$ is the collision term. In order to derive the fluid description of the plasma, the moments of the Boltzmann equation (B.e., Eq.2.7) are considered in order to switch from a microscopic to a macroscopic description:

$$\text{zeroth moment} \rightarrow \sum_j m_j \int d^3v\{B.e.\}_j \rightarrow \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) \tag{2.8}$$

$$\text{first moment} \rightarrow \sum_j m_j \int d^3v\mathbf{v}\{B.e.\}_j \rightarrow \rho_m \frac{d\mathbf{u}}{dt} = -\nabla \cdot \overline{\mathbf{P}} + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{2.9}$$

$$\text{second moment} \rightarrow \sum_j \frac{1}{2} m_j \int d^3v v^2\{B.e.\}_j \rightarrow \frac{d}{dt}\left(\frac{p}{\rho_m^\gamma}\right) = \text{heat flux/pressure tensor terms} \tag{2.10}$$

these equations together with Maxwell's equations are not yet usable, since there are more unknowns than equation. As is usual for fluid models, a closure is needed. In order to obtain the Ideal MHD model, firstly the pressure is considered to be a scalar, thus:

$$\overline{\mathbf{P}} = p\bar{\mathbf{I}} \rightarrow \rho_m \frac{d\mathbf{u}}{dt} = -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B}. \tag{2.11}$$

Then the heat flux is neglected in the second moment conservation equation, meaning that:

$$\frac{d}{dt}\left(\frac{p}{\rho_m}\right) = 0, \quad \gamma = 5/3 \tag{2.12}$$

Concerning Maxwell's equations, resistivity and displacement currents are neglected, as described previously, leading to the equations (2.4)-(2.6)

The complete derivation of the MHD model is reported, for example, in Ref.[3]. Due to the many assumptions required in order to derive the ideal MHD model starting from the complete kinetic-Maxwell system, several important phenomena in fusion plasmas cannot be treated within an ideal MHD framework. For instance, resistive instabilities and, more importantly for this thesis, resonant particle effects must be treated using so-called extended MHD models, which relax some approximations of the ideal model.

2.1.1 Normal modes and Energy principle

As briefly described before, the study of the equilibrium regards how the different forces acting on the plasma balances each other. Once an equilibrium configuration is obtained, its stability is studied by means of the linearized system: all quantities are expanded as $Q(\mathbf{r},t) = Q_0(\mathbf{r}) + Q_1(\mathbf{r},t)$ where $Q_1(\mathbf{r},t)$ is the small perturbation of the quantity with $|Q_1/Q_0| \ll 1$. All the perturbed quantities can then be written in terms of the *displacement vector* $\bar{\tilde{\xi}}(\mathbf{r},t)$ defined as:

$$\mathbf{u}_1 = \frac{\partial \bar{\tilde{\xi}}}{\partial t} \tag{2.13}$$

Considering the perturbation being introduced at t=0:

$$\bar{\tilde{\xi}}(\mathbf{r},0) = 0, \quad B_1(\mathbf{r},0) = 0, \quad \rho_1(\mathbf{r},0) = 0, \quad p_1(\mathbf{r},0) = 0 \tag{2.14}$$

$$\frac{\partial \bar{\tilde{\xi}}(\mathbf{r},0)}{\partial t} = \mathbf{u}_1(\mathbf{r},0) \neq 0 \tag{2.15}$$

Substituting the linearized mass and energy equations (2.1),(2.3) together with the Faraday's law (2.5) into the momentum equation (2.2) in terms of the displacement vector, it is possible to obtain:

$$\rho \frac{\partial^2 \bar{\tilde{\xi}}}{\partial t^2} = \mathbf{F}(\bar{\tilde{\xi}}) = \frac{1}{c}\mathbf{J} \times \mathbf{B}_1 + \frac{1}{c}\mathbf{J}_1 \times \mathbf{B} - \nabla p_1 \tag{2.16}$$

where the subscript 0 for equilibrium quantities has been dropped. **F** is called the force operator.
The linear stability can be investigated through a **normal modes** analysis. Since the coefficients of the force operator $\mathbf{F}(\bar{\xi})$ are time independent, it is possible to consider solution of the type: $\bar{\tilde{\xi}}(\mathbf{r},t) = \sum_q a_q \bar{\xi}_q(\mathbf{r}) exp(-i\omega_q t)$. The

right hand side of equation (2.16) does not present any time derivation meaning that we obtain for each decoupled q component:

$$-\omega_q^2 \rho \overline{\xi}_q = \mathbf{F}(\overline{\xi}_q) \tag{2.17}$$

Equation (2.17) is an eigenvalue problem for the eigenvalue ω_q^2 that can be solved after an appropriate choice of the boundary conditions for $\overline{\xi}$. Using the energy equation and Faraday's law, the force operator can be written explicitly in terms of $\overline{\xi}$:

$$\mathbf{F}(\overline{\xi}) = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}_1 + \frac{1}{4\pi}(\nabla \times \mathbf{B}_1) \times \mathbf{B} + \nabla(\overline{\xi} \cdot \nabla p + \gamma p \nabla \overline{\xi}) \tag{2.18}$$

In this way it is possible to show that the force operator $\mathbf{F}(\overline{\xi})$ is self-adjoint. This property has several consequences on the study of the plasma stability. Firstly, the eigenvalues ω_q^2 are real meaning that instability (i.e. $\mathcal{I}m(\omega_q) > 0$) can be obtained only if $\omega_q^2 < 0$. In addition, the eigenvectors $\overline{\xi}_q$ form a complete orthogonal set, meaning that an arbitrary initial perturbation will be dominated by the fastest growing mode instead of by a combination of different modes in the non orthogonal case.
The stability is obtained knowing how to avoid these fast modes for any perturbation. Ideal MHD stability analysis typically make use of the so-called **Energy Principle**: an equilibrium is stable if and only if the variation of the potential energy integral

$$\delta W = -\frac{1}{2} \int \overline{\xi}^* \cdot \mathbf{F}(\overline{\xi}) d^3x \tag{2.19}$$

is positive for any displacement $\overline{\xi}$ which is physically acceptable (i.e.,which satisfies appropriate boundary conditions).
This implies that the stability of an equilibrium can be determined by examining only the sign of δW, without analyzing the normal-mode equations.
There are two main advantages of the energy principle with respect to the full normal mode analysis. Firstly, physical intuition could lead to some hypothesis of the form of an unstable perturbation. This form can be then used as a trial function $\overline{\xi}_{trial}$ to evaluate δW. If $\delta W(\overline{\xi}^*_{trial}, \overline{\xi}_{trial}) < 0$, then the Energy principle guarantees that the equilibrium is unstable, since exists a physically acceptable displacement ($\overline{\xi}_{trial}$) for which δW is negative.
The second advantage is that it is possible to look for the absolute minimum of δW setting its variation with respect to $\overline{\xi}$ equal to zero. Even if this minimization procedure is slightly more difficult than the trial function approach, it is much less complex than the full normal-mode analysis.

2.2 Vertical instability: heuristics

As stated previously, among all the different ideal MHD modes, in this work we will focus on the axisymmetric toroidal modes resulting in rigid shift of the plasma column in the vertical direction. It is possible to understand and derive the dispersion relation for a vertical displacement perturbation that corresponds to a rigid shift by means of a simple model which is presented in the following. The plasma equilibrium is represented by three rectilinear current wires as shown in figure (2.1 (a)). The equilibrium presents the *plasma wire* at position $x = 0$ (x is the *vertical* direction) traversed by the current $I_p > 0$, and two external wires, fixed at their position $x = \pm l$, each carrying a current $I_{ext} > 0$. This model represents, on a very schematic and simple level, the equilibrium configuration of a straight tokamak with two magnetic X-points.

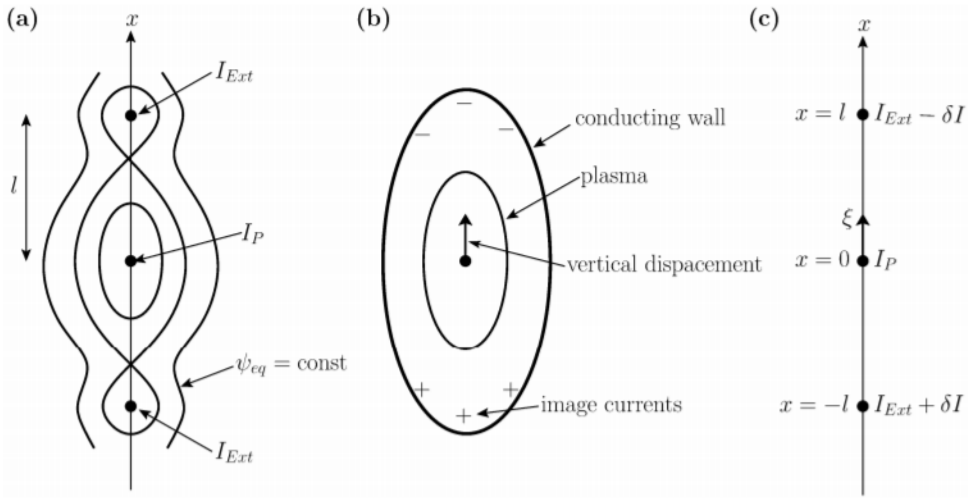

Figure 2.1: (a) Three-wire model of tokamak vertical stability and the corresponding magnetic flux-surfaces. (b) Stabilization of the vertical instability by eddy currents induced in a conducting wall surrounding the plasma. (c) Heuristic model of the stabilization of the vertical instability. [4]

Considering that the plasma wire is the only one free to move (only in the x direction) its equation of motion can be written (in c.g.s. units):

$$\mu \partial_{tt} x = \frac{4 I_p I_{ext}}{c^2} \frac{x}{l^2 - x^2} \tag{2.20}$$

where ∂_{tt} is the second time derivative, μ is the linear mass density and c the speed of light. Considering I_p and I_{ext} constants (i.e. neglecting self and mutual inductions) and small displacement $x \ll l$ the solution of equation (2.20) can be written as:

$$x = x_0 e^{\gamma t} \tag{2.21}$$

where x_0 is the initial displacement and the growth rate γ is:

$$\gamma = \frac{1}{l}\sqrt{\frac{4I_p I_{ext}}{\mu c^2}} \tag{2.22}$$

In this simple model there is no relation between I_p, I_{ext} and l. However recalling that the plasma wire mimics a vertically elongated plasma column with uniform current density a relationship is established as described in Ref.[9].

$$\frac{I_{ext}}{I_p} = \frac{b-a}{b+a}\frac{l^2}{a^2+b^2} \tag{2.23}$$

where a and b represents, respectively, minor and mayor semi-axis of the elliptical magnetic surface cross section within the plasma is confined in the xy plane. The growth rate (2.22) considering the relation (2.23) becomes:

$$\gamma = \left(\frac{b-a}{a+b}\right)^{1/2}\left(\frac{1}{a^2+b^2}\right)^{1/2}\frac{2I_p}{(\mu c^2)^{1/2}} \tag{2.24}$$

This last expression shows hoe the growth rate depends only on I_p, a and b, and is independent from both I_{ext} and l. It is possible to write γ in terms of more common plasma parameters considering a cylindrical plasma with elliptical cross section $\mu = \pi ab\rho_m$, where ρ_m is the volume mass density. Introducing the the ellipticity parameter $\epsilon = (b^2-a^2)/(b^2+a^2)$ and assuming that it is small, one gets: $a \simeq b$, $2I_p \simeq caB_p(a)$ and finally

$$\gamma = \frac{\epsilon^{1/2}v_A}{a} \tag{2.25}$$

where v_A is the Alfvén velocity defined as $B_p(a)/(4\pi\rho_m)^{1/2}$. The inverse of the growth rate is thus proportional to the characteristic Alfvén time, which is typically of the order of the microsecond. Considering for example a Deuterium plasma with $\epsilon \sim 0.2$, $a \sim 0.9m$, $B_p(a) \sim 1T$ and $n = 10^{20}m^{-3}$ plasma number density, we get $\gamma^{-1} \sim 1.3\mu s$. This is indeed a very fast growth rate related with a dangerous instability.
The ideal vertical instability is stabilized thanks to the presence of a perfect conducting wall not too far away from the plasma. Figure (2.1 (b)) shows

a sketch of the mechanism that suppress the vertical instability. When the plasma current is displaced from its equilibrium position, image currents are induced at the wall. Their sign is such that the corresponding forces oppose the motion of the plasma wire. This can be modeled back in the simple heuristic case as shown in figure (2.1 (c)). The image currents induced in the wall are mimicked by a current δI that is carried with opposite sign in the two external wires. In the perfectly conducting limit, the two currents can be considered as driven by an induced electromagnetic field proportional to $\partial_t x$:

$$\partial_t \delta I = \frac{D I_{ext}}{l} \partial_t x \tag{2.26}$$

where D is a dimensionless constant. Considering the tokamak problem the parameter D will depend on the actual geometry of the wall.
Including the the effects of the currents $\pm\delta I$, the equation of motion describing the plasma wire position in the x direction reads:

$$\mu c^2 \partial_{tt} x = 4 I_p I_{ext} \frac{x}{l^2 - x^2} - 4 I_p \delta I \frac{l}{l^2 - x^2} \tag{2.27}$$

$$x \ll l \rightarrow \mu c^2 \partial_{tt} x = 4 I_p I_{ext} (1 - D) \frac{x}{l^2} \tag{2.28}$$

The stability is obtained when $D > 1$. When the vertical instability is suppressed by the image currents in the wall ($D > 1$), an initial perturbation of the plasma in the x direction will evolve into an oscillatory motion with characteristic frequency of the order of the inverse of the Alfvén time, more precisely:

$$\omega = \gamma \sqrt{D - 1} \tag{2.29}$$

In Ref.[10] the geometrical parameter D was derived analytically in the case of uniform plasma current distribution and confocal elliptical plasma and wall cross-sections:

$$D = \frac{b + a}{b - a} \left(\frac{b + a}{b_w + a_w} \right)^2 \tag{2.30}$$

where a and b are the minor and major semi-axes of the elliptical plasma cross section, while a_w and b_w are the minor and major semi-axes of the wall elliptical cross-section.

2.2.1 δW_{MHD}

As stated in the equation (2.25), the ideal vertical instability has a very fast growth rate, meaning that it is potentially very dangerous for the machine

operation. Indeed, vertical instability causes so-called vertical displacement events (VDE), a sudden termination of the plasma discharge with consequent large electromagnetic stresses on the confinement device. The stabilization of this kind of instabilities is thus a crucial aspect of a tokamak configuration.
Considering an accurate ideal MHD analysis of the vertical stability, which takes into account the stabilization feedback currents ([10],[11],[4]), it is possible to write the variation of the potential energy integral as:

$$\delta W_{MHD} = \frac{1}{2}(V\rho_m)\xi_0^2 \frac{\epsilon_{el} v_A^2}{a^2}(D-1) \tag{2.31}$$

where: $\epsilon = (b^2 - a^2)/(b^2 + a^2)$ is the ellipticity parameter, $v_A = B_p(a)/(4\pi\rho_m)^{1/2}$ is the Alfvén speed, ρ_m is the volume mass density and V is the plasma volume.
Equation 2.31 clearly show that the stability condition $\delta W_{MHD} > 0$ is obtained only if $D > 1$ accordingly to the criterion founded with the heuristic analysis.

Chapter 3

Hybrid Kinetic MHD

Plasma particles with super thermal energies are called energetic particles. These particles, with energies in the MeV range, are produced in tokamak experiments by heating systems such as NBI and ICRH. In future projects, on the other hand, α particles will be generated with $3.5MeV$ energy by D-T fusion reactions. The presence of energetic particles populations introduce a particular kind of instabilities related with resonance interaction between periodic motions of these particles and macroscopic modes of the thermal plasma. This is the case of fishbones oscillations ([6],[7]) and of Toroidal Alfvén Eigenmodes (TAE) ([12]), where instability is indeed caused by the resonance between plasma modes and energetic particles orbits. The resonance mechanism is related the "Landau Damping", a well known physical effect which can be described only by a kinetic treatment of the plasma. This effect results into the damping (or growth) of macroscopic plasma waves due to collisionless transfer of energy between particles with resonant frequencies and plasma waves. Due to its many simplifications, the ideal MHD model is not able to treat effects related with the presence of energetic particles. In order to investigate resonance interaction between macroscopic modes of the plasma and energetic particles populations, a different kind of model has been used: the so-called *Hybrid Kinetic MHD*. The Hybrid kinetic MHD model consists in the ideal MHD equations for the treatment of the thermal plasma with the addiction of the kinetic Vlasov equation in order to deal with energetic particles contributions.

3.1 Landau resonance

In this first section the "Landau resonance" effect is briefly introduced. The Landau resonance is a resonant interaction between particles in the tails of a maxwellian distribution and so-called space-charge waves. The study of this

effect had important consequences on the study of plasma stability. Firstly, it showed that the fluid model fails to capture this kind of interactions. More importantly, the study carried out by Landau indicated the rigorous procedure to follow in order to deal with resonant interactions, summarized by the "Landau prescription" that will be used later in this work.
Space-charge waves, also called Langmuir waves, are longitudinal electrostatic plasma oscillations.

3.1.1 Fluid theory

The Elecron-fluid equations

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{3.1}$$

$$m_e n_e \left(\frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla \right) \mathbf{v}_e = -\nabla p_e - e n_e \mathbf{E} - \frac{e n_e}{c} \mathbf{v}_e \times \mathbf{B} + \overline{R}_{e\to i} \tag{3.2}$$

together with the Maxwell's equations and the fluid closure defined as:

$$\left(\frac{\partial}{\partial t} \mathbf{v}_e \cdot \nabla \right) \left(\frac{p_e}{n_e^\gamma} \right) = 0, \quad \gamma = 3 \tag{3.3}$$

describe a closed fluid model, where $\overline{R}_{e\to i} = m_e n_e \nu_{ei} (\mathbf{v}_i - \mathbf{v}_e)$ describes electrons-ions collisions with ν_{ei} electron-ion collision frequency. The ions are taken to be immobile and unperturbed, while the plasma equilibrium considered is uniform and unmagnetized, with $\mathbf{B}_0 = \mathbf{v}_{e0} = \mathbf{E}_0 = 0$, $n_e = n_i = n_0 = const.$ and $p_{e0} = const. = n_0 T_{e0}$. Introducing then the perturbation and considering longitudinal waves we obtain:

$$n_e = n_0 + \tilde{n}, \quad p_e = p_{e0} + \tilde{p}, \quad \mathbf{v}_e = \tilde{\mathbf{v}}_e \tag{3.4}$$

$$\mathbf{E} = E e^{-i\omega t + ikz} \hat{e}_z \tag{3.5}$$

As a consequence of the last relation we have $\nabla \times \mathbf{E} = 0$, meaning that $\mathbf{B} = 0$. Starting from the linearized Electron-fluid perturbed equations is easy to obtain:

$$- i\omega \tilde{n}_e - i n_0 k \tilde{\mathbf{v}}_{ez} = 0 \tag{3.6}$$

$$- i(\omega + i\nu_{ei}) \tilde{\mathbf{v}}_{ez} = -\frac{ik}{m_e n_0} \tilde{p}_e - \frac{e}{m_e} \mathbf{E} \tag{3.7}$$

Deriving the expressions for $\tilde{n}_e$,$\tilde{p}_e$ and $\mathbf{E}$ in terms of $\tilde{\mathbf{v}}_{ez}$ from the Equation (3.6), the closure equation and Maxwell's equations, it is possible to rewrite

Eq.(3.7) as:

$$\omega(\omega + i\nu_{ei})\tilde{\mathbf{v}}_{ez} = \frac{4\pi n_0 e^2}{m_e}\tilde{\mathbf{v}}_{ez} + \gamma k^2 \frac{T_{e0}}{m_e}\tilde{\mathbf{v}}_{ez} \tag{3.8}$$

Setting $T_{e0}/m_e = v_{the}^2$, with v_{the} the thermal velocity of the electrons, the dispersion relation becomes:

$$\omega(\omega + i\nu_{ei}) = \omega_p^2 + \gamma k^2 v_{the}^2 \tag{3.9}$$

where $\omega_p^2 = 4\pi n_0 e^2/m_e$ is the plasma frequency. Neglecting collisional effects and considering the known result for space-charge waves of $\gamma = 3$, the following dispersion relation, called Bohm-Gross dispersion relation, is obtained:

$$\omega^2 = \omega_p^2 + 3k^2 v_{the}^2 \tag{3.10}$$

On the other hand, the collisional dispersion relation can be studied perturbatively considering $\nu_{ei} \ll \omega_p$, resulting into: $\omega \approx \pm\sqrt{\omega_p^2 + 3k^2 v_{the}^2} - i\nu_{ei}/2$ Therefore, the damping rate is determined solely by the electron-ion collision frequency and is independent from the plasma frequency.

3.1.2 Kinetic theory

The kinetic description of space-charge waves is based on the same uniform unmagnetized equilibrium and considering a maxwellian equilibrium distribution function for the electrons:

$$f_{e0}(\mathbf{v}) = \frac{n_0}{(2\pi T_{e0}/m_e)^{3/2}} exp(-\frac{m_e v^2}{2T_{e0}}) \tag{3.11}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. In order to determine the response of the equilibrium after the introduction of a perturbation, in the framework of kinetic theory, the system composed by the collisionless Vlasov equation (Eq.(3.12)) and Maxwell's equations is investigated.

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\cdot\nabla f_e - \frac{e}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_e}{\partial \mathbf{v}} = 0 \tag{3.12}$$

Considering electrostatic longitudinal waves, the magnetic field is zero at all times, and the only Maxwell's equation of interest is the Poisson's law:

$$\nabla\cdot\mathbf{E} = 4\pi e(n_{i0} - n_e) \tag{3.13}$$

where $n_{i0} = n_0 = const.$ and $n_e = \int d^3v f_e$.
Introducing the perturbation:

$$f_e = f_{e0} + \tilde{f}_e \tag{3.14}$$

$$\tilde{f}_e = f(\mathbf{v})e^{(-i\omega t + ikz)} \tag{3.15}$$

$$\tilde{\mathbf{E}} = Ee^{(-i\omega t + ikz)}\hat{e}_z \tag{3.16}$$

Considering the linearized perturbed equations

$$\begin{cases} \frac{\partial \tilde{f}_e}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f}_e - \frac{e}{m_e}\tilde{E}_z \frac{\partial f_{e0}}{\partial v_z} = 0 \\ \nabla \cdot (\tilde{E}_z \hat{e}_z) = -4\pi e \tilde{n}_e = -4\pi e \int d^3v \tilde{f}_e \end{cases} \tag{3.17}$$

it is possible to solve the first equation for $\tilde{f}_e$, then substituting $\tilde{f}_e$ into the second is easy to obtain:

$$-ik\tilde{E}_z = -i\frac{4\pi e^2}{m_e}\int \frac{\partial f_{e0}/\partial v_z}{\omega - kv_z} d^3v \tilde{E}_z \tag{3.18}$$

$$\rightarrow ikD(\omega, k)\tilde{E}_z = 0 \tag{3.19}$$

where

$$D(\omega, k) = 1 - \frac{4\pi e^2}{km_e}\int \frac{\partial f_{e0}/\partial v_z}{kv_z - \omega} d^3v \tag{3.20}$$

The solution is $D(\omega, k) = 0$, which determines relevant dispersion relation $\omega = \omega(k)$.
Unfortunately, in the dispersion relarion $D(\omega, k) = 0$, the denominator of the integrand vanishes for $v_z = \omega/k$, meaning that the integral is not defined, unless a prescription is given as to how to integrate around the pole.
Physically, this difficulty is caused by electrons moving with exactly the phase velocity of the wave $v_z = v_{ph} = \omega/k$. If such electrons are not present in the equilibrium plasma, f_{e0} vanishes together with $\partial f_{e0}/\partial v_z$ and the integral can be carried out.
However, when resonant electrons are present, the correct procedure is based on an initial value problem, i.e. introducing the perturbation at t=0 and making use of the Laplace transformation with respect to time. This procedure was worked out by Landau and provides a procedure on how to integrate around the pole, which gives physical meaning to the integral appearing in $D(\omega, k)$.
We rewrite $D(\omega, k)$ as:

$$D(\omega, k) = 1 - \frac{4\pi e^2}{k^2 m_e} n_{e0} I(\omega, k) \tag{3.21}$$

where

$$I(\omega,k)=\int \frac{\partial f_{e0}/\partial v_z}{v_z-\omega/k}dv_x dv_y dv_z \tag{3.22}$$

Considering the maxwellian equilibrium distribution function of Eq.(3.11) and carrying out the integration in v_x and v_y, $D(\omega,k)$ and $I(\omega,k)$ become:

$$D(\omega,k)=1-\frac{\omega_p^2}{k^2}I(\omega,k) \tag{3.23}$$

$$I(\omega,k)=\int_{-\infty}^{+\infty}\frac{\partial f_{eM}}{\partial v_z}\frac{dv_z}{v_z-\omega/k} \tag{3.24}$$

where f_{eM} is the one dimensional maxwellian distribution defined as:

$$f_{eM}(v_z)=\frac{n_0}{(2\pi T_{e0}/m_e)^{3/2}}exp(-\frac{m_e v_z^2}{2T_{e0}}) \tag{3.25}$$

The *Landau prescription* for the integration around the pole is as follows. If ω is a complex number such that $\omega=\omega_R+i\gamma$, then the integral $I(\omega,k)$ is well defined. However, if $\gamma\leq 0$, then $I(\omega,k)$ must be obteined by analytic continuation of $I(\omega,k)$ in the complex ω plane. This means that the path of integration must be always below the pole $v_z=\omega/k$ as showed in Figure (3.1). The rigorous procedure used to derive the prescription can be founded, for instance, in Ref.[5].

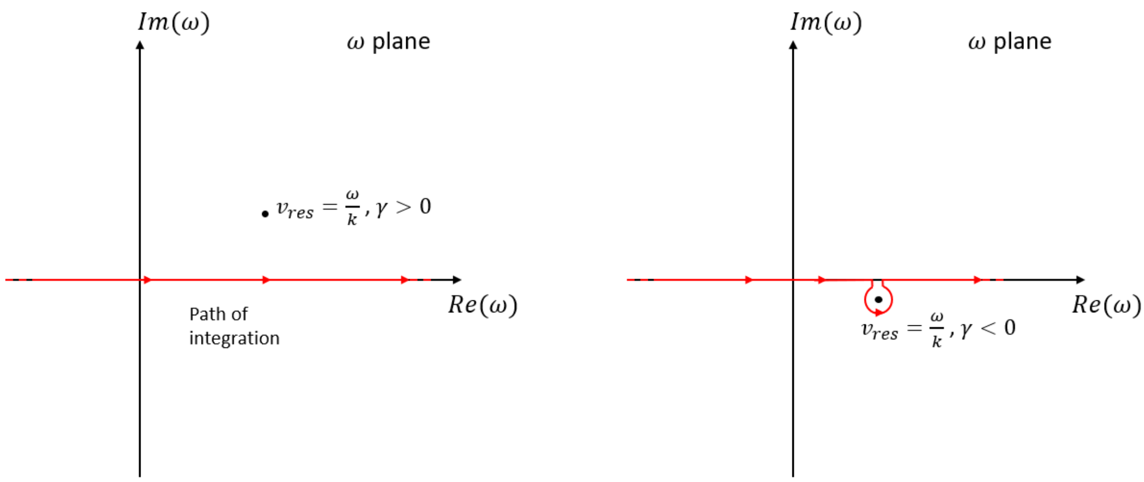

Figure 3.1: Path of integration according to the Landau prescription

For the limit $\gamma/\omega_R \ll 1$, which is also the case of major interest, is thus possible to approximate the integral $I(\omega, k)$ as:

$$I(\omega, k) = \fint_{-\infty}^{+\infty} \frac{\partial f_{eM}}{\partial v_z} \frac{dv_z}{v_z - \omega/k} + i\pi \left(\frac{\partial f_{eM}}{\partial v_z} \right) \Big|_{v_z = \omega_R/k} \tag{3.26}$$

where the $\fint$ represents the Cauchy principal value and the + sign is determined by the Landau prescription.
We will assume that $\omega/k \gg v_{th}$, meaning that the resonant electrons are in the tail of the maxwellian distribution. In this case the contribution of these electrons to the principal part of the integral would be negligible, therefore it is possible to evaluate it using kv_{th}/ω as small expansion parameter. The integral can be written as:

$$I(\omega, k) = \frac{k^2}{\omega^2} \left(1 + \frac{3k^2 v_{th}^2}{\omega^2} \right) + i\pi \left(\frac{\partial f_{eM}}{\partial v_z} \right) \Big|_{v_z = \omega_R/k} \tag{3.27}$$

Thus, the dispersion relation becomes:

$$D(\omega, k) = 1 - \frac{\omega_p^2}{\omega^2} \left(1 + \frac{3k^2 v_{th}^2}{\omega^2} \right) - i\pi \frac{\omega_p^2}{k^2} \left(\frac{\partial f_{eM}}{\partial v_z} \right) \Big|_{v_z = \omega_R/k} = 0 \tag{3.28}$$

Considering just the real contribution for a moment, remembering that $\omega/kv_{th} \gg 1$, it is possible to approximate $\omega^2 \approx \omega_p^2$ in the last term, obtaining:

$$\omega^2 = \omega_p^2 + 3k^2 v_{th}^2 \tag{3.29}$$

which is the Bohm-Gross dispersion relation obtained with the fluid theory. However, the resonant electron contribution introduces an imaginary part in the dispersion relation.
In the case of the maxwellian distribution is easy to solve the dispersion relation of Eq.(3.28) perturbatively considering $\omega = \omega_0 + \delta\omega$. The resonant electron contribution introduces the imaginary part of $\delta\omega$:

$$\mathcal{I}m(\delta\omega) = \gamma = -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{\omega_p}{(k\lambda_D)^3} exp[-\frac{1}{2(k\lambda_D)^2}] \tag{3.30}$$

where $\lambda_D = v_{th}/\omega_p$ is the Debye length.
Therefore, in the case of a maxwellian equilibrium distribution in the velocity phase space, the presence of resonant electrons introduces a collisionless damping of the space-charge waves. In particular this damping, which depend on the actual value of $k\lambda_D$, can be much stronger than the collisional damping

introduced in section 3.1.1.
More in general, depending on the sign of the derivative $\partial f_{e0}/\partial v_z$ of Eq.(3.22), the resonant contribution can result into either damping or growth.
Resonant electrons that move with a velocity equal to the phase velocity of the plasma wave feel a constant electric field instead of an oscillating one. Due to this these particles can be accelerated or decelerated by the wave. Electrons with energies slightly smaller that the phase velocity of the wave ($v_z \lesssim v_{ph}$) will increase their velocities and energy will be transferred from the wave to the kinetic energy of the particles. On the other hand, electrons with $v_z \gtrsim v_{ph}$ will be decelerated and will give their energy to the wave. In the maxwellian case, with $\partial f_{e0}/\partial v_z < 0$, there are more particle with velocity smaller than the phase velocity of the wave than larger, meaning that the net energy transfer will be from the wave to the particles resulting in the damping of the wave. The opposite happens, for instance, in the case of the so-called "bump on tail" instability where the equilibrium distribution takes the form showed in Figure (3.2).

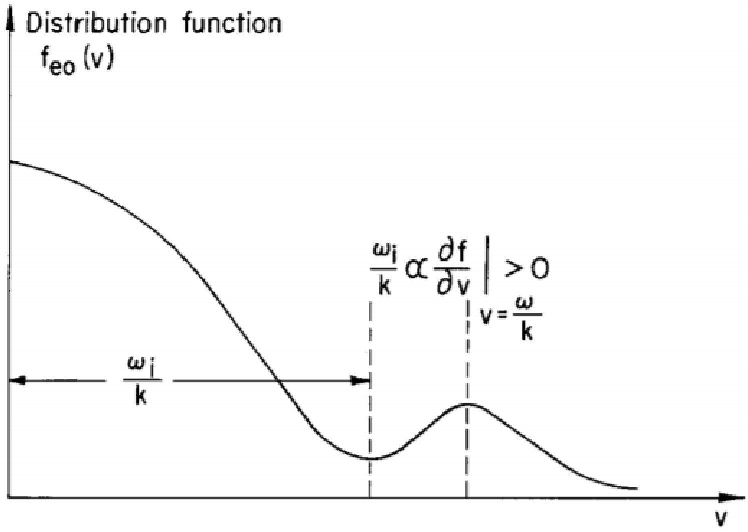

Figure 3.2: Bump on tail distribution function [5]

3.2 The problem of a fluid closure

The fluid closure used for the ideal MHD model is not valid when the energetic particles are present in the plasma. More in general, when these particles are added to the plasma there is no possible fluid closure. This is because the fluid closure represents a *local* relation between moments of the distribution such as ρ_m, $\mathbf{u}$ and p, as in the case of the ideal MHD closure:

$$\frac{\partial}{\partial t}\left(\frac{\rho_m}{p^\gamma}\right) + \mathbf{u}\cdot\nabla\left(\frac{\rho_m}{p^\gamma}\right) = 0 \tag{3.31}$$

In order to obtain a fluid closure one have to first make sure that fluid elements can be defined. Their dimension l must be small compared to other relevant scale length $l \ll L$ (where L is the macroscopic length of the system) and $l \ll \lambda$ (where λ is a generic wavelength of the system). In addiction fluid elements must maintain their coherence for times that are long compared with relevant time scales. In neutral gas described by fluid dynamics, collisions guarantee the coherence of fluid elements. Indeed, gas particles tend to stick together due to collisions with neighbours and their motion may be represented as the superposition of the local mass motion, plus an isotropic distribution of velocities. In this way $\mathbf{u}$ is provided with the physical meaning of the fluid velocity and $\overline{\mathbf{P}}$ is reduced to a scalar pressure p. For the simplest gas the continuity equation and the momentum equation contain the unknowns ρ_m, $\mathbf{u}$ and p, thus a third equation of state that connects ρ_m, $\mathbf{u}$ and p is required to close the fluid model and render the system solvable. Collisions can be described in terms of collisional mean-free-path λ_{mfp} and the collisional time τ_{coll}. Thus, collisional fluid closure is possible if one can choose the size of the fluid element l such that $\lambda_{mfp} \ll l \ll L$, and in addition $k\lambda_{mfp} \ll 1$ and $\omega\tau_{coll} \gg 1$.
In the case of fusion plasmas, on the other hand, collisions are very rare. Indeed, the relevant time scale for ideal MHD phenomena in fusion plasmas is the Alfvén time τ_A, which is typically of the order of the microsecond. On the other hand, a typical value of τ_{coll} for a thermonuclear plasma is of the order of $10^{-4}s$, meaning that collisions cannot guarantee the coherence of the fluid element since ideal MHD effects will act on much faster time scales. Thus, in order to define the fluid elements, mechanisms different from collisions must be considered.
In particular, the the coherence of fluid elements on the plane perpendicular to the magnetic field is guaranteed if the Larmor radius ρ_L is sufficiently small. In this way Larmor orbits will represent the fundamental "quasi-particles" in the system. Associated with the smallness of ρ_L is the conservation of the magnetic moment defined as $\mu_\perp = v_\perp^2/2B$. The Larmor orbit is a periodic motion with an associated frequency $\Omega_c = qB/mc$. Thus, $\mu_\perp$ is conserved as long as $k_{bot}\rho_L \ll 1$ and $\Omega_c/\omega \gg 1$. The conservation of the magnetic moments leads to the so-called CGL closure for a magnetized plasma. The conservation of μ_{bot} is related to the conservation of the magnetic flux through the Larmor orbit.
Considering then energetic particles, their motion contains three periodicities:

Larmor gyration (Ω_c), bounce or transit frequency of the particle orbit in the poloidal plane ($\omega_{b/t}$) and precessional drift motion on the toroidal direction (ω_D). Typically $\Omega_c \gg \omega_{b,t} \gg \omega_D$. Considering MHD perturbations that satisfy $\omega \ll n\omega_D$, the magnetic flux enclosed by the precessional drift orbit φ is conserved. In this situation the relevant "quasi-particle" of the system is the precessional drift orbit. However, this orbit has the size of the system itself meaning that, even for a perturbarion localized in space, the precessional drift orbit will react as a whole in order to conserve the magnetic flux φ. Thus, the condition $l \ll L$ cannot be satisfied, meaning that it is impossible to obtain a fluid closure in presence of energetic particles.

3.3 Hybrid kinetic model

The impossibility of a fluid closure when energetic particles are added in the plasma brought to the development of a different kind of model: the Hybrid kinetic-MHD.

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{U}) = 0 \tag{3.32}$$

$$\rho \frac{d\mathbf{U}}{dt} = \frac{1}{c}\mathbf{J} \times \mathbf{B} - \nabla p_c - \nabla \cdot \overline{\mathbf{P}}_h \tag{3.33}$$

$$\frac{d}{dt}\left(\frac{p_c}{\rho^\gamma}\right) = 0 \tag{3.34}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{3.35}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) \tag{3.36}$$

As suggested by its name, this model treats the plasma using both kinetic and fluid theories. In particular particles of the thermal plasma are treated using MHD, while terms related with energetic particles are described by the kinetic model. In particular the energetic particles pressure tensor can be written as:

$$\overline{\mathbf{P}}_h = p_{\perp h}\overline{\mathbf{I}} + (p_\parallel - p_\perp)_h \hat{e_\parallel}\hat{e_\parallel} \tag{3.37}$$

$$p_{\perp h} = m_h \int d^3v \frac{v_\perp^2}{2} f_h; \quad p_{\parallel h} = m_h \int d^3v v_\parallel^2 f_h \tag{3.38}$$

where $f_h(\mathbf{r}, \mathbf{v}, t)$ is the distribution function of the energetic particles described by the Vlasov equation:

$$\frac{\partial f_h}{\partial t} + \mathbf{v} \cdot \nabla f_h + \frac{q_h}{m_h}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_h}{\partial \mathbf{v}} = 0 \tag{3.39}$$

3.3.1 Drift-kinetic equation

The Vlasov equation for energetic particles showed in Eq. (3.39) can be simplified following the procedure outlined in Ref.[13].
Considering perturbations satisfying the ordering $k_\perp \rho_h \ll 1$ and $\omega/\Omega_h \ll 1$ being ρ_h and Ω_h respectively Larmor radius and Gyration frequency of energetic particles. In addition the different drifts related to the motion of the particles in the tokamak are assumed to be of the same order of magnitude. Thus, it is possible to have the $\mathbf{E} \times \mathbf{B}$ drift of the same order of the grad B and curvature drift setting $E/B = \mathcal{O}(\delta)$. Where $\delta = \rho_h/R_0 \ll 1$ is a small expansion parameter and R_0 is the toroidal major radius. The guiding center velocity correct to first order in δ is (dropping the subscript h for Ω and ρ):

$$\dot{\mathbf{R}} = v_\parallel \hat{e}_\parallel + \frac{1}{m\Omega}\hat{e}_\parallel \times (\mu \nabla B + m v_\parallel^2 \boldsymbol{\kappa} - Ze\mathbf{E}) = v_\parallel \hat{e}_\parallel + \mathbf{v}_D + \mathbf{v}_{\mathbf{E}\times\mathbf{B}} \tag{3.40}$$

where $\mathbf{R}$ denotes the guiding center position, the overdot indicates the time derivative, $v_\parallel$ is the velocity in the parallel direction described by the unit vector $\hat{e}_\parallel = \mathbf{B}/B$, $\mu = m v_\perp^2/2B$ is the magnetic moment, $\boldsymbol{\kappa} = (\hat{e}_\parallel \cdot)\hat{e}_\parallel$ is the curvature vector and Z is the charge number of the particle.
Due to the ordering assumed previously $\mathbf{v}_D \sim \mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \mathcal{O}(\delta)$ while $v_\parallel \hat{e}_\parallel = \mathcal{O}(1)$. Terms of order $\mathcal{O}(\delta^2)$ that have been neglected include different kind of particles drifts such as the so-called polarization drift and the drift $(v_\parallel/\Omega)\hat{e}_\parallel \times (\partial \hat{e}_\parallel)/\partial t$.
The acceleration of the particle in the parallel direction correct to order δ is

$$m\dot{v}_\parallel = -\mu \hat{e}_\parallel \cdot \nabla B + Ze\hat{e}_\parallel \cdot \mathbf{E} + m v_\parallel \boldsymbol{\kappa} \cdot \dot{\mathbf{R}} \tag{3.41}$$

Equations (3.40) and (3.41) are consistent with the guiding center Lagrangian formulation described by Littlejohn in Ref.[14]:

$$\mathcal{L} = \left(\frac{Ze}{c}\mathbf{A} + m v_\parallel \hat{e}_\parallel\right) \cdot \dot{\mathbf{R}} + \frac{1}{\Omega} y \dot{\alpha} - \frac{1}{2} m v_\parallel^2 - y - Ze\phi \tag{3.42}$$

where $\mathbf{A}$ and ϕ are electromagnetic potentials, $y = \mu B = m v_\perp/2$ represents the "perpendicular energy" and α is the gyroangle in the velocity space.
This Lagrangian is considered as a function of the variables

$$\mathcal{L} = \mathcal{L}(\mathbf{R}, v_\parallel, y, \alpha; \dot{\mathbf{R}}, \dot{v}_\parallel, \dot{y}, \dot{\alpha}) \tag{3.43}$$

in which α, $\dot{v}_\parallel$ and $\dot{y}$ do not appear.
Equations (3.40) and (3.41) can be easily obtained from the Euler-Lagrange equations correct to order δ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \tag{3.44}$$

where q_i, with i that ranges form 1 to 6, are the Lagrangian variables. In addiction also the following relations are derived form the Euler Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) = \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \rightarrow \frac{d}{dt}\left(\frac{y}{B}\right) = \frac{d\mu}{dt} = 0 \tag{3.45}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y} = 0 \rightarrow \dot{\alpha} = \Omega \tag{3.46}$$

The Vlasov equation can be written as:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{q}_i \frac{\partial f}{\partial q_i} = 0 \tag{3.47}$$

$$\rightarrow \frac{\partial f}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}} + \dot{y} \frac{\partial f}{\partial y} + \dot{\alpha} \frac{\partial f}{\partial \alpha} = 0 \tag{3.48}$$

where f is the distribution function in terms of the new variables: $f = f(\mathbf{R}, v_{\parallel}, y, \alpha; t)$.
It is possible to simplify Eq. (3.48) neglecting corrections of order δ. First, we point out that the term proportional to $\partial f/\partial \alpha$ is the largest. Setting $f = f_0 + f_1 + ...$ with $f_1/f_0 = \mathcal{O}(\delta)$ since all other terms of Eq. (3.48) are of order δ, to leading order one gets

$$\dot{\alpha} \frac{\partial f_0}{\partial \alpha} = 0 \tag{3.49}$$

meaning that f_0 is independent of α. Thus, to first order equation (3.48) becomes:

$$\frac{\partial f_0}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f_0 + \dot{v}_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} + \dot{y} \frac{\partial f_0}{\partial y} + \dot{\alpha} \frac{\partial f_1}{\partial \alpha} = 0 \tag{3.50}$$

Now, averaging over α the last term cancels out since $\oint d\alpha \dot{\alpha} \partial f_1/\partial \alpha = 0$ due to periodicity. The α-averaging of other terms is trivial since $\dot{\mathbf{R}}$,$\dot{v}_{\parallel}$ and $\dot{y}$ do not depend on α to first order in δ. Thus, the equation that will be considered in the following is the collisionless drift-kinetic equation:

$$\frac{\partial f_0}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f_0 + \dot{v}_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} + \dot{y} \frac{\partial f_0}{\partial y} = 0 \tag{3.51}$$

the subscript “0” will be dropped in the following sections.
Considering an axisymmetric equilibrium where the toroidal angle φ is an ignorable coordinate such that $\partial/\partial t = \partial/\partial \varphi = 0$. The equilibrium distribution

function F is then, in general, a function of three invariants of the particle motion and of the index σ:

$$F = F(\mathcal{E}, \mu, P_\varphi; \sigma) \tag{3.52}$$

where $\mathcal{E}$ is the total energy of the particle, μ is the magnetic moment and P_φ is the toroidal momentum and are defined as:

$$\mathcal{E} = \frac{1}{2} m v_\parallel^2 + y + Ze\phi(\mathbf{R}) \tag{3.53}$$

$$\mu = \frac{y}{B(\mathbf{R})} \tag{3.54}$$

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \varphi} = \frac{Ze}{c}\psi + mRv_\parallel \frac{B_\varphi}{B} \tag{3.55}$$

where ψ is the poloidal flux function, $R = |\mathbf{R}|$ is the distance between the guiding centre and the toroidal axis of symmetry and B_φ is the magnetic field in the toroidal direction.
For some choices of these three invariants two orbits of the particle may exist, meaning that F will have different values for the same $(\mathcal{E}, \mu, P_\varphi)$. The index σ is necessary to distinguish between the two orbits.

3.3.2 Solution of the linearized drift-kinetic equation

Considering the introduction of a perturbation in the system and denoting perturbed quantities with the subscript "(1)" the distribution function can be written in the the form $f = F + f^{(1)}$. Then, the drift-kinetic equation can be linearized, taking the form:

$$\frac{df^{(1)}}{dt} + \dot{\mathbf{R}}^{(1)} \cdot \nabla F + \dot{v}_\parallel^{(1)} \frac{\partial F}{\partial v_\parallel} + \dot{y}^{(1)} \frac{\partial F}{\partial y} = 0 \tag{3.56}$$

For clarity, for a generic quantity A, $(d/dt)A^{(1)} = (dA/dt)^{(1)} - \dot{\mathbf{R}} \cdot \nabla A$ differs from the notation used above where $\dot{A}^{(1)} = (dA/dt)^{(1)}$.
Using the equilibrium function $F = F(\mathcal{E}, \mu, P_\varphi; \sigma)$ and the equations (3.53)-(3.55) Eq. (3.56) becomes:

$$\frac{df^{(1)}}{dt} + \left(\dot{\mathbf{R}} \cdot \nabla P_\varphi + \dot{v}_\parallel^{(1)} \frac{\partial P_\varphi}{\partial v_\parallel}\right) \frac{\partial F}{\partial P_\varphi} + \left(Ze\dot{\mathbf{R}}^{(1)} \cdot \nabla \phi + \right.$$
$$\left. + m v_\parallel \dot{v}_\parallel^{(1)} + y^{(1)}\right) \frac{\partial F}{\partial \mathcal{E}} + \left(\frac{\dot{y}^{(1)}}{B} - \frac{y}{B}\dot{\mathbf{R}}^{(1)} \cdot \nabla B\right) \frac{\partial F}{\partial \mu} = 0 \tag{3.57}$$

After straightforward calculations, the coefficients that in Eq.(3.57) multiply derivatives of F can be rewritten in terms of the perturbed Lagrangian $\mathcal{L}^{(1)}$:

$$\frac{df^{(1)}}{dt} + \left(\frac{\partial \mathcal{L}^{(1)}}{\partial \varphi} - \frac{d}{dt}P_{\varphi}^{(1)}\right)\frac{\partial F}{\partial P_{\varphi}} - \left(\frac{\partial \mathcal{L}^{(1)}}{\partial t} + Ze\frac{d\phi^{(1)}}{dt}\right)\frac{\partial F}{\partial \mathcal{E}} + \mu\left(\frac{d}{dt}\frac{B^{(1)}}{B}\right)\frac{\partial F}{\partial \mu} = 0 \tag{3.58}$$

Thus, it is possible to write:

$$f^{(1)} = P_{\varphi}^{(1)}\frac{\partial F}{\partial P_{\varphi}} + Ze\phi^{(1)}\frac{\partial F}{\partial \mathcal{E}} - \mu\frac{B^{(1)}}{B}\frac{\partial F}{\partial \mu} + h^{(1)} \tag{3.59}$$

where $h^{(1)}$ is the so-called "nonadiabatic part" of $f^{(1)}$, which satisfies:

$$\frac{dh^{(1)}}{dt} = \frac{\partial F}{\partial \mathcal{E}}\frac{\partial \mathcal{L}^{(1)}}{\partial t} - \frac{\partial F}{\partial P_{\varphi}}\frac{\partial \mathcal{L}^{(1)}}{\partial \varphi} \tag{3.60}$$

In Eq.(3.60) it is possible to use the perturbed Lagrangian to leading order in δ showed in Eq.(3.61), since higher order corrections will lead to $\mathcal{O}(\delta^2)$ terms.

$$\mathcal{L}^{(1)} = \frac{Ze}{c}\mathbf{A}^{(1)} \cdot \dot{\mathbf{R}} - Ze\phi^{(1)} - \mu B^{(1)} + \mathcal{O}(\delta) \tag{3.61}$$

where $\mathbf{A}^{(1)}$ is the perturbed vector potential.
In order to solve Eq.(3.60) for $h^{(1)}$, the coordinate system $\mathbf{R} = (\psi, \theta, \varphi)$ is introduced, where ψ labels the equilibrium magnetic surfaces, θ is a generalized poloidal angle and φ is the previously defined toroidal angle. The perturbed Lagrangian is assumed to have the form:

$$\mathcal{L}^{(1)}(\mathbf{R}, t) = \hat{\mathcal{L}}^{(1)}(\psi, \theta)exp(-i\omega t - in\varphi) \tag{3.62}$$

where n is the toroidal mode number. Thus Eq. (3.60) has the form:

$$\frac{dh^{(1)}}{dt} = -i(\omega - n\omega_*)\frac{\partial F}{\partial \mathcal{E}}\mathcal{L}^{(1)} \tag{3.63}$$

where ω_* is defined as $(\partial F/\partial P_{\varphi})/(\partial F/\partial \mathcal{E})$ and is a constant of the unperturbed motion of the particle.
The formal solution of Eq. (3.63) is

$$h^{(1)} = -i(\omega - n\omega_*)\frac{\partial F}{\partial \mathcal{E}}\int_{-\infty}^{t} \mathcal{L}^{(1)}(\tau)d\tau \tag{3.64}$$

where the lower limit of integration is set according to the causality prescription and $\mathcal{L}^{(1)}(\tau) = \hat{\mathcal{L}}^{(1)}(\psi(\tau), \theta(\tau))exp(-i\omega\tau - in\varphi(\tau))$. The dependence of (ψ, θ, φ) on τ is described by the guiding center equations:

$$\dot{\psi} = \dot{\mathbf{R}} \cdot \nabla\psi, \quad \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla\theta, \quad \dot{\varphi} = \dot{\mathbf{R}} \cdot \nabla\varphi \tag{3.65}$$

with $\dot{\mathbf{R}}$ given in Eq.(3.40).
It is possible to separate $\varphi(\tau)$ into its secular and oscillating parts:

$$\varphi(\tau) = \langle\dot{\varphi}\rangle\tau + \tilde{\varphi}(\tau) \tag{3.66}$$

the brackets $\langle \quad \rangle$ indicates the average on a complete orbit in the poloidal cross section that can be either mirror-trapped or passing for energetic particles. The bounce or transit time is thus defined as $\tau_{b/t} = \oint d\tau$ and the orbit averaging as:

$$\langle A \rangle = \frac{1}{\tau_{b/t}} \oint A d\tau \tag{3.67}$$

Therefore, the quantity $\tilde{\mathcal{L}}^{(1)} = \hat{\mathcal{L}}^{(1)} exp(-in\tilde{\varphi})$ is a periodic function of τ that can be expanded in Fourier series:

$$\tilde{\mathcal{L}}^{(1)}(\tau) = \sum_{-\infty}^{\infty}{}_p \Upsilon_p(\mathcal{E}, \mu, P_\varphi; \sigma) exp(-ip\omega_{b/t}\tau) \tag{3.68}$$

where $\omega_{b/t} = 2\pi/\omega_{b/t}$ and the Fourier coefficients are defined as

$$\Upsilon_p(\mathcal{E}, \mu, P_\varphi; \sigma) = \oint \frac{d\tau}{\tau_{b/t}} \tilde{\mathcal{L}}^{(1)} exp(ip\omega_{b/t}\tau) \tag{3.69}$$

Then, considering again Eq. (3.64):

$$h^{(1)} = -i(\omega - n\omega_*) \frac{\partial F}{\partial \mathcal{E}} \sum_{-\infty}^{\infty}{}_p \Upsilon_p \int_{-\infty}^{t} exp[-i(\omega + n\langle\dot{\varphi}\rangle + p\omega_{b/t})\tau] = \tag{3.70}$$

$$= (\omega - n\omega_*) \frac{\partial F}{\partial \mathcal{E}} \sum_{-\infty}^{\infty}{}_p \Upsilon_p \frac{exp[-i(\omega + n\langle\dot{\varphi}\rangle + p\omega_{b/t})\tau]}{\omega + n\langle\dot{\varphi}\rangle + p\omega_{b/t}} \tag{3.71}$$

Equation (3.71)introduces in the perturbed distribution function $f^{(1)}$ a resonant denominator leading to the mode-particle resonance condition

$$\omega + n\langle\dot{\varphi}\rangle + p\omega_{b/t} = 0, \quad p = 0, \pm 1, \pm 2, ... \tag{3.72}$$

The mode investigated in this work presents the toroidal wavenumber n equal to zero. Thus, the resonant condition of interest will reduce to:

$$\omega + p\omega_{b/t} = 0, \quad p = 0, \pm 1, \pm 2, ... \tag{3.73}$$

Therefore, the periodic motion of energetic particles in the poloidal cross-section (passing orbit or trapped orbit) is the relevant motion that can be resonant with n=0 axisymmetric modes.

Chapter 4

Resonant instability

Stabilization of the ideal vertical instability by means of feedback currents results into an oscillatory behaviour of the plasma in the vertical direction as described in Chapter 2. When energetic particles are present in the plasma, a mode-particle resonant interaction between these vertical oscillations and the periodic motions of the fast particles is possible, leading to an excitation of the axisymmetric n=0 modes. This could lead to instabilities in the same way resonances between macroscopic plasma modes and the motion of energetic particles cause so-called fishbones oscillations ([6],[7]) and of Toroidal Alfvén Eigenmodes (TAE) ([12]).
Experimental results from the JET tokamak ([8]) shows that axisymmetric n=0 modes can be excited in presence of energetic particles. It is indeed possible that the excitation mechanism is the resonant interaction between periodic motion of energetic particles in the poloidal plane and plasma oscillations resulting from the vertical instability stabilization.
Considering JET experimental values, among the different periodic motions of energetic particles in tokamaks, the resonance of interest most likely involves passing particles, i.e. particles whose projection of the orbit on the poloidal cross section is a complete revolution around the toroidal axis.
In the following sections the dispersion relation describing the behaviour of n=0 axisymmetric modes in presence of energetic particles is derived within the hybrid Kinetic-MHD framework described in Chapter 3.

4.1 Particle orbits: Passing particles

In a tokamak equilibrium the guiding center of a particle follows a trajectory described, to leading order, by:

$$\dot{\mathbf{R}} = v_{\parallel}\hat{e}_{\parallel} + \frac{mc}{qB}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\hat{e}_{\parallel} \times \frac{\nabla B}{B} = v_{\parallel}\hat{e}_{\parallel} + \mathbf{v}_D \tag{4.1}$$

where $v_{\parallel}$ and $v_{\perp}$ are the velocity of the particle in the parallel and perpendicular directions, m is the particle mass, q the particle charge ,c the speed of light and B is the modulus of the magnetic field. The coordinate system considered is showed in figure (4.1), where φ is the toroidal angle, θ is the poloidal angle, R_0 is the tokamak major radius and r is the distance from the toroidal magnetic axis meaning that $R = R_0 + rcos(\theta)$.

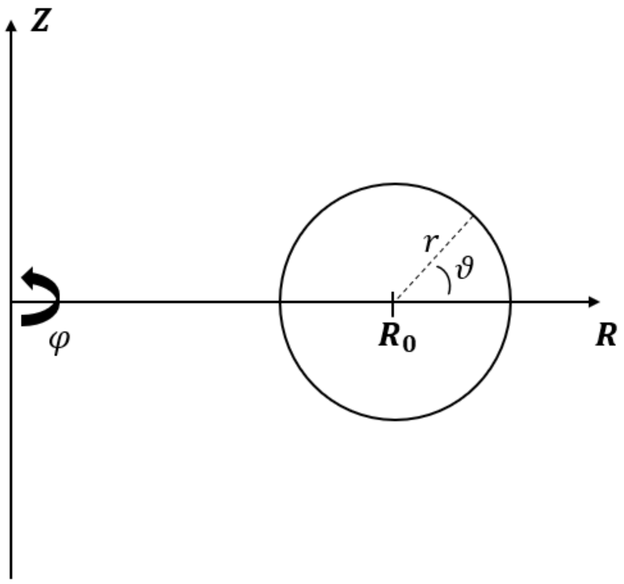

Figure 4.1: Schematic of the toroidal coordinate system.

The model used for **B** is:

$$\mathbf{B}(r,\theta) = \frac{1}{h}[B_{\varphi 0}(r)\hat{e}_{\varphi} + B_{\theta_0}(r)\hat{e}_{\theta}] \tag{4.2}$$

where $B_{\varphi 0}$ and B_{θ_0} are respectively toroidal and poloidal magnetic fields and h:

$$h(r,\theta) = \frac{R}{R_0} = 1 + \epsilon cos(\theta) \tag{4.3}$$

with ϵ a small parameter defined as $\epsilon = r/R_0$. The model (4.2) is an approximation valid up to terms of order ϵ and satisfies $\nabla \cdot \mathbf{B} = 0$ and $\nabla B/B \simeq (-B/R)\hat{e}_R$. Considering $B_{\varphi 0}$ = const. and that $B_{\theta_0}(r)/B_{\varphi 0} \sim \epsilon$, $\hat{e}_\parallel$ is mainly along the toroidal direction, meaning that $\mathbf{v}_D$ main contributions are in the Z direction.
It is also possible to approximate $B = (B_{\varphi 0}/h)(1 + \mathcal{O}(\epsilon^2))$. This means that a particle, moving along a field line, feels a varying magnetic field, more intense in the inside of the torus and weaker on the outside. This magnetic field modulation is of order ϵ:
$B_{max} - B_{min} = B(\theta = \pi) - B(\theta = 0) \simeq 2\epsilon B_{\varphi 0} + \mathcal{O}(\epsilon^2)$.
Introducing the parameter

$$\Lambda = \frac{\mu B_{\varphi 0}}{\mathcal{E}} = \frac{v_\perp^2}{v^2} h(r, \theta) \tag{4.4}$$

where $\mu = mv_\perp^2/2B$ is the magnetic moment and $\mathcal{E} = mv^2/2$ is the kinetic energy. Therefore Λ is a constant of the particle motion and varies in the interval

$$0 \leq \Lambda \leq 1 + \epsilon \tag{4.5}$$

Then, considering the expression for $v_\parallel$

$$v_\parallel = \pm v(1 - \frac{v_\perp^2}{v^2})^{1/2} = \pm v(1 - \frac{\Lambda}{h})^{1/2} \tag{4.6}$$

two classes of particles can be distinguished:

- Circulating (or passing) particles: when $0 \leq \Lambda \leq 1 - \epsilon$ the projection of the guiding centre motion in a poloidal cross section is a complete orbit around the magnetic axis.

- Trapped particles: in the case of $1 - \epsilon \leq \Lambda \leq 1 + \epsilon$ $v_\parallel$ changes sign during the orbit meaning that the projection of the motion in the poloidal plane is a so-called "banana orbit". The particle will reach a maximum value of θ, called bounce angle θ_b defined by $v_\parallel(\theta_b) = 0$.

The different orbits in the poloidal plane are schematized in figure (4.2).

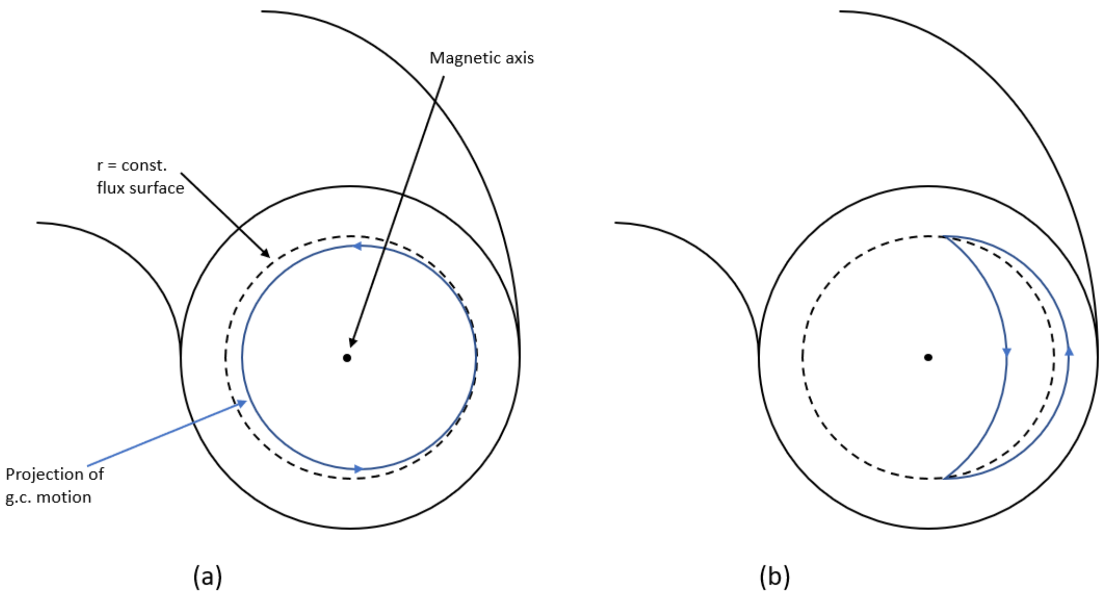

Figure 4.2: Scheme of the projection on the poloidal cross section of particle orbits in tokamaks: passing particles (a) and trapped particles (b)

We will now focus on the case of passing particles, which, due to the high frequency of the orbit in the poloidal plane, is most likely to be involved in the resonant interaction of interest.

4.1.1 Transit frequency of passing particles

The time a passing particle takes for completing an orbit in the poloidal plane is called transit time and is defined as:

$$\tau_t = \oint d\tau \approx \oint \frac{dl}{v_\parallel}(1 + \mathcal{O}(v_D/v_\parallel)) \tag{4.7}$$

where the integral is evaluated over a closed orbit.
It is possible to rewrite dl according to

$$\frac{rd\theta}{B_\theta} = \frac{Rd\varphi}{B_\varphi} = \frac{dl}{B} \rightarrow dl = \frac{rB}{B_\theta}d\theta \approx R_0 q(r) d\theta \tag{4.8}$$

where $q(r) = rB_\varphi/R_0B_\theta$ is the so-called safety factor.
Then, $q(r)$ becomes nearly constant over the orbit when the radial excursion

of a guiding center orbit from a given flux surface $r = r_0$ is assumed to be small: $r(t) = r_0 + \delta r$ with $|\delta r| \ll r_0$. Thus:

$$\tau_t \approx R_0 q \oint \frac{d\theta}{v_\parallel} = \frac{R_0 q}{v} \int_0^{2\pi} \frac{d\theta}{(1-\Lambda/h)^{1/2}} \tag{4.9}$$

Introducing now the parameter κ:

$$\begin{aligned} \kappa^2 &= \frac{1}{2} + \frac{1}{2\epsilon}(1-\Lambda) \\ (1-\Lambda) &= 2\epsilon\kappa^2 - \epsilon \end{aligned} \tag{4.10}$$

it is possible to rewrite the denominator of the integral of Eq. (4.9):

$$\begin{aligned} (1-\frac{\Lambda}{h}) &= \frac{1}{h}(1+\epsilon cos(\theta)-\Lambda) = (1-\Lambda+\epsilon cos(\theta)) + \mathcal{O}(\epsilon^2) \\ &= (2\epsilon\kappa^2 - \epsilon + \epsilon cos(\theta)) = 2\epsilon\kappa^2\left(1-\frac{1}{\kappa^2}\frac{1-cos(\theta)}{2}\right) \\ &= 2\epsilon\kappa^2\left(1-\frac{1}{\kappa^2}sin^2(\theta/2)\right) \end{aligned} \tag{4.11}$$

Substituting Eq. (4.11) back into Eq. (4.9) the transit time can be written as:

$$\tau_t = \frac{R_0 q}{v}\int_0^{2\pi}\frac{d\theta}{(1-\Lambda/h)^{1/2}} = \frac{R_0 q}{v}\frac{1}{(2\epsilon)^{1/2}\kappa}\int_0^{2\pi}\frac{d\theta}{\left(1-\frac{1}{\kappa^2}sin^2(\theta/2)\right)^{1/2}} \tag{4.12}$$

Then, changing integration variable $\theta = 2\phi$, $d\theta = 2d\phi$:

$$\tau_t = \frac{R_0 q}{v}\frac{1}{(2\epsilon)^{1/2}\kappa}2\int_0^{\pi}\frac{d\phi}{\left(1-\frac{1}{\kappa^2}sin^2(\phi)\right)} = \frac{R_0 q}{v}\frac{1}{(2\epsilon)^{1/2}\kappa}4\mathcal{K}(1/\kappa^2) \tag{4.13}$$

where $\mathcal{K}(x)$ is the "complete elliptic integral of the first kind".
Therefore, it is possible to define the transit frequency as:

$$\omega_t = \frac{2\pi}{\tau_t} = \frac{v\kappa}{2R_0 q}(2\epsilon)^{1/2}\frac{\pi}{\mathcal{K}(1/\kappa^2)} \tag{4.14}$$

4.2 Dispersion relation

In order to determine the stability of axisymmetric n=0 modes when energetic particles are present, one should consider the linearized momentum equation in terms of the displacement vector:

$$\rho_m \frac{\partial^2 \overline{\xi}}{\partial t^2} = \mathbf{F}(\overline{\xi}) = \mathbf{F}_{core} + \mathbf{F}_{hot} \tag{4.15}$$

where we have distinguished the different contributions arising from the thermal plasma and the energetic particles. The displacement vector of interest in the case of axisymmetric n=0 modes can be written as:

$$\overline{\xi} = \xi_0 e^{-i\omega t}\hat{e}_x \tag{4.16}$$

where $\hat{e}_x$ represent the vertical direction and ξ_0 is a constant.

In particular, the term $\mathbf{F}_{hot}$ contains the nonadiabatic term in the perturbed pressure tensor of energetic particles related to the mode-particle resonance described in Eq.(3.71). Due to the presence of this nonadiabatic term, $\mathbf{F}$ loses its self-adjoint property, meaning that it is impossible to find a necessary and sufficient stability criterion based only on the sign of the real part of δW, as in the ideal MHD case, and that a normal mode analysis is required. However, the stability of the axisymmetric n=0 modes in presence of energetic particles can be studied deriving the dispersion relation of the mode within a perturbative approach.

Multiplying Eq. (4.15) by $\overline{\xi}^*/2$ and integrating over the volume d^3x we define the potential energy integrals:

$$\delta I = -\delta W = -\delta W_{MHD} - \delta W_{hot} \tag{4.17}$$

$$\delta I = \frac{1}{2}\int d^3x \rho_m \overline{\xi}^* \cdot \frac{\partial^2 \overline{\xi}}{\partial t^2} = -\frac{1}{2}\rho_m \omega^2 |\overline{\xi}|^2 V \tag{4.18}$$

where $V = 2\pi^2 abR_0$ is the plasma volume, $\rho_m = const.$ has been assumed and

$$\delta W_{MHD} = -\frac{1}{2}\int d^3x \overline{\xi}^* \cdot \mathbf{F}_{core}(\overline{\xi}) \tag{4.19}$$

$$\delta W_{hot} = \frac{1}{2}\int d^3x \overline{\xi}^* \cdot (\nabla \cdot \overline{\mathbf{P}}_h^{(1)}) \tag{4.20}$$

where $\overline{\mathbf{P}}_h^{(1)}$ is the perturbed energetic particles pressure tensor.

The perturbative approach for the derivation of the dispersion relation is based on the use of the displacement vector defined in Eq. (4.16), which is obtained by ideal-MHD normal mode analysis. Therefore, the dispersion relation, obtained form Eq.(4.17), must coincide at its lowest order with the ideal-MHD dispersion relation obtained in absence of energetic particles. In other words, this means that zeroth-order terms of Eq.(4.15) represent a self-adjoint form related to a potential energy integral, whose minimization leads to the derivation of the ideal-MHD displacement vector $\overline{\xi}$ of Eq.(4.16). This can be obtained only considering $|\delta W_{hot}| \ll |\delta W_{MHD}|$.

The model used for δW_{MHD} is:

$$\delta W_{MHD} = \frac{1}{2}(V\rho_m)\xi_0^2 \frac{\epsilon_{el} v_A^2}{a^2}(D-1) \tag{4.21}$$

that has been discussed in Chapter 2. Thus, the dispersion relation in absence of energetic particles becomes:

$$\omega^2 = \frac{\epsilon_{el} v_A^2}{a^2}(D-1) = \omega_{MHD}^2 \tag{4.22}$$

The nonadiabatic term in the perturbed pressure tensor of energetic particles of δW_{hot} includes a mode-particle resonance contribution, leading to a frequency dependent imaginary part of δW. Due to the condition $|\delta W_{hot}| \ll |\delta W_{MHD}|$, only this imaginary part plays an important role in the stability of the mode. Thus, we will focus solely on the derivation of the imaginary part of δW_{hot}. Introducing a dimensionless real parameter $\lambda_{hot} \ll 1$, the imaginary part of δW_{hot} can be written as:

$$i\mathcal{I}m(\delta W_{hot}) = -\frac{1}{2}(V\rho_m)\xi_0^2\omega_{MHD}^2 i\lambda_{hot} \tag{4.23}$$

The dispersion relation (4.22) modified by the presence energetic particles becomes:

$$\omega^2 = \omega_{MHD}^2 + i\omega_{MHD}^2\lambda_{hot} \tag{4.24}$$

Setting

$$\omega = \omega_R + i\gamma \tag{4.25}$$

where $\gamma/\omega_R \sim \lambda_{hot} \ll 1$ and substituting (4.25) into (4.24) neglecting terms $\mathcal{O}(\lambda_{hot}^2)$, we obtain:

$$\omega_R^2 + 2i\omega_R\gamma = \omega_{MHD}^2 + i\omega_{MHD}^2\lambda_{hot} \tag{4.26}$$

$$\omega_R = \omega_{MHD} \tag{4.27}$$

$$\gamma = \frac{1}{2}\omega_{MHD}\lambda_{hot} \tag{4.28}$$

Therefore, the stability of the axisymmetric n=0 mode in presence of energetic particles depends on the sign of λ_{hot}, since $\xi \sim \xi_0 exp(-i\omega_{MHD}t)exp(\gamma t)$. Thus

$$\lambda_{hot} < 0 \tag{4.29}$$

is our stability condition.
In the following, analytical study of the hybrid kinetic-MHD model in the case of axisymmetric n=0 modes is carried out in order to derive the expression of λ_{hot}.

4.2.1 Derivation of δW_{hot}

The reduction of δW_{hot} follows a standard procedure that is outlined, for instance, in Ref.[13].

Since the general case of anisotropic equilibrium pressure tensor introduces more real terms to δW_{hot}, but does not affect its imaginary part, we will consider the simpler case of isotropic fast particle equilibrium pressure $\overline{\mathbf{P}}_{h,eq} = p_{h,eq}\overline{\mathbf{I}}$. Thus, the perturbed fast particle pressure tensor becomes anisotropic and can be written as:

$$\overline{\mathbf{P}}_h^{(1)} = p_\perp^{(1)}\overline{\mathbf{I}} + (p_\parallel^{(1)} - p_\perp^{(1)})\hat{e}_\parallel\hat{e}_\parallel \tag{4.30}$$

with

$$\begin{pmatrix} p_\perp^{(1)} \\ p_\parallel^{(1)} \end{pmatrix} = \int d^3v \begin{pmatrix} \mu B \\ mv_\parallel^2 \end{pmatrix} f^{(1)} \tag{4.31}$$

where $f^{(1)}$ is the energetic particles perturbed distribution function.

After straightforward algebra and considering perturbation that vanish at the plasma edge, δW_{hot} can be written as:

$$\delta W_{hot} = -\frac{1}{2}\int d^3xd^3v\mathcal{L}^{(1)*}f^{(1)} \tag{4.32}$$

Eq.(3.59) shows that $f^{(1)}$ presents an adiabatic and a nonadiabatic part, thus also δW_{hot} can be split into two terms:

$$\delta W_{hot} = \delta W_1 + \delta W_2(\omega) \tag{4.33}$$

In particular we are interested in the frequency dependent nonadiabatic part $\delta W_2(\omega)$, which contains the mode-particle resonance contribution. Considering Eq.(3.64) the nonadiabatic term of δW_{hot} can be written as:

$$\delta W_2(\omega) = \frac{1}{2}i\int d^3xd^3v(\omega - n\omega_*)\frac{\partial F}{\partial \mathcal{E}}\mathcal{L}^{(1)*}\int_{-\infty}^{t}\mathcal{L}^{(1)}(\tau)d\tau \tag{4.34}$$

Then, remembering the the expansion of $\mathcal{L}^{(1)}$ in harmonics of the orbit periodicity of Eq.(3.68):

$$\tilde{\mathcal{L}}^{(1)}(\tau) = \sum_{-\infty}^{\infty}{}_p\Upsilon_p(\mathcal{E},\mu,P_\varphi;\sigma)exp(-ip\omega_t\tau) \tag{4.35}$$

and carrying out the time integration, we obtain:

$$\delta W_2 = -\frac{1}{2}\int d^3xd^3v(\omega - n\omega_*)\frac{\partial F}{\partial \mathcal{E}}\sum_{-\infty}^{+\infty}{}_l\Upsilon_l^*e^{il\omega_t\tau}\sum_{-\infty}^{+\infty}{}_p\frac{\Upsilon_pe^{-ip\omega_t\tau}}{\omega + p\omega_t + n\langle\varphi\rangle} \tag{4.36}$$

In order to proceed with the calculation, is convenient to change the variables of the integration using the transformation $(\mathbf{x},\mathbf{v}) \rightarrow (P_\varphi, \varphi, \mathcal{E}, \tau, \mu, \alpha)$. The Jacobian of this transformation is a constant:

$$d^3x d^3v = (\frac{c}{Zem^2}) \sum_\sigma dP_\varphi d\varphi d\mathcal{E} d\tau d\mu d\alpha \tag{4.37}$$

After this change of variables, δW_2 now reads:

$$\delta W_2 = -\frac{1}{2}(\frac{c}{Zem^2}) \sum_\sigma \int dP_\varphi d\varphi d\mathcal{E} d\tau d\mu d\alpha (\omega - n\omega_*) \frac{\partial F}{\partial \mathcal{E}} \sum_{-\infty}^{+\infty}{}_l \Upsilon_l^* e^{il\omega_t\tau} \sum_{-\infty}^{+\infty}{}_p \frac{\Upsilon_p e^{-ip\omega_t\tau}}{\omega + p\omega_t + n\langle\varphi\rangle} \tag{4.38}$$

Carrying out the integration in τ, α and φ:

$$\int d\alpha = 2\pi \tag{4.39}$$

$$\int d\varphi = 2\pi \tag{4.40}$$

$$\int d\tau e^{i(l-p)\omega_t\tau} = \frac{2\pi}{\omega_b}\delta(l-p) \tag{4.41}$$

it is possible to write δW_2 as:

$$\delta W_2 = -\frac{2\pi^2 c}{Zem^2} \sum_\sigma \int dP_\varphi d\mathcal{E} d\mu \tau_t (\omega - n\omega_*) \frac{\partial F}{\partial \mathcal{E}} \sum_{-\infty}^{+\infty}{}_p \frac{|\Upsilon_p|^2}{\omega + p\omega_t + n\langle\varphi\rangle} \tag{4.42}$$

We are interested in axisymmetric modes characterized by toroidal wave number n=0. The harmonics that will give the largest contribution to the imaginary part of δW_2 are labeled with $p = \pm 1$ respectively for counterclockwise and clockwise particle orbits labeled by the index σ. Therefore, considering the contributions from the two orbits, and that $|\Upsilon_1|^2 = |\Upsilon_{-1}|^2$, we obtain:

$$\delta W_2 = -\frac{4\pi^2 c}{Zem^2} \int dP_\varphi d\mathcal{E} d\mu \tau_t \omega \frac{\partial F}{\partial \mathcal{E}} \frac{|\Upsilon_1|^2}{\omega - \omega_t} \tag{4.43}$$

Considering then the result of Eq.(4.14), it is possible to rewrite ω_t as

$$\omega_t = v f(r,\Lambda); \quad f(r,\Lambda) = \frac{\kappa}{2R_0 q}(2\epsilon)^{1/2} \frac{\pi}{\mathcal{K}(1/\kappa^2)} \tag{4.44}$$

meaning that δW_2 takes the form:

$$\delta W_2 = \frac{4\pi^2 c}{Zem^2} \int dP_\varphi d\mathcal{E} d\mu \tau_t \omega \frac{\partial F}{\partial \mathcal{E}} \frac{|\Upsilon_1|^2}{v - \omega/f(r,\Lambda)} \frac{1}{f(r,\Lambda)} \tag{4.45}$$

The expression of δW_2 can be further simplified by means of some change of variables. Firstly in the limit of $r \sim const.$ along the orbit P_φ can be written as:

$$P_\varphi \simeq \frac{Ze}{c}\Psi \tag{4.46}$$

where Ψ is the poloidal magnetic flux function. Since r $\sim$ constant also $q = \frac{rB_T}{R_0 B_p} \sim const. \rightarrow B_p = \frac{rB_T}{R_0 q}$, then:

$$\frac{d\Psi}{dr} = B_p \rightarrow \Psi = \frac{r^2 B_T}{2R_0 q} R_0 \simeq \frac{r^2 B_0}{2R_0 q} R_0 \tag{4.47}$$

Substituting 4.47 into 4.46 we obtain:

$$P_\varphi \simeq \frac{Ze}{c}\frac{B_0}{2q} r^2 \tag{4.48}$$

$$\rightarrow dP_\varphi = \frac{Ze}{c}\frac{B_0}{q} r dr \tag{4.49}$$

Then, since $\omega_t = vf(r, \Lambda)$ is convenient to use the relation between μ and Λ in order to use the latter as integration variable.

$$\Lambda = \frac{\mu B_0}{\mathcal{E}} \rightarrow \mu = \frac{\Lambda \mathcal{E}}{B_0} \tag{4.50}$$

$$d\mu = \frac{\mathcal{E}}{B_0} d\Lambda \tag{4.51}$$

Lastly we will consider the integration over v instead of $\mathcal{E}$ in order to explicit the pole in the expression 4.45.

$$\mathcal{E} = \frac{1}{2} m v^2 \quad \rightarrow \quad d\mathcal{E} = m v dv \tag{4.52}$$

Considering then this change of variables and substituting 4.46, 4.51 and 4.52 into 4.45 we obtain:

$$\delta W_2 = +\frac{2\pi^2}{q} \int dr d\Lambda dv r v^3 \tau_t \omega \frac{\partial F(r, \Lambda, v)}{\partial \mathcal{E}} \frac{|\Upsilon_1|^2}{v - \omega / f(r, \Lambda)} \frac{1}{f(r, \Lambda)} \tag{4.53}$$

The integration around the pole can be carried out following the "Landau prescription" of Eq.(3.26), accordingly to the relation:

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{G(u) du}{u - \omega / |k| - i\epsilon} = \fint_{-\infty}^{+\infty} \frac{G(u) du}{u - \omega / |k|} + \pi i \int_{-\infty}^{+\infty} G(u) \delta(u - \omega / |k|) \tag{4.54}$$

where $⨍$ is the Cauchy principal value. Since the real contribution of δW_{hot} are assumed to be much smaller than δW_{MHD} we will neglect the first integral of the expression 4.54 focusing our analysis on the imaginary part of δW_{hot}, which arises from the mode-particle resonant contributions. Thus we can write:

$$\delta W_2 = +(i\pi)\frac{2\pi^2}{q}\int drd\Lambda dvrv^3\tau_t\omega\frac{\partial F(r,\Lambda,v)}{\partial \mathcal{E}}\left|\Upsilon_1\right|^2\delta(v-\omega/f(r,\Lambda))\frac{1}{f(r,\Lambda)} \tag{4.55}$$

Now remembering that $\tau_t = 2\pi/\omega_t = 2\pi/vf(r,\Lambda)$ and carrying out the integration in v

$$\delta W_2 = +i\pi\frac{2\pi^2}{q}2\pi\int drd\Lambda r\left(\frac{\omega}{f(r,\Lambda)}\right)^3\frac{\partial F(r,\Lambda,v)}{\partial \mathcal{E}}\bigg|_{v=\omega/f(r,\Lambda)}\left|\Upsilon_1\right|^2\bigg|_{v=\omega/f(r,\Lambda)}\frac{1}{f(r,\Lambda)} \tag{4.56}$$

4.2.2 Fourier coefficient

The Fourier coefficient Υ_1 defined as

$$\Upsilon_1(\mathcal{E},\mu,P_\varphi) = \oint\frac{d\tau}{\tau_t}\tilde{\mathcal{L}}^{(1)}exp(i\omega_t\tau) \tag{4.57}$$

must be derived in order to proceed with the calculation of δW_{hot}.
We will focus our attention on the case of ideal-MHD perturbations that satisfy the constraint

$$\mathbf{B}^{(1)} = \nabla\times(\bar{\xi}_\perp\times\mathbf{B}) \tag{4.58}$$

that excludes the possibility of modes that give rise to magnetic reconnection. Considering a gauge where the perturbed vector potential perpendicular to the equilibrium magnetic field, as suggested by Eq.(4.58),

$$\mathbf{A}^{(1)} = \bar{\xi}_\perp\times\mathbf{B} \tag{4.59}$$

the perturbed Lagrangian correct up to order δ of Eq.(3.61) can be rewritten as:

$$\mathcal{L}^{(1)} = -mv_\parallel^2\bar{\xi}_\perp\cdot\boldsymbol{\kappa} - (B^{(1)}+\bar{\xi}_\perp\cdot\nabla B)\mu - (\phi^{(1)}+\bar{\xi}_\perp\cdot\nabla\phi)Ze + \mathcal{O}(\delta) \tag{4.60}$$

Assuming now that $\phi = 0$ at equilibrium, then from Eq.(4.59) and $E_\parallel^{(1)} = 0$ also $\phi^{(1)} = 0$ and $\mathbf{E}^{(1)} = (i\omega/c)\bar{\xi}_\perp\times\mathbf{B}$. Eliminating $B_\parallel^{(1)}$ through Eq.(4.58), the perturbed Lagrangian reads:

$$\mathcal{L}^{(1)} = -(mv_\parallel^2-\mu B)\bar{\xi}\cdot\boldsymbol{\kappa} + \mu B\nabla\cdot\bar{\xi}_\perp \tag{4.61}$$

In addiction, ideal-MHD perturbations satisfy the relation $\nabla \cdot \overline{\xi}_\perp + 2\overline{\xi}_\perp \cdot \boldsymbol{\kappa} = \mathcal{O}(\epsilon \xi_\perp / R)$ minimizing the so-called magnetic compression term in δW_{MHD}. Thus, we obtain:

$$\mathcal{L}^{(1)} \simeq -(mv_\parallel^2 + \mu B)\overline{\xi} \cdot \boldsymbol{\kappa} \tag{4.62}$$

Considering the large aspect ratio tokamak limit the curvature vector can be written as:

$$\kappa = -\frac{1}{R}\hat{e}_R - \frac{B_\theta^2}{rB_\varphi^2}\hat{e}_r \tag{4.63}$$

where $R = R_0 + rcos(\theta)$ and in cartesian coordinates ($\hat{e}_x$ being the vertical direction) $\hat{e}_R = \hat{e}_y$ and $\hat{e}_r = cos(\theta)\hat{e}_y + sin(\theta)\hat{e}_x$.
Remembering the definition of Λ and the expression for $v_\parallel$ from previous sections

$$\Lambda = \frac{\mu B_0}{\mathcal{E}} \rightarrow \mu B = \mathcal{E}\frac{\Lambda}{h} \tag{4.64}$$

$$v_\parallel^2 = v^2(1 - \frac{v_\perp^2}{v^2}) = v^2(1 - \frac{\mu B}{\mathcal{E}}) = \frac{2\mathcal{E}}{m}(1 - \frac{\Lambda}{h}) \tag{4.65}$$

the perturbed Lagrangian takes the form:

$$\mathcal{L}^{(1)} = -\mathcal{E}\left(2 - \frac{\Lambda}{h}\right)\overline{\xi} \cdot \boldsymbol{\kappa} \tag{4.66}$$

Considering now the displacement vector defined in Eq.(4.16) we obtain:

$$\mathcal{L}^{(1)} = \mathcal{E}\left(2 - \frac{\Lambda}{h}\right)\frac{B_\theta^2}{rB_\varphi^2}\xi_0 sin(\theta) \tag{4.67}$$

$$\approx \epsilon^2 \mathcal{E}\left(2 - \Lambda\right)\frac{\xi_0}{r}sin(\theta) \tag{4.68}$$

where in the last step $B_\theta / B_\varphi \sim \epsilon^2$ was considered and correction of higher order introduced by h were neglected.
Therefore, the Fourier coefficient Υ_1 can be written as:

$$\Upsilon_1 = \oint \frac{d\tau}{\tau_t}\mathcal{L}^{(1)} * e^{i\omega_t \tau} \tag{4.69}$$

$$= \epsilon^2 \mathcal{E}(2 - \Lambda)\frac{\xi_0}{r}\oint \frac{d\tau}{\tau_t} sin(\theta) e^{i\omega_t \tau} \tag{4.70}$$

where the last integral represents the average on the particle orbit of $sin(\theta)e^{i\omega_b \tau}$. Thus, it is possible to consider only $\langle isin(\theta)sin(\tau\omega_t)\rangle$ (where $\langle \quad \rangle$ represents the orbit averaging), since the term $sin(\theta)cos(\tau\omega_t)$ vanishes when averaged over the orbit. Remembering that τ_t is described in Eq.(4.13), we can consider $\tau \sim \tau_t f(\theta)$ where $f(\theta)$ is an almost linear function of θ. In this case $\tau\omega_t$ reduces to $\alpha f(\theta)$ where α is a numerical factor and the orbit averaging reads:

$$i\int_0^{2\pi} sin(\theta) * sin(\alpha f(\theta))d\theta = ia_1 \tag{4.71}$$

which can be solved numerically in order to determine the numerical coefficient a_1.
In the particular case in which $v_{\parallel} = v$ and $v_{\perp} = 0 \rightarrow \Lambda = 0$ we have that:

$$\tau_t = \oint d\tau = R_o q\int_0^{2\pi} \frac{d\theta}{v} = 2\pi\frac{R_0 q}{v} \tag{4.72}$$

$$\tau = R_0 q\int \frac{d\theta}{v} = \frac{R_0 q}{v}\theta \tag{4.73}$$

$$\rightarrow \omega_t\tau = \frac{2\pi}{\tau_t}\tau = \theta \tag{4.74}$$

Meaning that in this particular case the integral can be easily solved.

$$i\int_0^{2\pi} sin(\theta) * sin(\theta)d\theta = i\pi \tag{4.75}$$

Thus, the Fourier coefficient Υ_1 can be written as: The coefficient $\Upsilon^{(1)}$ then can be written as:

$$\Upsilon^{(1)} = \epsilon^2\mathcal{E}(2-\Lambda)\frac{\xi_0}{r}(ia_1) \tag{4.76}$$

while, in the case $\Lambda = 0$, it can be further simplified into: $\Upsilon^{(1)} = \epsilon^2\mathcal{E}(2\pi i)\xi_0/r$
Considering $\epsilon = r/R_0$, it is possible to write:

$$|\Upsilon_1|^2\Big|_{v=\omega/f(r,\Lambda)} = \frac{r^2\xi_0^2}{4R_0^2}m^2(2-\Lambda)^2\left(\frac{\omega}{f(r,\Lambda)}\right)^4 a_1^2 \tag{4.77}$$

Substituting Eq.(4.77) into the Eq.(4.56) for δW_2, we obtain:

$$\delta W_2 = +i\pi\frac{4\pi^3}{q}\frac{\xi_0^2 m^2}{4R_0^4}\int drd\Lambda r^3(2-\Lambda)^2 a_1^2\left(\frac{\omega}{f(r,\Lambda)}\right)^7\frac{1}{f(r,\Lambda)}\frac{\partial F(r,\Lambda,v)}{\partial\mathcal{E}}\Big|_{v=\omega/f(r,\Lambda)} \tag{4.78}$$

It is clear from Eq.(4.78) that the sign δW_2, which is strictly related with the sign of λ_{hot}, will depend on the sign of the derivative of the equilibrium distribution function with respect to the energy.

4.2.3 Equilibrium distribution function

In order to proceed with the analytical derivation of λ_{hot} defined in Eq.(4.23), the following equilibrium distribution function is assumed:

$$F(r, \Lambda, v) = \mathcal{C}H(r_h - r)\delta(\Lambda)exp\left[-\left(\frac{v - v_0}{\alpha v_0}\right)^2\right] \tag{4.79}$$

where $\mathcal{C}$ is a normalization constant and $H(r_h - r)$ is a step function describing a constant uniform distribution of energetic particles in r up to a distance r_h from the magnetic axis. With $\delta(\Lambda)$ we mimic the case of an external heating mechanism which accelerates particles mainly in the direction of the field lines, resulting in energetic particles with $v_\parallel \gg v_\perp \rightarrow v \simeq v_\parallel$. Lastly we have assumed a maxwellian distribution in velocity centered in v_0 and width determined by the parameter $\alpha < 1$.
More general, and thus more realistic, equilibrium distribution functions make the analytical procedure too complex and require numerical studies. The treatment of such cases is beyond the scope of the present work, and will be the subject of more detailed studies in the immediate near future.
In order to determine the normalization constant $\mathcal{C}$ we consider the number density of energetic particles defined as:

$$\int d^3vF = n_h \tag{4.80}$$

Changing integration variable and exploiting the equilibrium distribution function defined in Eq.(4.79), we obtain:

$$\pi\mathcal{C}\int dvv^2exp\left[-\left(\frac{v - v_0}{\alpha v_0}\right)^2\right] = n_h \tag{4.81}$$

$$\rightarrow \mathcal{C} = \frac{n_h\pi^{-3/2}}{v_0^3\alpha(1 + \alpha^2/2)} \tag{4.82}$$

Considering then the derivative $\partial F/\partial\mathcal{E}$ that appears in Eq.(4.78), straightforward calculations lead to:

$$\frac{\partial F}{\partial\mathcal{E}} = -\mathcal{C}H(r_h - r)\delta(\Lambda)\frac{1}{\frac{1}{2}mv_0^2\alpha^2}\left(\frac{v - v_0}{v}\right)exp\left[-\left(\frac{v - v_0}{\alpha v_0}\right)^2\right] \tag{4.83}$$

Therefore, substituting Eq.(4.83) into Eq.(4.78) and defining $v^* = \omega/f(r, \Lambda)$

the resonant velocity, δW_2 can be written as:

$$\delta W_2 = - i\pi \frac{4\pi^3}{q} \frac{\xi_0^2 m^2}{4R_0^4} \int drd\Lambda r^3 (2-\Lambda)^2 a_1^2 \left(\frac{\omega}{f(r,\Lambda)} \right)^7 \frac{1}{f(r,\Lambda)}$$
$$\mathcal{C} H(r_h - r)\delta(\Lambda) \frac{1}{\frac{1}{2} m v_0^2 \alpha^2} \left\{ \left(\frac{v - v_0}{v} \right) exp \left[- \left(\frac{v - v_0}{\alpha v_0} \right)^2 \right] \right\} \Big|_{v=\omega/f(r,\Lambda)} \tag{4.84}$$

Then, carrying out the integration in Λ and r, substituting the definition of $\mathcal{C}$ of Eq.(4.82) and remembering that in the $\Lambda = 0$ case $a_1 = \pi$, we obtain:

$$\delta W_2 = - i\pi \frac{\pi^3}{q} \frac{\xi_0^2 m^2}{R_0^4} r_h^4 \pi^2 \left(\frac{\omega}{f(r,\Lambda)} \right)^7 \frac{1}{f(r,\Lambda)} \frac{n_h \pi^{-3/2}}{v_0^3 \alpha (1 + \alpha^2/2)}$$
$$\frac{1}{\frac{1}{2} m v_0^2 \alpha^2} \left\{ \left(\frac{v - v_0}{v} \right) exp \left[- \left(\frac{v - v_0}{\alpha v_0} \right)^2 \right] \right\} \Big|_{v=\omega/f(r,\Lambda)} \tag{4.85}$$

Now, considering that $\mathcal{I}m(\delta W_2) = \mathcal{I}m(\delta W_{hot})$, we want to rewrite δW_2 as shown in Eq.(4.23) in order to obtain the expression of λ_{hot}. After straightforward algebra, is easy to show that λ_{hot} can be written as $\lambda_{hot}\,(n_h/n_c, r_h/a, \alpha; v^*/v_0)$, and takes the form:

$$\lambda_{hot} = \sigma \frac{n_h}{n_c} \frac{r_h^4}{a^4} \frac{a^2 q^2}{R_0^2} \Lambda_h(\alpha, \hat{v}^*) \tag{4.86}$$

$$\Lambda_h(\alpha, \hat{v}^*) = \frac{1}{(\alpha^2/2 + 1)\alpha^3} \left(\frac{v^*}{v_0} \right)^5 (1 - \frac{v_0}{v^*}) exp \left[- \frac{1}{\alpha^2} \left(\frac{v^*}{v_0} - 1 \right)^2 \right] \tag{4.87}$$

where $a \simeq b$ is the minor radius of the torus, $v^* = \omega/f(r,\Lambda)$ is the resonant velocity, n_c is the number density of the core plasma and $\sigma = 2\pi^{7/2}$ is a numerical factor.

4.3 Results and Discussion

It is clear from Eq.(4.86) and Eq.(4.87) that the sign of λ_{hot}, and thus the mode stability, depends on the ratio v^*/v_0. In particular the unstable situation characterized by $\lambda_{hot} > 0$ is obtained when $v^*/v_0 > 1$. Figure (4.3) shows the behaviour of λ_{hot} as a function of v^*/v_0 for different values of α and with other parameters fixed at typical values for a tokamak experiment. In the instability region where $v^*/v_0 > 1$, λ_{hot} exhibit a maximum. This maximum called λ_{hot}^{max} represents the worst-case instability, in fact, considering the relation 4.28, is clear that the maximum lambda is related with the fastest growth rate.

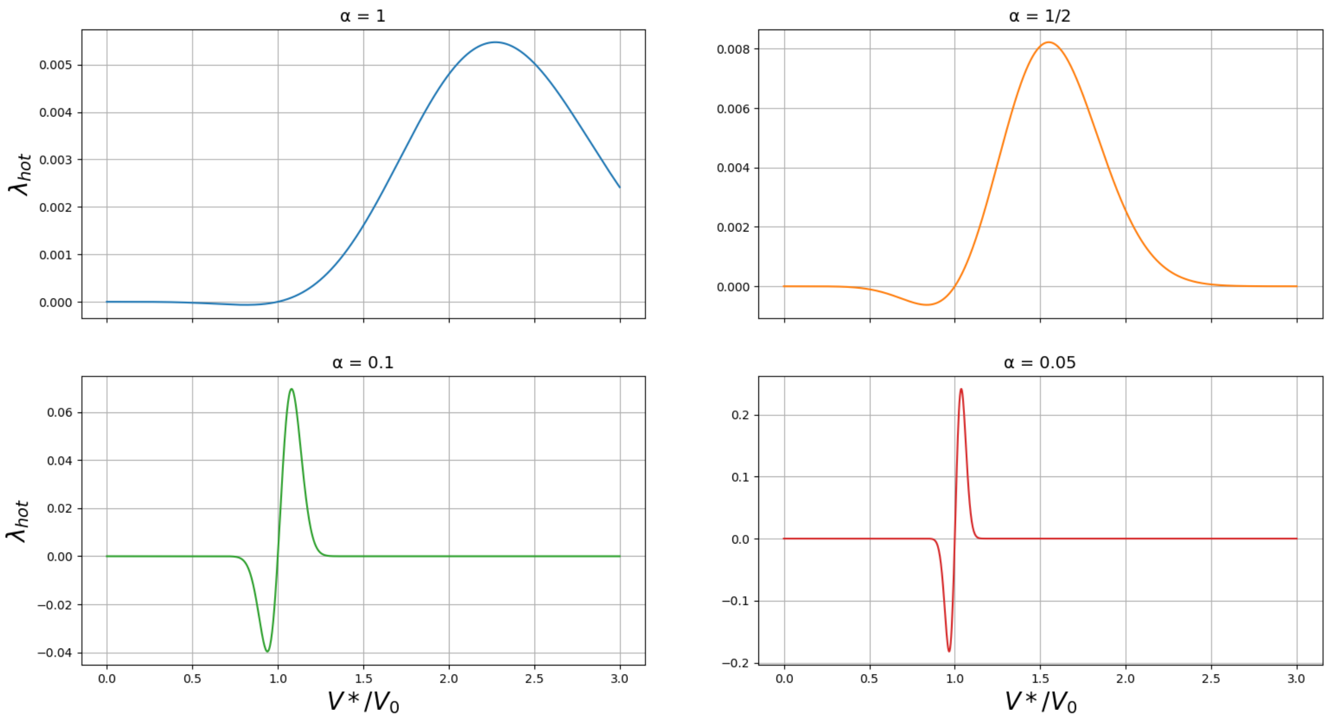

Figure 4.3: λ_{hot} for different values of α with: $a = 0.9$, $R_0 = 3$, $r_h = 0.3$, $q = 1$, $n_h/n_c = 10^{-2}$.

On the other hand, λ_{hot} can become very small. Therefore, the consideration of different damping mechanism, such as collisions or thermal plasma Landau damping, leads to an estimation of a lower threshold for λ_{hot}. In particular, regarding the exponential in Eq.(4.87), a new criterion for the instability could be introduced, and together with the instability condition $\lambda_{hot} > 0$ reads:

$$1 < \frac{v^*}{v_0} < 1 + \hat{c}\alpha \tag{4.88}$$

where $\hat{c} \sim 2$ is a numerical factor determined considering 10^{-2} as the minimum acceptable value for the exponential $exp\left[-1/\alpha^2\left(v^*/v_0 - 1\right)^2\right]$.

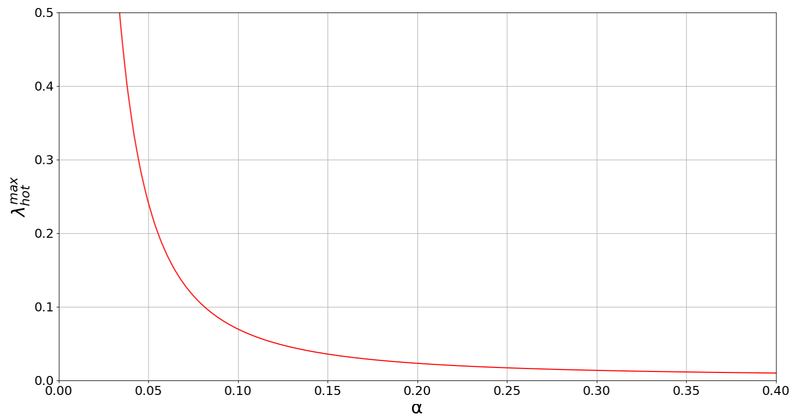

Figure 4.4: Maximum value of λ_{hot} as a function of α with: $a = 0.9$, $R_0 = 3$, $r_h = 0.3$, $q = 1$, $n_h/n_c = 10^{-2}$.

In Figure (4.4) the value of λ_{hot}^{max} is plotted as a function of α. When α decreases, λ_{hot}^{max} tends to diverge due to the denominator appearing in the function Λ_{hot} of Eq.(4.87). However, when α becomes very small, the criterion shown in Eq.(4.88) means that the range of unstable λ_{hot} values becomes very narrow, as is also clear from Figure (4.3), meaning that obtaining an instability becomes very improbable. Furthermore, when α becomes very small Figure (4.4) shows also that λ_{hot} would become very large, meaning that the perturbative procedure that we have used for the derivation of the dispersion relation would be no longer valid.
The inverse of the growth rate defined as

$$\gamma^{-1} = \frac{2}{\omega_{MHD}\lambda_{hot}} \tag{4.89}$$

represents the time scale relevant for the instability.
As described in Chapter 2, $1/\omega_{MHD}$ is a fraction of the Alfvén time. In particular, for a Deuterium plasma with $\epsilon_{el} \sim 0.2$, $a \sim 0.9m$,$B_p(a) \sim 1T$, $n = 10^{20}m^3$ plasma number density and the geometrical factor $D \sim 1.5$ we

obtain $\omega_{MHD}^{-1} \sim 1.8\mu s$. Considering then the maximum value of λ_{hot} with $\alpha = 0.1$, as showed in Figure (4.3), $\lambda_{hot}^{max} \sim 0.07$ and the inverse of the growth rate becomes $\gamma^{-1} \sim 5.1 \times 10^{-5}s$. On the other hand, the minimum value allowed for λ_{hot} considering $v^*/v_0 = 1 + \hat{c}\alpha$ is $\lambda_{hot}^{min} \sim 0.007$, corresponding to $\gamma^{-1} \sim 5.1 \times 10^{-4}s$.
However, this unstable growth will be mitigated by other damping mechanisms. The collisional damping, introduced in section 3.1.1 in the case of space-charge waves, could be one of such mechanisms. Considering the result of the simpler case of space-charge waves, we guess that also in this case the collisional damping will depend only on the electron-ion collision frequency ν_{ei}. In particular we will take into account $\gamma_{coll} = -a\nu_{ei}$ as the damping rate due to collisions. The numerical factor a is considered to be of order one and can be derived with an accurate analysis of the collisional damping in the case of axisymmetric n=0 modes, that will be the subject of future studies. Therefore, the competition between the resonant interaction and collisional effects defines the behaviour of mode. In particular, the evolution of the perturbation becomes:

$$\xi = \xi_0 e^{-i\omega_{MHD}t} e^{\gamma_{tot}t} \tag{4.90}$$

$$\gamma_{tot} = \gamma_{res} + \gamma_{coll} \tag{4.91}$$

where $\gamma_{res} = \omega_{MHD}\lambda_{hot}/2$ and $\gamma_{coll} = -a\nu_{ei}$. The inverse of the electron-ion collision frequency is typically of the order of $100\mu s$ for plasmas of fusion interest. Thus, for small values of λ_{hot}, the instability could be already suppressed by the collisional damping. On the other hand, it is possible to define a critical density ratio $(n_h/n_c)_{crit}$ at which effects of collisions prevents the instability to arise even with $\lambda_{hot} = \lambda_{hot}^{max}$. Considering same plasma parameters as before, it is easy to determine that $\gamma_{res}(\lambda_{hot}^{max}) = -\gamma_{coll}$ when $n_h/n_c \sim 5 \times 10^{-3} = (n_h/n_c)_{crit}$. More accurate studies are required in order validate this simple argument. However, it is reasonable to guess that it will be possible to determine a critical density ratio below which the resonant instability is balanced by the collsional damping.

4.3.1 Instability saturation

A stability criterion for the excitation of the n=0 axisymmetric mode in presence of energetic particles has been derived for the simple equilibrium of Eq.(4.79). However, the potential impact of this instability on the system is yet to be determined.
In fact, the linear theory predicts an unstable growth for all time when the criterion of Eq.(4.88) is satisfied. This is of course not true, and non-linear

mechanism of instability saturation must be considered.
In order to understand the instability saturation mechanism, it is possible to consider the simpler but similar case of the "Bump on tail" instability. This instability, briefly introduced in the beginning of Chapter 3, describes the growth of space-charge waves due to Landau resonance when the equilibrium distribution function presents $\partial f_{e0}/\partial v_z > 0$ near the phase speed of the wave $v_z \approx v_{ph} = \omega/k$. This instability arises due to a collisionless transfer of energy between wave and particles. In fact, a particle with velocity slightly smaller that the phase velocity of the wave will be accelerated and energy will be transferred from the wave to the particle. On the other hand, a particle slightly faster than v_{ph} will give energy to the wave as it is decelerated. In the bump on tail instability case, there are more fast particles than slower, meaning that the wave will increase its total energy.
Linear stability theory of this instability predicts unstable growth for all time as in our case ([5]). However, it is clear that as the instability grows in time particles are decelerated, so that less particles will have a velocity $v_z \gtrsim v_{ph}$. Non-linear investigation of this phenomenon shows that, as the time goes on, the distribution function tends to develop a plateau in the resonant region, as shown in Figure (4.5).

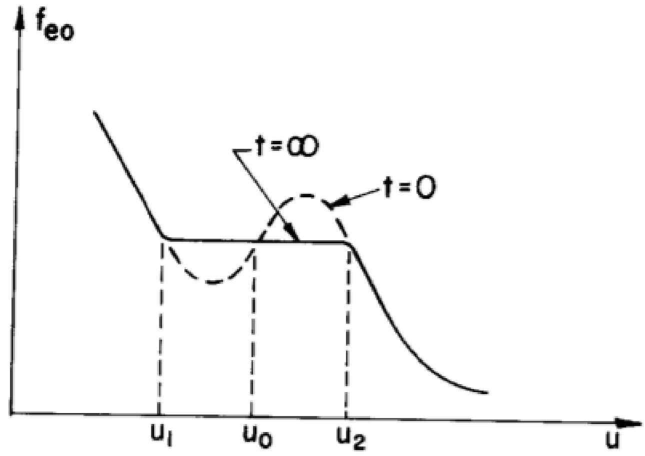

Figure 4.5: Evolution of the distribution function showing the flattening of the distribution. At $t \to \infty$ the distribution is no longer unstable. [5]

A similar behaviour is expected in our case, and the non-linear analysis of the instability will be carried out in future work.
The study of the instability saturation is fundamental for the determination of the impact on the plasma of the instability itself. If the saturation mecha-

nism is very fast, the instability could have a very little effect on the plasma. On the other hand, if the instability will grow for a long time before reaching its saturation, the amplitude of plasma oscillation could become so large that deteriorates the confinement performances.

Chapter 5

Conclusions

In this thesis an analytical study of the stability of the axisymmetric n=0 modes in presence of energetic particles has been carried out. The main goal was to determine if the experimentally observed excitation of toroidally axisymmetric modes in presence of energetic particles could be proven to be the result of resonant interaction between the periodic motions of the energetic particles and the vertical oscillation of the plasma column that arise due to the stabilization of ideal vertical instabilities. Indeed, Ref.[8] describes the observation of an axisymmetric n=0 mode at the JET tokamak experiment. During this experiment an energetic particles population composed by Deuterium with energies ranging from $100keV$ to $1MeV$ was generated by a combination of two heating techniques, NBI (neutral beam injection) and ICRH (ion cyclotron resonance heating). The presence of energetic particles excited the axisymmetric n=0 mode which showed an observed frequency f_{obs} of $325kHz$.

In the context of magnetic confinement nuclear fusion experiments, low density plasmas are confined using strong magnetic fields in order to reach the so-called thermonuclear ignition, i.e. when nuclear fusion reactions provide enough energy to the plasma to balance energy losses and to self-sustain the high-temperature plasma conditions required for thermonuclear burn. Nowdays the most promising and successful design for magnetic confinement fusion experiments is the so-called Tokamak. The performances of a tokamak device are strictly related to the concepts of plasma equilibrium and stability. In particular, the study of plasma stability determines if a magnetic geometry can provide a confinement time long enough for fusion reactions to take place and produce a net gain in energy, since the presence of instabilities will tend to disassemble the confined plasma.
Modern day tokamak experiments and designs often present non-circular plasma cross-section due to the use of magnetic divertor configurations and plasma

shaping. As described in Chapter 2, the elongation of the plasma cross-section is related to dangerous displacements of the plasma column in the vertical direction. Due to the very fast growth rate of this instability, vertical displacements are stabilized by means of external feedback currents. The result of this stabilization process is the oscillation of the plasma column in the vertical direction accordingly to $\omega_{MHD}^2 = (D-1)\epsilon_{el} v_A^2/a^2$.
Since the ideal MHD model is inadequate for the description of kinetic effects and the presence of energetic particles makes it impossible to find a fluid closure, the hybrid-kinetic MHD model must be used in order to study the resonant interaction between vertical oscillations and energetic particles. The analysis of the drift-kinetic equation describing the perturbed distribution function showed that the particle periodic motion on the poloidal plane is the motion of interest for the resonant interaction. Considering experimental results from JET, the resonant interaction between these modes and energetic particles involves *passing particles*, i.e. particles with $0 \leq \Lambda = \mu B_0/\mathcal{E} \leq 1-\epsilon$ (μ is the magnetic moment, B_0 is the toroidal magnetic field and $\mathcal{E}$ is the energy of the particle). The projection on the poloidal cross-section of these particles orbits is a complete revolution around the magnetic axis with frequency

$$\omega_t = \frac{2\pi}{\tau_t} = \frac{v\kappa}{2R_0 q}(2\epsilon)^{1/2}\frac{\pi}{\mathcal{K}(1/\kappa^2)} \qquad (5.1)$$
$$\text{where} \quad \kappa^2 = \frac{1}{2} + \frac{1}{2\epsilon}(1-\Lambda)$$

Where $\epsilon = r/R_0$, $\mathcal{K}(x)$ is the complete elliptic integral of the first kind and v is the modulus of the energetic particles velocity. The parameter Λ can also be written as $(1+\epsilon cos(\theta))v_\perp^2/v^2$ where θ is the poloidal angle and $v_\perp$ is the velocity of the particle in the direction perpendicular to the magnetic field line. The presence of a resonant interaction introduces an imaginary part in the mode dispersion relation in a way similar as in the Landau damping case described in Chapter 3. Therefore, the stability of the mode has been investigated studying its dispersion relation in presence of energetic particles. In particular the following dispersion relation for axisymmetric n=0 modes in presence of energetic particles has been derived using a perturbative approach.

$$\omega = \omega_{MHD} + i\frac{\omega_{MHD}\lambda_{hot}}{2} \qquad (5.2)$$

At its lowest order this dispersion relation coincides with the ideal MHD dispersion relation in absence of energetic particles. The small parameter λ_{hot} takes into account the resonant interaction contributions.
A simple equilibrium distribution function for the energetic particles has been

considered in order to proceed with analytical calculation of λ_{hot} and a stability threshold for the instability excitation has been derived. The inverse of the growth rate γ^{-1} representing the relevant time scale for the instability has been showed to be very fast, being of the order of $10^{-5} - 10^{-4}s$. Collisional damping has been considered as a possible competing mechanism. A simple argument showed that it is possible to define a critical density ratio $(n_h/n_c)_{crit}$, below which the collisional damping prevails over the resonant instability.
Further studies are required in order to determine instability threshold in more general situation. In particular, in order to investigate this instability considering more realistic equilibrium distribution function for energetic particles numerical analysis is required. Furthermore, competing damping mechanisms, such as the collisional damping, and non-linear analysis of the instability saturation mechanism must be studied extensively in order to determine the impact of this instability on the plasma.
In conclusion, in this work we proved that instability can arise when energetic particles are present in elliptical plasmas due to the resonant interaction between periodic motion of so-called "passing particles" and plasma vertical oscillations caused by the stabilization of the ideal vertical instability by means of feedback currents. In particular, it is reasonable that this instability is the mechanism that caused the excitation of n=0 axisymmetric modes in presence of energetic particles observed at the JET tokamak experiment.

Bibliography

[1] J. P. Freidberg, *Plasma Physics and Fusion Energy.* Cambridge University Press, 2007.

[2] https://www.euro-fusion.org/news/detail/detail/News/jet-operation-at-high-current/.

[3] J. P. Freidberg, *Ideal MHD.* Cambridge University Press, 2014.

[4] F.Porcelli, R.Fitzpatrick, and A. Yolbarsop, "in preparation,"

[5] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics.*

[6] L. Chen, R. B. White, and M. N. Rosenbluth, "Excitation of internal kink modes by trapped energetic beam ions," *Phys. Rev. Lett.*, vol. 52, pp. 1122–1125, Mar 1984.

[7] B. Coppi and F. Porcelli, "Theoretical model of fishbone oscillations in magnetically confined plasmas," *Phys. Rev. Lett.*, vol. 57, pp. 2272–2275, Nov 1986.

[8] H. J. C. Oliver, S. E. Sharapov, B. N. Breizman, and L.-J. Zheng, "Axisymmetric global alfvén eigenmodes within the ellipticity-induced frequency gap in the joint european torus," *Physics of Plasmas*, vol. 24, no. 12, p. 122505, 2017.

[9] F. Porcelli and A. Yolbarsop, "Analytic equilibrium of "straight tokamak" plasma bounded by a magnetic separatrix," *Physics of Plasmas*, vol. 26, no. 5, p. 054501, 2019.

[10] G. Laval, R. Pellat, and J. S. Soule, "Hydromagnetic stability of a current-carrying pinch with noncircular cross section," *The Physics of Fluids*, vol. 17, no. 4, pp. 835–845, 1974.

[11] R. Coelho and F. Porcelli, "Ideal stability of an elliptical plasma column in the presence of external feedback currents," *Physics of Plasmas*, vol. 10, no. 4, pp. 930–933, 2003.

[12] L. Chen and F. Zonca, "Physics of alfvén waves and energetic particles in burning plasmas,," *Rev. Mod. Phys.*, vol. 88, p. 015008, Mar 2016.

[13] F. Porcelli, R. Stankiewicz, W. Kerner, and H. L. Berk, "Solution of the drift-kinetic equation for global plasma modes and finite particle orbit widths," *Physics of Plasmas*, vol. 1, no. 3, pp. 470–480, 1994.

[14] R. G. Littlejohn, "Variational principles of guiding centre motion," *Journal of Plasma Physics*, vol. 29, no. 1, p. 111–125, 1983.