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Master Thesis

Analytical solution for the one group kinetic  
diffusion equation for the reflected reactor



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# 1-INTRODUCTION

The main purpose of this work is the presentation of a method for the analytical resolution of a classic reactor kinetics problem, concerning the heterogeneous reactor and in particular the reflected one, using the diffusion model with the approximation to one energy group. The elaborate is substantially divided into two main parts.

The first one recalls the diffusion theory, treating neutrons as monokinetics, and the reactor kinetic theory with the aim of building step by step the equation of the type of problem whose analytical solution is to be found. In particular, section 2, which has the task of briefly recalling the diffusion model, begins with the introduction of the quantities involved and the derivation of the diffusion equation passing through the continuity equation and Fick's law. This equation is formulated for several cases, starting from a purely diffusive medium, moving to a multiplying medium to finish with the specific case of the reflex reactor. For all the cases of diffusion equation exposed the analytical solution for simple geometric cases is reported, without however solving the problems by addressing specific numerical examples, with the dual purpose of investigating how the system behaves from the mathematical-physical point of view, case for case, and to see how useful the solutions found to better understand the resolution of the problem of the reflected reactor in the reactor kinetic theory. The following section introduces the nuclear reactor kinetic equation theory, initially describing the importance of delayed neutrons and then deriving the expression of the point reactor kinetics equations. Furthermore, still not using specific numerical examples, these equations are solved for a homogeneous reactor, in a different way from how the problem will be treated later.

In the second part of the work, the last two sections, is thus faced the problem of reflected reactor in the reactor kinetics, and therefore in the presence of delayed precursors. In this part, the first section is purely theoretical, therefore after introducing the type of problem and the geometry in which it is intended to be solved, the Laplace transform technique is used as a method of resolution, as in the aforementioned (*Corno and Ravetto, 1976*) and (*Corno et al., 2008*). Finally, having dealt with all the necessary theory and found the analytical solution to the problem, in the last section we proceed with various numerical examples. The practical examples performed subsequently concern several initial states, the presence or absence of a concentration of delayed precursors in equilibrium with the initial concentration, the change in the properties of materials and the presence of an external source.

## 2-INTRODUCTION TO THE DIFFUSION MODEL

Before presenting in detail the problem addressed is better to start treating the problem starting with the simplest cases possible to take into consideration and therefore to recall some basic definitions<sup>1</sup>.

### 2.1-MAIN PHYSICAL QUANTITIES INVOLVED

The fundamental information necessary to describe the motion of a neutron are its position  $\vec{r} \equiv (x, y, z)$  and its speed  $\vec{v} \equiv (v_x, v_y, v_z)$ , through which it is possible to describe the neutron field at the instant  $t$  in a portion of the space  $V$  as the function:

$$f(x, y, z, v_x, v_y, v_z, t) dx dy dz dv_x dv_y dv_z \quad (2.1)$$

Function  $f$  that describes therefore the number of neutrons that have coordinates between  $x$  and  $x + dx$ ,  $y$  and  $y + dy$ ,  $z$  and  $z + dz$  and the velocity components comprised between  $v_x$  and  $v_x + dv_x$ ,  $v_y$  and  $v_y + dv_y$ ,  $v_z$  and  $v_z + dv_z$ .

In the other way, using a parallelepiped  $d\vec{r}$  that has opposite vertices in  $(x, y, z)$  and  $(x + dx, y + dy, z + dz)$ , then of volume  $dx dy dz$ , and in the same way  $d\omega$  denotes a parallelepiped that has opposite vertices in  $(v_x, v_y, v_z)$  and  $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ , hence of volume  $dv_x dv_y dv_z$ , it is possible to write the expression (2.1) more briefly:

$$N(\vec{r}, \vec{v}, t) d\vec{r} d\omega \quad (2.2)$$

The (2.2) is called the distribution function and represents the number of neutrons that at time  $t$  are in the volume element  $d\vec{r}$  around  $\vec{r}$  and have velocities belonging to the velocity space  $d\omega$  around  $\vec{v}$ .

It is possible to write the speed as the product between its module and its versor:

$$\vec{v} = v \vec{\Omega} \quad (2.3)$$

Furthermore the  $d\omega$  element can be expressed in spherical coordinates  $(v, \vartheta, \varphi)$ :

$$d\omega = v^2 dv d\Omega = v^2 \sin\vartheta dv d\vartheta d\varphi \quad (2.4)$$

The distribution function (2.2) then becomes:

$$N(\vec{r}, v \vec{\Omega}, t) v^2 d\vec{r} dv d\Omega = v(\vec{r}, v \vec{\Omega}, t) d\vec{r} dv d\Omega \quad (2.5)$$

The formula (2.5) takes the name of angular density and indicates the number of neutrons that, at time  $t$ , are in the volume element  $d\vec{r}$  in the neighborhood of  $\vec{r}$ , have module of the velocity between  $v$  and  $v + dv$  and have versor speed contained in the

solid angle  $d\Omega$  in the neighborhood of  $\vec{\Omega}$ . Furthermore, if instead of the velocity module we used the expression of kinetic energy, with  $m$  equal to the mass of the neutron, the velocity  $v$  and its differential would be:

$$v = \sqrt{\frac{2E}{m}} \quad \wedge \quad dv = \frac{dE}{\sqrt{2mE}} \quad (2.6)$$

Thus applying a change of variable to the angular density, we obtain:

$$v(\vec{r}, v, \vec{\Omega}, t) d\vec{r} dv d\Omega = v(\vec{r}, \sqrt{2E/m}, \vec{\Omega}, t) d\vec{r} \frac{dE}{\sqrt{2mE}} d\Omega = n(\vec{r}, E, \vec{\Omega}, t) d\vec{r} dE d\Omega \quad (2.7)$$

The density thus obtained represents the number of neutrons which, at time  $t$ , are found in the volume element  $d\vec{r}$  around  $\vec{r}$ , have an energy between  $E$  and  $E + dE$  and the direction contained in the solid angle  $d\Omega$  around  $\vec{\Omega}$ . Wanting to get the amount of interactions that neutrons have, it is necessary to study the path that they travel. Being able to express the path of a single particle like  $v \cdot dt$ , for all the particles the path traveled is:

$$dx = v n(\vec{r}, E, \vec{\Omega}, t) d\vec{r} dE d\Omega dt = \Phi(\vec{r}, E, \vec{\Omega}, t) d\vec{r} dE d\Omega dt \quad (2.8)$$

In this way I introduce a new physical quantity, the flux  $\Phi(\vec{r}, E, \vec{\Omega}, t)$ , which indicates the sum of the lengths traveled by the particles per unit of volume, of solid angle, of energy, of time. To construct the balance of particles it is necessary to introduce beyond the flux, which describes what happens to the neutrons within the volume  $d\vec{r}$ , the current term  $\vec{j}$  that is useful to obtain a balance of the particles that cross the faces of the volume taken into consideration.

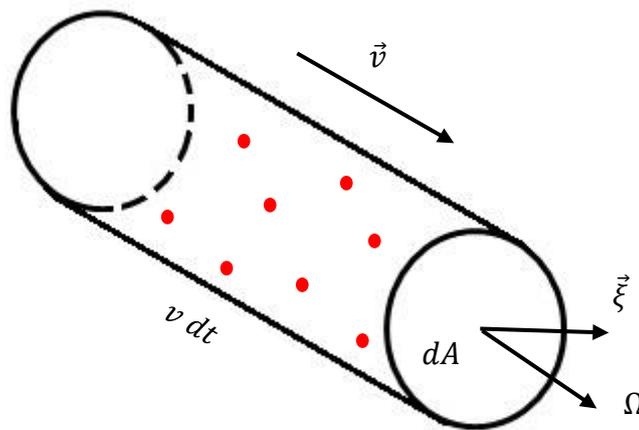


Fig. 1. Particles contained in the cylinder with  $dA \cdot v dt$  volume.

As shown in the figure Fig. 1, taking an infinitesimal section  $dA$ , of normal versor  $\vec{\xi}$ , and orienting it in such a way that it has a positive and a negative face to differentiate the incoming particles from the outgoing ones, the current term  $\vec{j}$  takes into account the neutrons oriented according to the direction  $\Omega$  that travel through space  $v \cdot dt$ .

$$v n(\vec{r}, E, \vec{\Omega}, t) \vec{\Omega} \xi dA dt dE d\Omega = \vec{\Omega} \Phi(\vec{r}, E, \vec{\Omega}, t) dE d\Omega \xi dA dt = \vec{j}(\vec{r}, E, \vec{\Omega}, t) \xi dE d\Omega dA dt \quad (2.9)$$

Taking in account the final term in (2.9), referring to unitary energy intervals and integrating all possible directions, we obtain the number of neutrons that pass through the area unit in the time unit and in particular it indicates the net total of particles passing through  $dA$  in the positive sense per unit time.

## 2.2-THE CONTINUITY EQUATION

Before constructing a balance equation for neutrons, it is necessary to define the simplified hypotheses used in the diffusion model:

- All thermal neutrons have the same energy, we are in the monokinetic case: the speed is equal to the average of the thermal group while the macroscopic cross sections are the average sections of the thermal group. ;
- Isotropic field: each particle has equal probability of moving in any direction after a scattering collision ;
- The medium in which the particles move is homogeneous so the macroscopic cross sections are constant or very regular functions of  $\vec{r}$  ;
- The sources present in the medium are isotropic and constant or very regular functions of  $\vec{r}$  ;

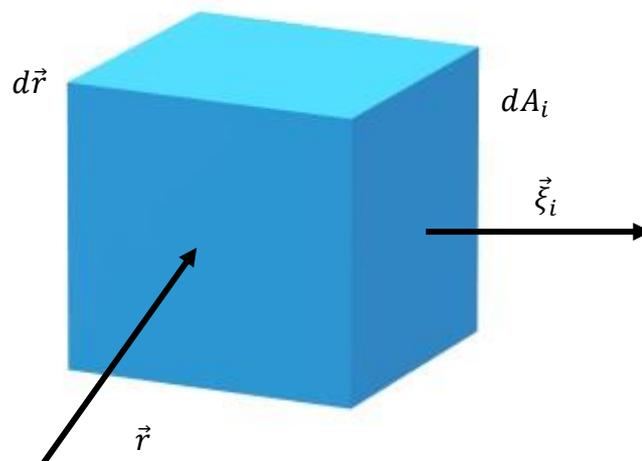


Fig. 2. Infinitesimal reference volume  $d\vec{r}$ .

With the purpose of constructing a balance sheet we take a small volume  $d\vec{r}$ , as in Fig. 2, surrounded by  $dA_i$  surfaces, each having its normal  $\vec{\xi}_i$ . The variation over time of the neutrons inside the element of considered volume is equal to the particles that enter less those that come out, less those absorbed by the medium, plus those coming from any external sources. Taking therefore as reference what happened at time  $t$  and at the successive instant  $t + dt$ :

$$n(\vec{r}, t + dt) - n(\vec{r}, t) = \frac{\partial n(\vec{r}, t)}{\partial t} = - \sum_i \vec{J}(\vec{r}, t) \cdot dA_i \vec{\xi}_i - \Sigma_a(\vec{r}) \Phi(\vec{r}, t) d\vec{r} dt + S(\vec{r}, t) d\vec{r} dt \quad (2.10)$$

As regards the current term, the surface integral is reduced to a volume integral by means of the Gauss theorem, as in (2.11).

$$\sum_i \vec{J}(\vec{r}, t) \cdot dA_i \vec{\xi}_i = \nabla \cdot \vec{J}(\vec{r}, t) d\vec{r} dt \quad (2.11)$$

In conclusion, recalling the relation (2.8) and introducing the current term derived from (2.11), we obtain the continuity equation from (2.10):

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = -\nabla \cdot \vec{J}(\vec{r}, t) - \Sigma_a(\vec{r}) \Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.12)$$

It was obtained an equation in which it is essential to find a relationship between current and flow: the Fick's law will therefore be shown below with this purpose.

### 2.3-FICK'S LAW

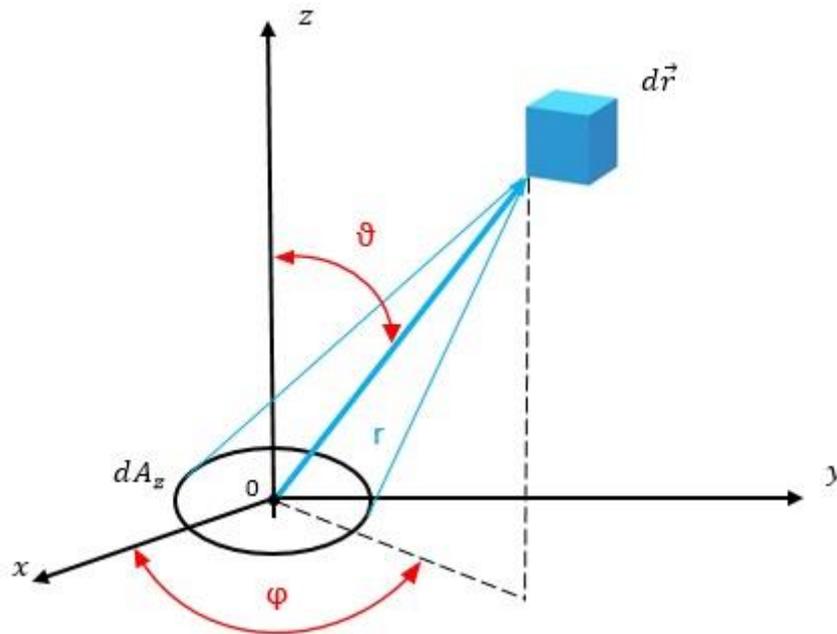


Fig. 3. Reference system, in Cartesian and polar coordinates, to find  $J_z$ .

As mentioned above, it is necessary to find a relationship between the current and the flux, so as to solve the equation of continuity. Starting from the same hypotheses used in the previous paragraph, to determine the current  $\vec{J}$  it is necessary to find all three of its components  $J_x, J_y, J_z$ . As shown in Fig. 3, we start with  $J_z$ , that is the net current, the upwards current minus the downward current, which crosses the area  $dA_z$ , having normal parallel to the axis  $z$ .

The particles that pass through  $dA_z$  come from every part of the medium, so it is best to start by analyzing what comes from a small part of space. Taking an infinitesimal element of volume,  $d\vec{r}$ , from this small volume, which emits isotropically as from hypothesis, comes new neutrons since it is a source neutrons and neutrons since it is a scattering source, therefore the particles that take flight from  $d\vec{r}$  identify a so defined emission density:

$$\psi(\vec{r}, t) d\vec{r} dt = [\Sigma_s(\vec{r}) \Phi(\vec{r}, t) + S(\vec{r}, t)] d\vec{r} dt \quad (2.13)$$

In particular the particles having the right direction to reach  $dA_z$  are that fraction of them which pass through the solid angle below which the reference area is seen by the volume  $d\vec{r}$ . Then we define the particles that from the small volume reach the area as:

$$\psi(\vec{r}, t) \frac{dA_z \cos\vartheta}{4\pi r^2} d\vec{r} dt \quad (2.14)$$

However during the journey to arrive at  $dA_z$  the neutrons will have a certain probability of undergoing scattering or absorption, moreover if emitted at time  $t$  they will arrive at time  $t' = t + r/v$ , so it is necessary to write the emission density at time  $t - r/v$ , because we want to calculate  $J_z(0, t)$ . In conclusion (2.14) becomes:

$$\psi\left(\vec{r}, t - \frac{r}{v}\right) e^{-\Sigma_t r} \frac{dA_z \cos\vartheta}{4\pi r^2} d\vec{r} dt \quad (2.15)$$

In which  $\Sigma_t(\vec{r})$  is the total macroscopic cross section and it is the sum between the absorption cross section and the scattering one. Now I want to write the total of neutrons per unit of area and per unit of time, the exact definition of the current, that come from the upper half-space  $z > 0$ , which is those particles that pass through the origin 0 from top to bottom:

$$J_z^-(0) = \int_{V^+} \psi\left(\vec{r}, t - \frac{r}{v}\right) \frac{1}{4\pi r^2} \cos\vartheta e^{-\Sigma_t r} d\vec{r} \quad (2.16)$$

Recall that it is possible to express the Cartesian coordinates in the polar ones since  $d\vec{r} = r^2 \sin\vartheta dr d\vartheta d\varphi$ , then the previous integral can be expressed as in (2.17).

$$J_z^-(0) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^\infty dr \cdot \psi\left(\vec{r}, t - \frac{r}{v}\right) e^{-\Sigma_t r} \cos\vartheta \sin\vartheta \quad (2.17)$$

At this point, a new type of hypothesis is made: the system under examination is a collision dominated system, so  $\Sigma_a$  is very small compared to  $\Sigma_s$ , so that there are no sources of anisotropy. Moreover the system is so scattering that it is impossible that the neutrons arrive at the origin coming from places far away: this means that the current at time  $t$  depends on places so nearby that they depend on times very close to  $t$ , and therefore similar to it. Then the (2.17) becomes:

$$J_z^-(0) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^\infty dr \cdot \psi(\vec{r}, t) e^{-\Sigma_t r} \cos\vartheta \sin\vartheta \quad (2.18)$$

Precisely because we have a collision dominated system, and therefore  $\Phi(\vec{r}, t)$  and  $S(\vec{r}, t)$  are regular functions in space, it is possible to develop  $\psi(\vec{r}, t)$  in McLaurin series near the origin with very few terms:

$$\psi(\vec{r}, t) = \psi(0, t) + x \left(\frac{\partial\psi}{\partial x}\right)_0 + y \left(\frac{\partial\psi}{\partial y}\right)_0 + z \left(\frac{\partial\psi}{\partial z}\right)_0 \quad (2.19)$$

Writing  $x$ ,  $y$  and  $z$  in spherical coordinates:

$$x = r \sin\vartheta \cos\varphi \quad ; \quad y = r \sin\vartheta \sin\varphi \quad ; \quad z = r \cos\vartheta \quad (2.20)$$

And substituting in (2.19):

$$\psi(\vec{r}, t) = \psi(0, t) + (r \sin\vartheta \cos\varphi) \left(\frac{\partial\psi}{\partial x}\right)_0 + (r \sin\vartheta \sin\varphi) \left(\frac{\partial\psi}{\partial y}\right)_0 + (r \cos\vartheta) \left(\frac{\partial\psi}{\partial z}\right)_0 \quad (2.21)$$

By replacing and integrating it in (2.18), the integrals from 0 to  $2\pi$  cancel the contributions of  $\left(\frac{\partial\psi}{\partial x}\right)_0$  and  $\left(\frac{\partial\psi}{\partial y}\right)_0$ , and therefore we obtain:

$$\begin{aligned} J_z^-(0) &= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\vartheta \int_0^\infty dr \cdot \left[ \psi(0, t) + (r \cos\vartheta) \left(\frac{\partial\psi}{\partial z}\right)_0 \right] e^{-\Sigma_t r} \cos\vartheta \sin\vartheta = \\ &= \frac{1}{4\Sigma_t} \psi(0, t) + \frac{1}{6\Sigma_t^2} \left(\frac{\partial\psi}{\partial z}\right)_0 \end{aligned} \quad (2.22)$$

In a similar way the current is obtained which from the lower half-space passes through  $dA_z$ :

$$J_z^+(0) = \frac{1}{4\Sigma_t} \psi(0, t) - \frac{1}{6\Sigma_t^2} \left(\frac{\partial\psi}{\partial z}\right)_0 \quad (2.23)$$

And the net current  $J_z(0, t)$  will be:

$$J_z(0, t) = -\frac{1}{3\Sigma_t^2} \left( \frac{\partial \psi}{\partial z} \right)_0 \quad (2.24)$$

However, according to the hypotheses adopted, I want the sources to be negligible, so:

$$J_z(0, t) = -\frac{\Sigma_s}{3\Sigma_t^2} \left( \frac{\partial \Phi}{\partial z} \right)_0 \quad (2.25)$$

Repeating the same procedure, for  $x$  and  $y$  components you get:

$$J_x(0, t) = -\frac{\Sigma_s}{3\Sigma_t^2} \left( \frac{\partial \Phi}{\partial x} \right)_0 \quad ; \quad J_y(0, t) = -\frac{\Sigma_s}{3\Sigma_t^2} \left( \frac{\partial \Phi}{\partial y} \right)_0 \quad (2.26)$$

Since the choice of the origin is arbitrary, these formulas are valid for any  $\vec{r}$  point provided that it is well internal to the  $dV$  volume and in conclusion putting together the three different vector components we finally get Fick's law:

$$\vec{J}(\vec{r}, t) = -\frac{\Sigma_s}{3\Sigma_t^2} \nabla \Phi(\vec{r}, t) \quad (2.27)$$

The Fick's law thus obtained which expresses the proportionality of the current to the gradient of its concentration according to a coefficient called diffusion coefficient:

$$D(\vec{r}) = \Sigma_s(\vec{r})/3\Sigma_t^2(\vec{r}) [cm] \quad (2.28)$$

As previously mentioned, however, the absorption cross section is very small so the scattering cross section and the total one are very similar and consequently it is possible to simplify the expression of the diffusion coefficient as follows:

$$D \cong 1/3\Sigma_t \quad (2.29)$$

## 2.4-THE DIFFUSION EQUATION

Going back to the equation of continuity and putting it together with Fick's law, we obtain the system:

$$\begin{cases} \frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = -\nabla \cdot \vec{J}(\vec{r}, t) - \Sigma_a(\vec{r})\Phi(\vec{r}, t) + S(\vec{r}, t) \\ \vec{J}(\vec{r}, t) = -D(\vec{r})\nabla \Phi(\vec{r}, t) \end{cases} \quad (2.30)$$

Now replacing the second equation in the first one is obtained the diffusion equation, a differential equation of the second order respect the space with partial derivatives:

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\vec{r})\nabla \Phi(\vec{r}, t)) - \Sigma_a(\vec{r})\Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.31)$$

Which in extended form is written:

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = \frac{\partial}{\partial x} D(\vec{r}) \frac{\partial \Phi(\vec{r}, t)}{\partial x} + \frac{\partial}{\partial y} D(\vec{r}) \frac{\partial \Phi(\vec{r}, t)}{\partial y} + \frac{\partial}{\partial z} D(\vec{r}) \frac{\partial \Phi(\vec{r}, t)}{\partial z} - \Sigma_a(\vec{r}) \Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.32)$$

However, bearing in mind that the medium in which the particles are diffused is homogeneous, the diffusion coefficient can be considered constant and it be carried out obtaining a laplacian operator applied to the flux:

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = D \nabla^2 \Phi(\vec{r}, t) - \Sigma_a(\vec{r}) \Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.33)$$

Assuming the absorption cross section is also constant in space, we define the diffusion length as:

$$L^2 = \frac{D}{\Sigma_a} [m^2] \quad (2.34)$$

Therefore the diffusion equation (2.33) can be written in the form:

$$\frac{1}{vD} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = \nabla^2 \Phi(\vec{r}, t) - \frac{1}{L^2} \Phi(\vec{r}, t) + \frac{S(\vec{r}, t)}{D} \quad (2.35)$$

Furthermore, by introducing the diffusion time  $t_d$  and the free mean path  $\ell_a$ , with  $v$  indicating the average speed on the thermal spectrum:

$$\ell_a = \frac{1}{\Sigma_a} \quad ; \quad t_d = \frac{\ell_a}{v} = \frac{1}{v\Sigma_a} \quad (2.36)$$

Then it is possible to write the equation of diffusion in another form again:

$$t_d \frac{\partial \Phi(\vec{r}, t)}{\partial t} = L^2 \nabla^2 \Phi(\vec{r}, t) - \Phi(\vec{r}, t) + \frac{S(\vec{r}, t)}{\Sigma_a} \quad (2.37)$$

In the various forms found the diffusion equation is a partial differential equation that has an infinite number of solutions and therefore needs to set conditions in order to find, among all possible solutions, the one that represents the physical quantity sought: the flux  $\Phi(\vec{r}, t)$ .

These conditions, different depending on the physics of the considered problem, are:

- a) Initial condition. At a certain time  $t$ , defined as initial, the flow is known:  
 $\Phi(\vec{r}, t = 0) = \Phi_0(\vec{r})$  ;
- b) Conditions at the border. No neutron flux on the external boundary of the domain:  
 $\Phi(\partial V, t > 0) = 0$  ;

- c) Conditions at the interfaces, in the typical case of change of materials in heterogeneous reactors. These conditions are expressed by the continuity of the flow and the current at the interface.

Regarding the solution of the diffusion equation, since it represents a flow, it must be real, not negative and defined throughout the considered domain.

### 2.4.1-THE BOUNDARY CONDITIONS

As mentioned above, Fick's law has no validity when applied far from the border, however in many cases the medium in which the neutrons diffuse has an external surface and therefore also the diffusion equation is not applicable. More accurate methods show that if we assume that the flux  $\Phi(\vec{r}, t)$  calculated with the diffusion equation vanishes at a certain distance  $d$  beyond the boundary, within the medium considered the solution found is very close to being equal to that exact.

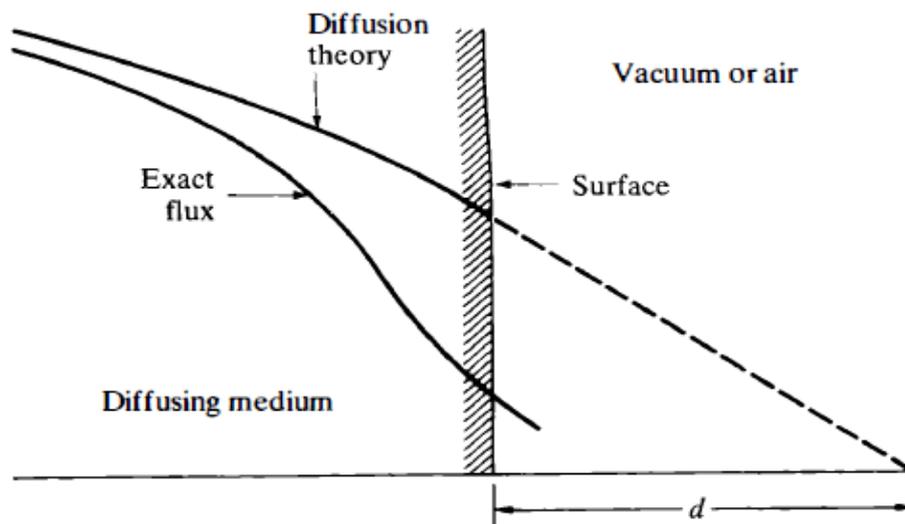


Fig. 4. Representation of the extrapolation distance  $d$ .

This approximation certainly cannot be defined physical, since in the void the particles do not behave as inside the medium, but it is very useful from the mathematical point of view as it allows us to define the flow within the medium with a good degree of accuracy. The parameter  $d$ , shown in Fig.4, is called extrapolation distance and for many cases of common interest it has the following formula:

$$d = 0.71 \lambda_{tr} \quad (2.38)$$

In which  $\lambda_{tr}$  is the transport mean free path of the medium:

$$\lambda_{tr} = 3 D \quad (2.39)$$

In conclusion, since for non-gaseous materials the diffusion coefficient is of the order of centimeters and often even less, the extrapolation distance will be small compared to the dimensions of the reactor: therefore it is often possible in the resolution of the diffusion equation to assume that the flow vanishes at the actual physical frontier of the system.

## 2.4.2-THE SOLUTIONS OF THE DIFFUSION EQUATION

Let us now the solutions of the diffusion equation for some simple cases with the purpose of seeing the forms of the solutions from the mathematical point of view and how the boundary and initial conditions are used.

### Infinite Planar Source

First we take an infinite diffusive medium in which we insert an infinite plane source located at  $x = 0$  that emits  $S$  neutrons for  $\text{cm}^2/\text{s}$  such as not to cause changes in the flow along the other directions  $y$  and  $z$ . With the simplifications so adopted it was obtained one-dimensional, symmetrical (with respect to axis  $x$ ) and stationary system so that the diffusion equation becomes:

$$D \frac{d^2\Phi(x)}{dx^2} - \Sigma_a \Phi(x) + S(x) = 0 \quad (2.40)$$

It is possible to express the source term in (2.40) as the product between the constant value  $S_0$ , the intensity of the source in  $x = 0$ , and the Dirac delta:

$$D \frac{d^2\Phi(x)}{dx^2} - \Sigma_a \Phi(x) + S_0 \delta(x) = 0 \quad (2.41)$$

Thanks to the symmetry of the system under consideration, the source is an even function, it is possible to solve the equation for only one half of the plane, so let's start from  $x > 0$ . The boundary conditions must now be imposed. As for the first, the medium is dissipative, so the limit tending to infinity of the flux will be zero:

$$\lim_{x \rightarrow \infty} \Phi(x) = 0 \quad (2.42)$$

For the second condition needed to solve the equation, I must go to see what value the current takes in  $0^+$ , point in which there are three contributions: the particles that go from right to left, the ones that go from left to right and the source.

Thanks to the perfect symmetry of the system the two particle flows cancel each other out and therefore the only contribution is the source, which assumed to be isotropic so

that it emits with the same intensity to the right and to the left of the origin, so the situation at  $0^+$  is:

$$\lim_{x \rightarrow 0^+} J(x) = S_0/2 \quad (2.43)$$

Taking into account the fact that the contribution of the source from 0 to  $+\infty$  is null, the equation (2.41) becomes homogeneous and its solution is in the form:

$$\Phi(x) = A \cdot e^{\frac{x}{L}} + B \cdot e^{-\frac{x}{L}} \quad (2.44)$$

Applying the first of the two boundary conditions, the positive exponential term vanishes in agreement with what one would expect from a dissipative system which therefore cannot diverge:

$$\lim_{x \rightarrow \infty} \Phi(x) = A \cdot \lim_{x \rightarrow \infty} e^{\frac{x}{L}} + B \cdot 0 = 0 \Rightarrow A = 0 \quad (2.45)$$

To apply the second boundary condition I derive the flux and use the Fick's law:

$$\Phi'(x) = -\frac{B}{L} e^{-\frac{x}{L}} \Rightarrow J(x) = -D\Phi'(x) = \frac{DB}{L} e^{-\frac{x}{L}} \quad (2.46)$$

For  $x = 0$  the second constant  $B$  is obtained:

$$J(0) = \frac{S_0}{2} = \frac{DB}{L} \Rightarrow B = \frac{S_0 L}{2D} \quad (2.47)$$

The solution of the differential equation for  $x > 0$  was thus obtained:

$$\Phi(x) = \frac{S_0 L}{2D} e^{-\frac{x}{L}} \quad (2.48)$$

And for the whole domain considered:

$$\Phi(x) = \frac{S_0 L}{2D} e^{-\frac{|x|}{L}} \quad (2.49)$$

## Point Source

Taking always an infinite medium, in this case a source  $S$  emits isotropically at the center of a system of spherical coordinates, then the Laplacian operator will be expressed in spherical coordinates and the diffusion equation becomes:

$$D \frac{1}{r} \frac{d^2}{dr^2} (r\Phi(r)) - \Sigma_a \Phi(r) + S_0 \frac{\delta(r)}{4\pi r^2} = 0 \quad (2.50)$$

Solving outside the source, and making the substitution  $r\Phi(r) = y(r)$ , we obtain the well-known equation in (2.51).

$$D \frac{d^2 y(r)}{dr^2} - \Sigma_a y(r) = 0 \quad , \quad r \neq 0 \quad (2.51)$$

Of which the solution is known:

$$y(r) = A \cdot e^{\frac{r}{L}} + B \cdot e^{-\frac{r}{L}} \quad (2.52)$$

$$\Phi(r) = \frac{y(r)}{r} = A \cdot \frac{e^{\frac{r}{L}}}{r} + B \cdot \frac{e^{-\frac{r}{L}}}{r} \quad (2.52')$$

The medium is dissipative, for  $r$  tending to infinity the flux will cancel, therefore the positive exponential is to be eliminated as in the previous case:

$$\Phi(r) = B \cdot \frac{e^{-\frac{r}{L}}}{r} \quad (2.53)$$

In this case, to calculate the outgoing and the original current, we take a small sphere of  $\varepsilon$  radius surrounding the origin, obtaining the following condition:

$$\lim_{\varepsilon \rightarrow 0} J(\varepsilon) \cdot 4\pi\varepsilon^2 = S_0 \quad (2.54)$$

Now we derive the flow and apply Fick's law with the aim of finding the current:

$$J(r) = -D\Phi'(r) = BD \left( \frac{1}{Lr} + \frac{1}{r^2} \right) e^{-\frac{r}{L}} \quad (2.55)$$

Applying the boundary condition (2.54) to this last expression we find the constant B:

$$\lim_{\varepsilon \rightarrow 0} BD \left( \frac{1}{L\varepsilon} e^{-\frac{\varepsilon}{L}} + \frac{1}{\varepsilon^2} e^{-\frac{\varepsilon}{L}} \right) \cdot 4\pi\varepsilon^2 = 4\pi BD = S_0 \quad \Rightarrow \quad B = \frac{S_0}{4\pi D} \quad (2.56)$$

Returning to (2.53), the solution is:

$$\Phi(r) = \frac{S_0}{4\pi D} \cdot \frac{e^{-\frac{r}{L}}}{r} \quad (2.57)$$

## Bare Slab

For the third case I consider an infinite slab of thickness  $2a$  with at the center a source  $S$  emitting neutrons per  $cm^2/s$  as illustrated in figure Fig.5.

The initial equation is equal to (2.41), in the case of the infinite flat source, however there is a different initial condition since the medium is always dissipative but the flow must be zeroed at the extrapolated surfaces. So the boundary conditions are:

$$\Phi(a + d) = \Phi(-a - d) = 0 \quad (2.58)$$

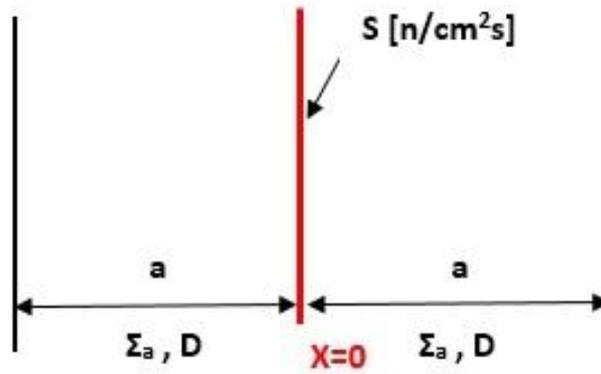


Fig. 5. Representation of bare slab under study.

Starting from the solution of the right half-plane, since symmetry is always valid, the solution is:

$$\Phi(x) = A \cdot e^{\frac{x}{L}} + B \cdot e^{-\frac{x}{L}} \quad (2.59)$$

Applying the boundary condition at  $a + d$ :

$$\Phi(a + d) = A \cdot e^{\frac{(a+d)}{L}} + B \cdot e^{-\frac{(a+d)}{L}} = 0 \quad (2.60)$$

From which it is easily obtained:

$$A = -B \cdot e^{\frac{2(a+d)}{L}} \quad (2.61)$$

Returning to the solution, replacing this in (2.60):

$$\Phi(x) = B \left[ e^{-\frac{x}{L}} - e^{\left(\frac{x}{L} - \frac{2(a+d)}{L}\right)} \right] \quad (2.62)$$

To find the second constant we apply the condition (2.43), therefore we begin with obtaining the derivative of (2.62) and applying Fick's law:

$$\Phi'(x) = -\frac{B}{L} \left[ e^{\left(\frac{x}{L} - \frac{2(a+d)}{L}\right)} + e^{-\frac{x}{L}} \right] \quad (2.63)$$

$$J(x) = -D\Phi'(x) = \frac{DB}{L} \left[ e^{\left(\frac{x}{L} - \frac{2(a+d)}{L}\right)} + e^{-\frac{x}{L}} \right] \quad (2.63')$$

So going to study the behavior of the current in  $0^+$  we obtain the constant  $B$ :

$$J(0) = \frac{DB}{L} \left[ e^{\left(-\frac{2(a+d)}{L}\right)} + 1 \right] = \frac{S_0}{2} \Rightarrow B = \frac{S_0 L}{2D} \left[ e^{\left(-\frac{2(a+d)}{L}\right)} + 1 \right]^{-1} \quad (2.64)$$

The solution for  $x > 0$  will be:

$$\Phi(x) = \frac{S_0 L}{2D} \left[ \frac{e^{-\frac{x}{L}} - e^{\left(\frac{x}{L} - \frac{2(a+d)}{L}\right)}}{e^{\left(-\frac{2(a+d)}{L}\right)} + 1} \right] \quad (2.65)$$

And for the whole considered domain, the flux will have the expression:

$$\Phi(x) = \frac{S_0 L}{2D} \left[ \frac{e^{-\frac{|x|}{L}} - e^{\left(\frac{|x|}{L} - \frac{2(a+d)}{L}\right)}}{e^{\left(-\frac{2(a+d)}{L}\right)} + 1} \right] \quad (2.66)$$

## 2.5-DIFFUSION OF NEUTRONS IN A MULTIPLICATIVE SYSTEM

Using a multiplicative medium, and not only a diffuse one, it is necessary to add the term fission to the diffusion equation, the term that expresses the multiplication of neutrons. Starting from the equation (2.31):

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\vec{r}) \nabla \Phi(\vec{r}, t)) + v \Sigma_f(\vec{r}) \Phi(\vec{r}, t) - \Sigma_a(\vec{r}) \Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.67)$$

In which  $\Sigma_f$  is the macroscopic cross section of fission and  $v$  indicates the number of neutrons produced for each fission reaction. Analyzing a specific situation, stationary system, no sources term and properties of materials constant in space, the previous equation becomes as (2.68).

$$D \nabla^2 \Phi(\vec{r}) + v \Sigma_f \Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) = 0 \quad (2.68)$$

I thus obtained a homogeneous differential equation of the second order in space having infinite solutions all the same, unless a constant of proportionality.

With the aim of finding a useful result we introduce the parameter  $1/k$ , then (2.68) becomes:

$$D \nabla^2 \Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) + \frac{1}{k} v \Sigma_f \Phi(\vec{r}) = 0 \quad (2.69)$$

There are three possible cases regarding the value that the eigenvalue  $k$  can take:

- $k > 1$  : supercritical case, the system produces more than it absorbs and loses ;
- $k = 1$  : critical case, the system feeds itself ;
- $k < 1$  : subcritical system, the system tends to turn off ;

Now I define the geometric buckling  $B^2$  as:

$$B^2 = \frac{1}{D} \left( \frac{1}{k} v \Sigma_f - \Sigma_a \right) \quad (2.70)$$

In this way it is possible to rewrite the equation (2.69) in the following way:

$$D \nabla^2 \Phi(\vec{r}) = -B^2 \Phi(\vec{r}) \quad (2.71)$$

Having found another way of writing the term of leakage, it is now possible to express the equation of diffusion (2.69) as:

$$-DB^2 \Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) + \frac{1}{k} v \Sigma_f \Phi(\vec{r}) = 0 \quad (2.72)$$

The (2.72) is called one-group reactor equation and can be solved by the constant  $k$ :

$$k = \frac{v \Sigma_f \Phi(\vec{r})}{DB^2 \Phi(\vec{r}) + \Sigma_a \Phi(\vec{r})} = \frac{v \Sigma_f}{DB^2 + \Sigma_a} \quad (2.73)$$

From the physical point of view, the expression of  $k$  has as numerator the number of neutrons produced by fissions in the current generation, while the denominator is the number of neutrons absorbed and escaped from the system in the previous one.

It is easy to observe that in the case of an infinite medium in which the particles diffuse, the term of leakage would not necessarily be present in the previous formula and therefore the value  $k_\infty$  and another important relation are obtained:

$$k_\infty = \frac{v \Sigma_f}{\Sigma_a} \implies v \Sigma_f = k_\infty \Sigma_a \quad (2.74)$$

Returning to the one-group reactor equation, in case of critical case that means  $k = 1$  :

$$-DB^2 \Phi(\vec{r}) + (1 - k_\infty) \Sigma_a \Phi(\vec{r}) = 0 \quad (2.75)$$

Dividing by  $D$  all the members:

$$-B^2 \Phi(\vec{r}) + \frac{k_\infty - 1}{L^2} \Phi(\vec{r}) = 0 \quad (2.76)$$

Finally it is possible find a solution to the one-group reactor equation by using the geometric bucking  $B^2$ :

$$B^2 = \frac{k_\infty - 1}{L^2} \quad (2.77)$$

Therefore, the equation (2.77) determines the conditions under which a bare reactor is in critical state in the case in which material characteristics have been set and the

reactor must be sized or the reverse problem, vary the characteristics of the materials according to the dimensions that the reactor must have.

Returning to the time-dependent problem we have a differential equation with partial derivatives of which the initial and boundary conditions are known. We rewrite the (2.67), with its initial and boundary conditions, assuming constant the properties of the materials, thus using the laplacian operator:

$$\frac{1}{v} \frac{\partial \Phi(\vec{r}, t)}{\partial t} = D \nabla^2 \Phi(\vec{r}, t) + v \Sigma_f \Phi(\vec{r}, t) - \Sigma_a \Phi(\vec{r}, t) + S(\vec{r}, t) \quad (2.78)$$

$$\Phi(\vec{r}, t = 0) = \Phi_0(\vec{r}) \quad ; \quad \Phi(\vec{r}_s, t) = 0 \quad (2.78')$$

First you can think of rewriting the flux  $\Phi(\vec{r}, t)$  through a series that best represents the function: the solution of the differential equation can then be expressed as a series of spatial functions known in any instant of time. Therefore each solution to the problem will take the form:

$$\Phi(\vec{r}, t) = \sum_{n=1}^{\infty} A_n(t) \varphi_n(\vec{r}) \quad (2.79)$$

$A_n(t)$  is a coefficient of development in series, while  $\varphi_n(\vec{r})$  is a function used for the development chosen a priori in such a way that it is easier to find the coefficients  $A_n(t)$ . In particular I want to use a basis of functions that are orthonormalized and therefore respect the following condition: (2.80)

$$(\varphi_n, \varphi_m) = \int \varphi_n(\vec{r}) \cdot \varphi_m(\vec{r}) d\vec{r} = \delta_{nm}$$

Within the differential equation (2.78) I go on to put the development in series:

$$\frac{1}{v} \frac{\partial}{\partial t} \sum_{n=1}^{\infty} A_n(t) \varphi_n(\vec{r}) = D \nabla^2 \sum_{n=1}^{\infty} A_n(t) \varphi_n(\vec{r}) + (v \Sigma_f - \Sigma_a) \sum_{n=1}^{\infty} A_n(t) \varphi_n(\vec{r}) + S(\vec{r}, t) \quad (2.81)$$

Regarding the source term, let's see in detail if it is possible to express it in terms of series:

$$S(\vec{r}, t) \cdot \varphi_m(\vec{r}) = \left( \sum_{n=1}^{\infty} S_n(t) \varphi_n(\vec{r}) \right) \cdot \varphi_m(\vec{r}) \quad (2.82)$$

Both members are integrated in the space and I bring out the terms that are only a function of time, getting the following expression.

$$\int d\vec{r} S(\vec{r}, t) \cdot \varphi_m(\vec{r}) = \sum_{n=1}^{\infty} S_n(t) \int d\vec{r} \varphi_n(\vec{r}) \cdot \varphi_m(\vec{r}) = S_m(t) \quad (2.82')$$

The various components of the source were found in this way: making them integral in space and multiplying them by  $\varphi_m$ . I practice the same procedure for each term belonging to equation (2.81):

$$\sum_{n=1}^{\infty} A_n(t) \int d\vec{r} \varphi_n(\vec{r}) \cdot \varphi_m(\vec{r}) = A_m(t)$$

$$\frac{1}{v} \sum_{n=1}^{\infty} \frac{dA_n(t)}{dt} \int d\vec{r} \varphi_n(\vec{r}) \cdot \varphi_m(\vec{r}) = \frac{1}{v} \frac{dA_m(t)}{dt} \quad (2.83)$$

$$\sum_{n=1}^{\infty} A_n(t) \nabla^2 \varphi_n(\vec{r}) = - \sum_{n=1}^{\infty} A_n(t) B_n^2 \int d\vec{r} \varphi_n(\vec{r}) \cdot \varphi_m(\vec{r}) = - B_m^2 A_m(t)$$

Let's recompose the starting equation:

$$\frac{1}{v} \frac{dA_m(t)}{dt} = -DB_m^2 A_m(t) + (v\Sigma_f - \Sigma_a)A_m(t) + S_m(t) \quad (2.84)$$

$$\frac{dA_m(t)}{dt} = v(v\Sigma_f - \Sigma_a - DB_m^2)A_m(t) + vS_m(t) = \alpha_m A_m(t) + vS_m(t) \quad (2.84')$$

I thus found a simple decay equation whose solution is:

$$A_m(t) = A_m(0)e^{\alpha_m t} + v \int_0^t dt' S_m(t') e^{\alpha_m(t-t')} \quad (2.85)$$

With  $A_m(0)$  which represents the initial state of  $A_m(t)$ , in fact:

$$\Phi(\vec{r}, t = 0) = \Phi_0(\vec{r}) = \sum_{n=1}^{\infty} A_n(0) \varphi_n(\vec{r}) \Rightarrow \int \Phi_0(\vec{r}) \varphi_m(\vec{r}) d\vec{r} = A_m(0) \quad (2.86)$$

In conclusion, the solution will be written as:

$$\Phi(\vec{r}, t) = \sum_{m=1}^{\infty} \left( A_m(0)e^{\alpha_m t} + v \int_0^t dt' S_m(t') e^{\alpha_m(t-t')} \right) \varphi_m(\vec{r}) \quad (2.87)$$

Regarding  $\varphi_m$  functions to be used, in addition to taking functions that respect the condition of orthonormality, I want the functions thus obtained with the laplacian operator not to be too complicated and also to respect the boundary conditions.

$$\begin{cases} \nabla^2 \varphi(\vec{r}) = -B^2 \varphi(\vec{r}) \\ \varphi(\vec{r}_0) = 0 \end{cases} \quad (2.88)$$

The system (2.88) represents the Helmholtz problem: let's look at the resolution of this problem for the simple case of bare slab.

### Bare Slab

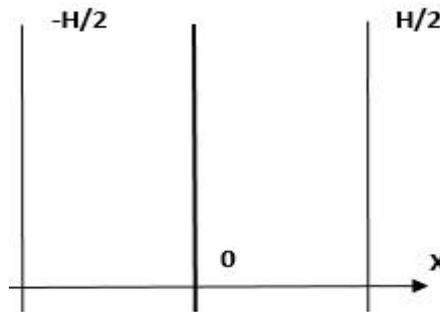


Fig. 6. Representation of bare slab under study.

First of all we see what form the starting equation must have, whose boundary conditions are the cancellation at the two borders  $H/2$  and  $-H/2$ . Assuming the coefficient multiplying the  $\varphi(x)$  function to be positive, the solution would be an exponent composition:

$$\frac{d\varphi(x)}{dx} = a\varphi(x) \Rightarrow \varphi(x) = A \cdot e^{\sqrt{a}x} + B \cdot e^{-\sqrt{a}x} \quad (2.89)$$

However, this type of solution can be canceled in one or the other desired point but not in  $H/2$  and  $-H/2$  at the same time. So the only possible alternative is to choose a negative coefficient that leads to a solution that is a combination of sine and cosine:

$$\frac{d\varphi(x)}{dx} = -B^2 \varphi(x) \Rightarrow \varphi(x) = A \cdot \cos(Bx) + C \cdot \sin(Bx) \quad (2.90)$$

I now impose boundary conditions in order to find constants:

$$\begin{cases} A \cdot \cos\left(\frac{BH}{2}\right) + C \cdot \sin\left(\frac{BH}{2}\right) = 0 \\ A \cdot \cos\left(\frac{BH}{2}\right) - C \cdot \sin\left(\frac{BH}{2}\right) = 0 \end{cases} \quad (2.91)$$

Solving the linear system (2.91) the rank of the matrix must be equal to 1, so:

$$\begin{vmatrix} \cos\left(\frac{BH}{2}\right) & \sin\left(\frac{BH}{2}\right) \\ \cos\left(\frac{BH}{2}\right) & \sin\left(\frac{BH}{2}\right) \end{vmatrix} = 0 \Rightarrow \cos\left(\frac{BH}{2}\right) \cdot \sin\left(\frac{BH}{2}\right) = 0 \quad (2.91')$$

It is possible to find the solution in several ways, let's start to see what happens canceling the cosine function, seeing first where the function is canceled and then replacing in one of the two system equations (2.91) with the aim of finding a solution for  $\varphi(x)$  :

$$\begin{aligned} \cos\left(\frac{BH}{2}\right) = 0 &\Rightarrow \frac{B_n H}{2} = \frac{2n-1}{2}\pi, \quad n = 0,1,2, \dots \\ A \cdot 0 + C \cdot \sin\left(\frac{B_n H}{2}\right) &= 0 \Rightarrow C = 0 \end{aligned} \quad (2.92)$$

In this case the solution found is in the form:

$$\varphi_n(x) = A \cdot \cos\left(\frac{2n-1}{H}\pi x\right); \quad \varphi_1(x) = A \cdot \cos\left(\frac{\pi x}{H}\right) \quad (2.92')$$

The solution for  $n = 1$  is called fundamental eigenfunction and is among the solutions found the most particular because it is the only positive on the whole domain. We now reset the sine function and proceed as previously done:

$$\begin{aligned} \sin\frac{BH}{2} = 0 &\Rightarrow \frac{B_m H}{2} = m\pi, \quad m = 0,1,2, \dots \\ A \cdot \cos\frac{B_m H}{2} + C \cdot 0 &= 0 \Rightarrow A = 0 \end{aligned} \quad (2.93)$$

With the cancellation of the sine function the solution has the form:

$$\begin{aligned} \varphi_m(x) &= C \cdot \sin\left(\frac{2m}{H}\pi x\right) \\ \varphi_1(x) &= C \cdot \sin\left(\frac{2\pi x}{H}\right) \end{aligned} \quad (2.93')$$

Going back to the previous case, cancellation of cosine, let's now check for which values of  $A$  the orthonormality condition is respected:

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} dx \cdot \varphi_n^2(x) = A^2 \int_{-\frac{H}{2}}^{\frac{H}{2}} \cos^2\left(\frac{2n-1}{H}\pi x\right) dx = 1 \Rightarrow A = \sqrt{\frac{2}{H}} \quad (2.94)$$

In conclusion, the following normalized orthogonal set of solutions was found:

$$\varphi_n(x) = \sqrt{\frac{2}{H}} \cdot \cos\left(\frac{2n-1}{H}\pi x\right) \quad (2.95)$$

## 2.6-REFLECTED REACTOR

The amount of fuel inside a reactor can decrease if the reactor core is surrounded with non-multiplying scattering material, an unfueled region of moderator. This material acts precisely as a reflector because it sends back part of those neutrons that would have escaped from the reactor back into the core. Therefore a reflector has the property of decreasing the neutron leakage from the reactor and allows a given fuel/moderator system to become critical when the dimensions of the core are significantly smaller than those required for a bare reactor, thus allowing fuel savings. As for the diffusion equation, to describe a reflector system, it will be necessary to divide the reactor into two different zones and write an equation for the core and one for the reflector. Let's see how the two equations are, the one concerning the reflector is the one with "(R)" as apices.

$$\partial\Phi(\vec{r}, t)/\partial t = vD\nabla^2\Phi(\vec{r}, t) + v(k_\infty - 1)\Sigma_a\Phi(\vec{r}, t) + vS(\vec{r}, t) \quad (2.96)$$

$$\partial\Phi^{(R)}(\vec{r}, t)/\partial t = vD\nabla^2\Phi^{(R)}(\vec{r}, t) - v\Sigma_a\Phi^{(R)}(\vec{r}, t) + vS^{(R)}(\vec{r}, t) \quad (2.96')$$

To find the solutions of the two flows,  $\Phi$  and  $\Phi^{(R)}$ , in addition to the two initial conditions and the border conditions, which will only concern the flux of the reflector, it is also necessary to impose the conditions at the interface, that is the area where the properties of the material change and the slowing down properties of the reflector are generally different from those of the core. In particular, the two conditions to be imposed on the interface concern the continuity of the flux and the current.

Now we rewrite (2.96) and (2.96 ') in stationary case and without the presence of external sources, in the specific geometric case of the infinite slab having the core and the reflector thick  $H_C$  and  $H_R$  respectively, setting the origin of the coordinate in the middle of the system and taking into account only the positive  $x$  values, given precisely the symmetry with respect to the center of the system.

$$D\nabla^2\Phi(x) + (k_\infty - 1)\Sigma_a\Phi(x) = 0 \quad (2.97)$$

$$D^{(R)}\nabla^2\Phi^{(R)}(x) - \Sigma_a^{(R)}\Phi^{(R)}(x) = 0 \quad (2.97')$$

Recalling the definition (2.77) of the geometric buckling in a critical case and the definition of the diffusion length, the two equations become:

$$\nabla^2\Phi(x) + B_C^2\Phi(x) = 0 \quad (2.98)$$

$$\nabla^2\Phi^{(R)}(x) - \frac{1}{L^{(R)}}\Phi^{(R)}(x) = 0 \quad (2.98')$$

The two solutions will respectively be a combination of sine and cosine, in which I cancel the term sine since I cannot have values of the negative flux, and a combination of sine and hyperbolic cosine, (2.99) and (2.99').

$$\Phi(x) = A \cos(B_C x) \quad (2.99)$$

$$\Phi^{(R)}(x) = A' \cosh\left(\frac{x}{L^{(R)}}\right) + C' \sinh\left(\frac{x}{L^{(R)}}\right) \quad (2.99')$$

Now applying the boundary condition to the reflector flux, that is the cancellation at the border  $H = H_C + H_R$  :

$$\Phi^{(R)}(H) = A' \cosh\left(\frac{H}{L^{(R)}}\right) + C' \sinh\left(\frac{H}{L^{(R)}}\right) = 0 \Rightarrow A' = -C' \tanh\left(\frac{H}{L^{(R)}}\right) \quad (2.100)$$

Entering the constant  $A'$  in (2.99 '):

$$\begin{aligned} \Phi^{(R)}(x) &= -C' \frac{\sinh\left(\frac{H}{L^{(R)}}\right)}{\cosh\left(\frac{H}{L^{(R)}}\right)} \cosh\left(\frac{x}{L^{(R)}}\right) + C' \sinh\left(\frac{x}{L^{(R)}}\right) = \\ &= \frac{C'}{\cosh\left(\frac{H}{L^{(R)}}\right)} \left[ -\sinh\left(\frac{H}{L^{(R)}}\right) \cosh\left(\frac{x}{L^{(R)}}\right) + \cosh\left(\frac{H}{L^{(R)}}\right) \sinh\left(\frac{x}{L^{(R)}}\right) \right] = \\ &= \frac{C'}{\cosh\left(\frac{H}{L^{(R)}}\right)} \sinh\left(\frac{H-x}{L^{(R)}}\right) = C \sinh\left(\frac{H-x}{L^{(R)}}\right) \end{aligned} \quad (2.101)$$

Two conditions are now applied to the interface, with the aim of finding the constants  $A$  and  $C$  , the continuity of the flux and of the current:

$$\begin{cases} \Phi(H_C^-) = \Phi^{(R)}(H_C^-) \\ D \frac{d\Phi(x, t)}{dx} \Big|_{x=H_C^-} = D^{(R)} \frac{d\Phi^{(R)}(x, t)}{dx} \Big|_{x=H_C^+} \end{cases} \quad (2.102)$$

$$\begin{cases} A \cos(B_C H_C) = C \sinh\left(\frac{H-H_C}{L^{(R)}}\right) = C \sinh\left(\frac{H_R}{L^{(R)}}\right) \\ A D B_C \sin(B_C H_C) = C \frac{D^{(R)}}{L^{(R)}} \cosh\left(\frac{H-H_C}{L^{(R)}}\right) = C \frac{D^{(R)}}{L^{(R)}} \cosh\left(\frac{H_R}{L^{(R)}}\right) \end{cases} \quad (2.102')$$

Finally dividing the second equation of the system (2.102') by the first one we find the following transcendental equation:

$$\frac{D^{(R)}}{L^{(R)}} \coth\left(\frac{H_R}{L^{(R)}}\right) = D B_C \tan(B_C H_C) \quad (2.103)$$

This transcendent equation is the critical equation for a reflected reactor in infinite slab geometry in a one-group approximation. Moreover, this equation, having all the properties of the two areas involved, core and reflector, provides the critical value of the thickness  $H_C$  corresponding to the thickness  $H_R$  of the reflector and it anticipates, in the form in which it was obtained, the characteristic equation that we will see later.

### 3-INTRODUCTION TO NUCLEAR REACTOR KINETICS

In the previous paragraphs, the situation in which the reactor is critical was mainly treated. Critical reactor means having a constant power level due to the equilibrium between the production rate of neutrons generated by fission and those lost due to absorption of the materials constituting the reactor and due to leakage. Any type of deviation from this equilibrium causes the neutron population, and consequently the power, to be time-dependent: in this chapter the behavior of neutrons is treated in case of non-critical reactor and therefore the nuclear reactor kinetics is introduced<sup>2</sup>.

First of all, let us remember that reactor physics is a very diverse field in which we often deal with physical phenomena that evolve over different time scales. For example, there are the fast ready neutrons ( $\sim 10^{-3} \text{ ms}$ ), ready thermal ones ( $\sim 1 \text{ ms}$ ) and delayed ones ( $1 \div 100 \text{ s}$ ) which moving at different speeds and being emitted with different times and all contribute differently to the neutron balance. Again, there are time scales concerning thermal phenomena and the check of control bars and neutron poisons ( $1 \div 100 \text{ s}$ ) or the temporal scale of transmutation of fission products (*weeks*  $\div$  *years*). The variety of physical phenomena and operative time scales means that the neutron population's time dependence can be divided into three main classes of problems:

- **Short time problems:** whose time scale goes from a few seconds to a tens of minutes, are those problems that arise during power transients and safety analysis calculations and where fuel depletion is ignored. A concrete example may be the demand for steam that can suddenly change due to a change in the turbine load. In a BWR reactor this change leads to a pressure variation inside the vessel, while in a PWR reactor in an alteration of the temperature in the primary coolant system: in both cases the neutron moderation rate changes and consequently also the multiplication factor  $k$ . As will be explained further below, these problems are solved by taking on that the shape of flux does not change over time assuming that the reactor acts as a point, hence the name point kinetics.

<sup>2</sup> The general references of this chapter are (Lamarsh and Baratta, 1975) and (Duderstadt and Hamilton, 1976).

- Intermediate time problems: whose time scale is of the order of a few hours up to one/two days, also in this case the fuel depletion is ignored, but the change in the characteristics of the fission products due to the radioactive decay is taken into consideration. In fact, since fission products, which are produced in large quantities, have a large cross-section of absorption of thermal neutrons, their variation in time must necessarily be taken into account as they cause a variation of the multiplication factor  $k$ .
- Long time or depletion problems: whose involve time periods of days to months. These problems concern the variation of the flow over long periods of time and therefore it is necessary to know the depletion of fuel and the distribution of both the flux and the fuel inside the reactor.

### 3.1-THE POINT REACTOR KINETICS MODEL

Until now in the preceding paragraphs, neutrons were considered in their totality as prompt and in the case of a thermal reactor, for example, the average time between the birth and the disappearance of neutrons is in the order of  $10^{-3} \div 10^{-4}$ . However a small fraction of neutrons (about 0.75%) are emitted by fission products with an appreciable delay, about  $0.1 \div 55$  s : it is thanks to this kind of neutrons, which have the ability to accumulate, if it is possible to control a power reactor according to human reaction times, as we will see later. The delayed neutrons can be grouped into six main families, each characterized by a different decay constant  $\lambda_i$  and a different fraction of delayed neutrons  $\beta_i$ , as can be seen in the table Tab. 1 below.

Group	$\lambda_i$ ( $s^{-1}$ )	$\beta_i$ (pcm)
1	0.0124	21,5
2	0.0305	142,4
3	0.111	127,4
4	0.301	256,8
5	1.14	74,8
6	3.01	27,3

**Tab. 1.** Main properties of the neutron precursors families.

The one-speed diffusion model seen previously can also be used to describe the reactor kinetics, implementing the equation with the presence of delayed neutrons. In particular in the nuclear reactor kinetics, the one-speed diffusion model is considerably simplified by the point reactor kinetics model, which presupposes a separation between amplitude and shape with the assumption that the spatial dependence of the

flux in the reactor can be described by a single spatial mode shape, the fundamental mode. Going then to study the temporal behavior of the reactor as long as the shape does not change, it is possible to remove the spatial dependence of the diffusion model and arrive at a description involving only ordinary differential equations in time.

### 3.1.1-THE IMPORTANCE OF DELAYED NEUTRONS IN REACTOR KINETICS

Let us remember the solution of the diffusion equation in the non-stationary case is a superposition of spatial eigenstate, here we rewrite it in the case where an external source is not present:

$$\Phi(\vec{r}, t) = \sum_{n=1}^{\infty} A_n(0)e^{\alpha_n t} \varphi_n(\vec{r}) \quad (3.1)$$

Considering long times, the solution approaches an asymptotic form, and therefore I can consider the only fundamental eigenfunction and eliminate the summation term:

$$\Phi(\vec{r}, t) \cong A_1(0)e^{\alpha_1 t} \varphi_1(\vec{r}) \quad (3.2)$$

Let us now analyze the time eigenvalue  $\alpha_1$  in detail:

$$\begin{aligned} \alpha_1 &= v(v\Sigma_f - \Sigma_a - DB_1^2) = v\Sigma_a \left( v \frac{\Sigma_f}{\Sigma_a} - 1 - \frac{D}{\Sigma_a} B_1^2 \right) = \\ &= v\Sigma_a (k_\infty - 1 - L^2 B_1^2) = v\Sigma_a (1 + L^2 B_1^2) \left( \frac{k_\infty}{1 + L^2 B_1^2} - 1 \right) \end{aligned} \quad (3.3)$$

From (3.3) it is possible to extrapolate the following two quantities, useful for what will be dealt with:

$$l = \frac{1}{v\Sigma_a(1 + L^2 B_1^2)} \Rightarrow \text{mean lifetime of neutron in reactor} \quad (3.4)$$

$$k = \frac{k_\infty}{1 + L^2 B_1^2} \Rightarrow \text{effective multiplication factor} \quad (3.4')$$

Returning to (3.2) it is possible to rewrite it in the following manner

$$\Phi(\vec{r}, t) \cong A_1 \exp(\alpha_1 t) \varphi_1(\vec{r}) = A_1 \exp \left[ \left( \frac{k-1}{l} \right) t \right] \varphi_1(\vec{r}) \quad (3.5)$$

About how long this asymptotic behavior takes to establish itself, it is possible to determine it by assuming that the reactor operates in critical conditions,  $\alpha_1 = 0$ , and then estimating  $\alpha_n$ . Recalling being the geometric buckling for a slab equal to  $B_n = n^2(\pi/H)^2$ , then:

$$\alpha_n = -vD(B_n^2 - B_1^2) = -vD(n^2 - 1)(\pi/H)^2 \quad (3.6)$$

Taking typical values for a thermal reactor,  $H \sim 300 \text{ cm}$   $\wedge$   $v \sim 3 \cdot 10^5 \text{ cm/s}$   $\wedge$   $D = 1 \text{ cm}$ ,  $\alpha_n$  is of the order of  $100 \div 1000 \text{ s}^{-1}$  and this implies that its contribution vanishes very quickly. Therefore, not taking into account the contribution made to the solution by higher-order eigenfunctions, it is possible to write the flux using only the fundamental eigenfunction, separating shape and amplitude:

$$\Phi(\vec{r}, t) = v n(t) \varphi_1(\vec{r}) \quad (3.7)$$

Replacing this flow form in the one-speed diffusion equation, we find that  $n(t)$  satisfies the relation:

$$\frac{dn(t)}{dt} = \left( \frac{k-1}{l} \right) n(t) \quad (3.8)$$

In this case  $n(t)$  represents the total number of neutrons present inside the reactor at the instant " $t$ ". Given the arbitrary normalization of  $n(t)$ , it is also possible to scale this dependent variable in order to express the total instantaneous power  $P(t)$  generated inside the reactor at the instant " $t$ " as:

$$n(t) \Rightarrow P(t) = w_f v \Sigma_f n(t) \quad (3.8')$$

Where  $w_f$  represents the usable energy released by fission.

We can see how useful it is to express the time dependence of the reactor with the power since  $P(t)$  is an easily monitored variable. Equation (3.8) somewhat recalls the point reactor kinetics equations, which will later be derived. Returning instead to (3.7), inserting it in the solution of (3.8), shows us how the temporal behavior of the neutron flux can be expressed as the product between shape factor  $\varphi_1(\vec{r})$  and time-dependent amplitude factor  $n(t)$  :

$$\Phi(\vec{r}, t) = v n(t) \varphi_1(\vec{r}) = v n_0 \exp \left[ \left( \frac{k-1}{l} \right) t \right] \varphi_1(\vec{r}) \quad (3.9)$$

This description of the behavior of a reactor is characterized by the time constant:

$$T = \frac{l}{k-1} \Rightarrow \text{reactor period} \quad (3.10)$$

However, something very important was forgotten in the description of this model. Assuming the term fission equal to  $v \Sigma_f \Phi(\vec{r}, t)$  it has been assumed that all the neutrons produced are ready, but we know that a small fraction ( $\beta \sim 0.7\%$ ) of these is emitted with a certain delay. In fact, remembering that the mean lifetime of a neutron in a thermal reactor is of the order of  $10^{-4}$  sec, it is clear that the reactor period

predicted in this way is too small to allow an effective control on the reactor. It is possible to estimate the influence that delayed neutrons have on the temporal behavior of the reactor noting that the effective lifetime of these neutrons is given by their prompt neutron lifetime  $l$  plus the additional delay time  $\lambda_i^{-1}$  that characterizes their precursors. Ultimately, by weighing the ready and delayed neutrons according to their respective fractions, the average lifetime which characterizes all fission neutrons is expressed in the following manner:

$$\langle l \rangle = (1 - \beta)l + \sum_{i=1}^6 \beta_i \left[ \frac{1}{\lambda_i} + l \right] \quad (3.11)$$

Using the data of the delayed neutrons listed in Tab.1, the average lifetime is typically equal to  $\langle l \rangle \sim 0.1$  sec, much greater if only the ready neutrons were considered, which is in fact of the order of  $l \sim 10^{-4} \div 10^{-6}$  sec, as seen before. In conclusion we note that even such a small fraction of delayed neutrons is fundamental since it considerably increases the reactor period  $T$  and allows a reactor to be controlled according to human intervention times.

### 3.1.2-DERIVATION OF THE POINT REACTOR KINETICS EQUATIONS

To get started we need to build the balance equation that relates only delayed neutrons. In this respect we define the precursor atomic number density, i.e. the number of expected precursors of the  $i$ -th family present within the volume  $d\vec{r}$  and that always decays by emitting a delayed neutron:

$$C_i(\vec{r}, t) d\vec{r} \quad (3.12)$$

The balance is constructed taking into account the number of precursors per unit of time that decay in the volume  $d\vec{r}$  (3.13) and the number of precursors produced per unit of time in the volume  $d\vec{r}$  (3.14).

$$\lambda_i C_i(\vec{r}, t) d\vec{r} \quad (3.13)$$

$$\beta_i \nu \Sigma_f \Phi(\vec{r}, t) d\vec{r} \quad (3.14)$$

Of the aforementioned terms, neither depends on space since fission products have very limited mobility and therefore the equation of retreated neutrons will have no space-dependent term and will need only initial conditions to be resolved. Thus the balance equation for delayed neutrons is as follows:

$$\frac{dC_i(\vec{r}, t)}{dt} = -\lambda_i C_i(\vec{r}, t) + \beta_i \nu \Sigma_f \Phi(\vec{r}, t) \quad (3.15)$$

As regards the diffusion equation, the contribution of the delayed neutrons is introduced directly to the fission source, indicating with  $\beta$  the total fraction of delayed neutrons:

$$S_f(\vec{r}, t) = (1 - \beta)v\Sigma_f\Phi(\vec{r}, t) + \sum_{i=1}^6 \lambda_i C_i(\vec{r}, t) \quad (3.16)$$

In this way we obtain a system composed of two differential equations:

$$\begin{cases} \frac{1}{v} \frac{d\Phi(\vec{r}, t)}{dt} = D\nabla^2\Phi(\vec{r}, t) - \Sigma_a\Phi(\vec{r}, t) + (1 - \beta)v\Sigma_f\Phi(\vec{r}, t) + \sum_{i=1}^6 \lambda_i C_i(\vec{r}, t) \\ \frac{dC_i(\vec{r}, t)}{dt} = -\lambda_i C_i(\vec{r}, t) + \beta_i v\Sigma_f\Phi(\vec{r}, t) \quad ; \quad i = 1, \dots, 6 \end{cases} \quad (3.17)$$

As mentioned before, the simplifying hypothesis that characterizes the point reactor kinetics model consists in considering constant the spatial evolution, the shape  $\varphi_1(\vec{r})$ , of our system but not its evolution over time, the amplitude  $n(t)$ , so as to be able to write the flow and the concentration of precursors such as:

$$\Phi(\vec{r}, t) = vn(t)\varphi_1(\vec{r}) \quad (3.18)$$

$$C_i(\vec{r}, t) = C_i(t)\varphi_1(\vec{r}) \quad (3.19)$$

By replacing (3.18) and (3.19) in the system (3.17) you get:

$$\begin{cases} \frac{dn(t)}{dt} = vn(t)[D\nabla^2 - \Sigma_a + (1 - \beta)v\Sigma_f] + \sum_{i=1}^6 \lambda_i C_i(t) \\ \frac{dC_i(t)}{dt} = -\lambda_i C_i(t) + v\beta_i v\Sigma_f n(t) \quad ; \quad i = 1, \dots, 6 \end{cases} \quad (3.20)$$

Now let's rewrite this system in a more compact way, introducing the terms  $a$  and  $b$  as in the system (3.21).

$$\begin{cases} \frac{dn(t)}{dt} = a n(t) + \sum_{i=1}^6 \lambda_i C_i(t) \\ \frac{dC_i(t)}{dt} = b n(t) - \lambda_i C_i(t) \quad ; \quad i = 1, \dots, 6 \end{cases} \quad (3.21)$$

To arrive at the final form of point kinetics equations we recall the definitions given above for the lifetime of neutrons in reactor  $l$  and the effective multiplication factor  $k$ , adding two new quantities, the reactivity, (3.22), and the effective lifetime of prompt neutrons, (3.22').

$$\rho = \frac{k-1}{k} \Rightarrow \text{reactivity} \quad (3.22)$$

$$\Lambda = \frac{l}{k} \Rightarrow \text{effective lifetime of prompt neutrons} \quad (3.22')$$

Let's analyze the terms  $a$  and  $b$  of the system (3.21):

$$\begin{aligned} a &= v[D\nabla^2 - \Sigma_a + (1-\beta)v\Sigma_f] = v[(1-\beta)v\Sigma_f - \Sigma_a - B^2D] = \\ &= v\Sigma_a \left[ v \frac{\Sigma_f}{\Sigma_a} - v\beta \frac{\Sigma_f}{\Sigma_a} - 1 - B^2 \frac{D}{\Sigma_a} \right] = v\Sigma_a [k_\infty - \beta k_\infty - 1 - L^2 B^2] = \\ &= v\Sigma_a (1 + L^2 B^2) \left[ \frac{k_\infty}{1 + L^2 B^2} - \beta \frac{k_\infty}{1 + L^2 B^2} - 1 \right] = \frac{1}{l} [k - \beta k - 1] = \\ &= \frac{k}{l} \left( \frac{k-1}{k} - \beta \right) = \frac{\rho - \beta}{\Lambda} \end{aligned}$$

$$b = v\beta v\Sigma_f = v\beta v\Sigma_f \frac{\Sigma_a(1 + L^2 B^2)}{\Sigma_a(1 + L^2 B^2)} = \beta \frac{v\Sigma_f}{\Sigma_a} \frac{v\Sigma_a(1 + L^2 B^2)}{(1 + L^2 B^2)} = \beta \frac{k}{l} = \frac{\beta}{\Lambda}$$

The point reactor kinetics equations are thus obtained:

$$\begin{cases} \frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{i=1}^6 \lambda_i C_i(t) \\ \frac{dC_i(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda_i C_i(t) \quad ; \quad i = 1, \dots, 6 \end{cases} \quad (3.23)$$

A set of coupled differential equations was therefore obtained which describe both the temporal behavior of the neutron population and the decay of delayed neutrons precursors within the reactor.

## 3.2-SOLUTION OF THE POINT REACTOR KINETICS EQUATIONS

### 3.2.1-SOLUTION WITH ONE EFFECTIVE DELAYED GROUP

In this paragraph we analyze the very simple case in which the reactor operates in steady-state conditions  $P_0$  at  $t = 0$ , instant in which the reactivity is changed to a non-zero value  $\rho_0$ . Furthermore the delayed neutrons are all grouped in a single effective delayed group, with the consequence that the fraction  $\beta$  and the average decay constant  $\lambda$  are expressed as:

$$\beta = \sum_{i=1}^6 \beta_i \quad (3.24)$$

$$\lambda = \langle \lambda \rangle \equiv \left[ \frac{1}{\beta} \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \right]^{-1} \quad (3.25)$$

In this simplified case the point reactor kinetics equations can be expressed as:

$$\begin{cases} \frac{dP(t)}{dt} = \frac{\rho_0 - \beta}{\Lambda} P(t) + \lambda C(t) \\ \frac{dC(t)}{dt} = \frac{\beta}{\Lambda} P(t) - \lambda C(t) \end{cases} \quad (3.26)$$

We say that at the instant  $t = 0$  the power has a stationary value  $P_0$ , so that it is possible to find the value for the same instant of the precursors. In fact, for  $t < 0$  is valid the relation:

$$\frac{dP}{dt} = \frac{dC}{dt} \Rightarrow C_0 = \frac{\beta}{\lambda \Lambda} P_0 \quad (3.27)$$

Both the initial conditions necessary to solve the problem are thus available. There are various ways to solve the system (3.26), here we try an elementary approach looking for an exponential solution:

$$P(t) = P e^{st} \quad ; \quad C(t) = C e^{st} \quad (3.28)$$

Substituting these solutions in the initial system we find the following algebraic equations:

$$\begin{cases} sP = \left( \frac{\rho_0 - \beta}{\Lambda} \right) P + \lambda C \\ sC = \frac{\beta}{\Lambda} P - \lambda C \end{cases} \quad (3.29)$$

This set of coupled algebraic equations has a solution if and only if:

$$\left[ s - \left( \frac{\rho_0 - \beta}{\Lambda} \right) \right] (s + \lambda) - \frac{\lambda \beta}{\Lambda} = 0 \Rightarrow \Lambda s^2 + (\lambda \Lambda + \beta - \rho_0) s - \rho_0 \lambda \quad (3.30)$$

A second degree equation has been obtained for the parameter  $s$ , which can therefore take two distinct values:

$$s_{1,2} = \frac{1}{2\Lambda} \left[ -(\lambda \Lambda + \beta - \rho_0) \pm \sqrt{(\lambda \Lambda + \beta - \rho_0)^2 - 4\lambda \Lambda \rho_0} \right] \quad (3.31)$$

The solutions of power and concentration of the precursors therefore they can be expressed as follows in (3.32) and (3.33).

$$P(t) = P_1 \exp(s_1 t) + P_2 \exp(s_2 t) \quad (3.32)$$

$$C(t) = C_1 \exp(s_1 t) + C_2 \exp(s_2 t) \quad (3.33)$$

To find the coefficients of equations (3.32) and (3.33) it is necessary to apply the initial conditions to the system (3.29) by obtaining four equations, one for each coefficient to be found. As we will thus have cumbersome solutions, let us analyze a more simple case in which  $(\lambda\Lambda + \beta - \rho_0)^2 \gg 4\lambda\Lambda\rho_0$  and  $|\rho_0| \ll \beta$  for which  $s_1$  and  $s_2$  assume the values:

$$s_1 \cong \frac{\lambda\rho_0}{\beta - \rho_0} \quad ; \quad s_2 \cong -\frac{\beta - \rho_0}{\Lambda} \quad (3.34)$$

Values corresponding to a power  $P_0$  equal to:

$$P(t) \cong P_0 \left\{ \left( \frac{\beta}{\beta - \rho_0} \right) \exp \left[ \left( \frac{\lambda\rho_0}{\beta - \rho_0} \right) t \right] - \left( \frac{\rho_0}{\beta - \rho_0} \right) \exp \left[ - \left( \frac{\beta - \rho_0}{\Lambda} \right) t \right] \right\} \quad (3.35)$$

The power  $P(t)$  trend is shown in Figure Fig.7 with both positive and negative insertion of reactivity equal to  $|\rho_0| = 0.0025$  and with the others reactor parameters  $\beta = 0.0075$ ,  $\lambda = 0.08 \text{ sec}^{-1}$ ,  $\Lambda = 10^{-3} \text{ sec}^{-1}$ . From the reported trends it is clear that after a rapid transient ( $s_2^{-1} \sim 0.2 \text{ s}$ ) the response of the reactor becomes exponential with a period equal to  $T = s_1^{-1} \sim 25 \text{ s}$ , which we identify as the stable reactor period.

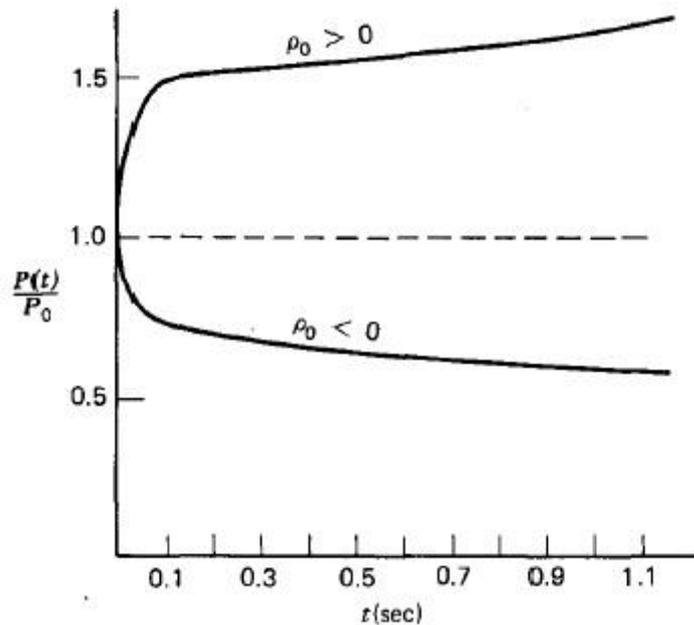


Fig. 7. Reactor power time behavior after reactivity insertion.

### 3.2.2-THE INHOUR EQUATION

The equation that defines the time constant  $s_1$  and  $s_2$  in terms of  $\rho_0$ ,  $\beta$  and  $\lambda$  can be written differently. In fact, using the definitions of the reactivity and the effective lifetime of prompt neutrons seen previously, the equation (3.30) can be rewritten as:

$$\rho_0 = \frac{sl}{sl + 1} + \frac{sl}{sl + 1} \frac{\beta}{s + \lambda} \quad (3.36)$$

This equation identifies the decay constant  $s$  for each constant reactivity value  $\rho_0$ . Generalizing, and therefore supposing to work with all six main families of precursors, we obtain:

$$\rho_0 = \frac{sl}{sl + 1} + \frac{sl}{sl + 1} \sum_{i=1}^6 \frac{\beta_i}{s + \lambda_i} \equiv \rho(s) \quad (3.37)$$

This last equation is known as inhour equation and it is convenient to study its roots by adopting graphic techniques. In the figure below the right-hand side of the equation (3.37) is plotted for various values of  $s$ .

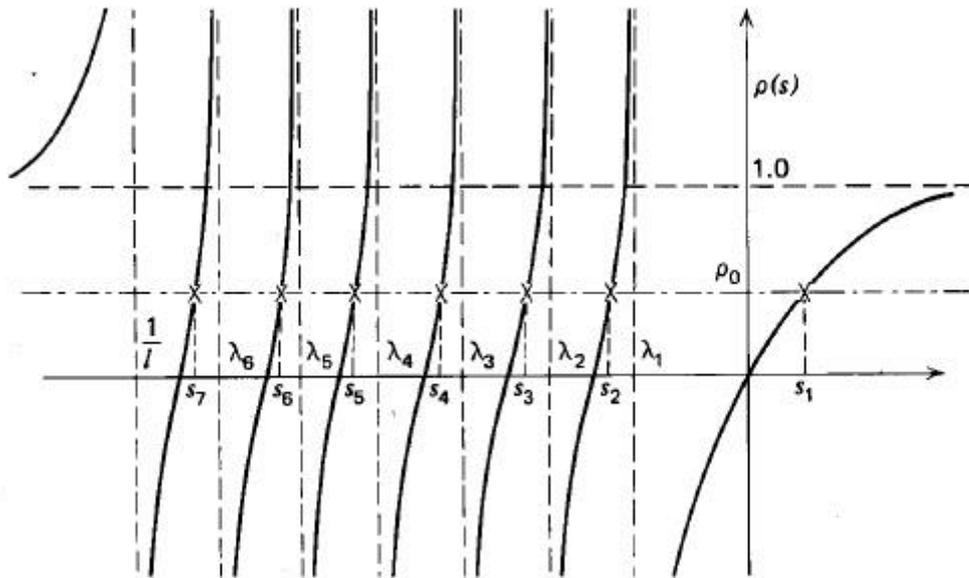


Fig. 8. Graphical determination of the roots to inhour equation.

The intersections of the curves found with the straight line that identifies the desired reactivity value  $\rho = \rho_0$ , give us the seven decay constant  $s_j$  that characterize the temporal behavior of the system:

$$P(t) = \sum_{i=1}^7 P_j \exp(s_j t) \quad (3.38)$$

Also from the graph it can be seen how the decay constant more to the right  $s_0$  can be identified as the reciprocal of the period of the reactor  $s_1 = T$ . All the remaining roots, those for  $j > 1$ , are considered as transients that are extinguished as soon as the reactivity  $\rho_0$  is inserted inside the reactor. We note how the reactivity is included between the two extremes:

$$\infty < \rho = \frac{k-1}{k} < 1 \quad (3.39)$$

Three main limit cases are identified:

- Critical:  $\rho_0 = 0 \Rightarrow s_1 = 0$
- Supercritical:  $\rho_0 \rightarrow 1 \Rightarrow s_1 \rightarrow +\infty$
- Subcritical:  $\rho_0 \rightarrow -\infty \Rightarrow s_1 \rightarrow -\lambda_1$

The last of the three is particularly interesting as it makes us understand that no matter how much negative reactivity is inserted into the reactor as it will not be able to switch off any faster than its period  $T = \lambda_1^{-1}$ , determined by the longest-lived delayed neutron precursors.

There are also three other interesting cases corresponding to three different reactivity insertions:

A) Small reactivity insertions,  $\rho_0 \ll \beta$

It is assumed that the magnitude of  $s_0$  is so small as to be

$$|s_0| \ll \lambda_1 < \lambda_2 \dots < l^{-1} \quad (3.40)$$

It is therefore reasonable to assume the reduced inhour equation as follows:

$$\rho_0 \cong s_1 l + s_1 \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \quad (3.41)$$

Hence the reactor period is given in this case by:

$$T = \frac{1}{s_1} = \frac{1}{s_0} \left[ l + \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \right] \cong \frac{\langle l \rangle}{\rho_0} \cong \frac{\langle l \rangle}{k-1} \quad (3.42)$$

Therefore for small insertions of reactivity the reactor period  $T$  is determined by the average neutron lifetime  $\langle l \rangle$  which also includes the delayed neutrons.

B) Large positive reactivity insertion,  $\rho_0 \gg \beta$

In this case it is possible to take  $s_1 \gg \lambda_1$  in writing the inhour equation:

$$\rho_0 = \frac{s_1}{s_1 + l^{-1}} + \frac{l^{-1}}{s_1 + l^{-1}} \sum_{i=1}^6 \beta_i = \frac{s_1 + \beta l^{-1}}{s_1 + l^{-1}} \quad (3.43)$$

Calculating the reactor period, we find:

$$T = \frac{1}{s_1} \cong \frac{l}{k(\rho_0 - \beta)} \cong \frac{l}{k - 1} \quad (3.44)$$

The result obtained completely ignores the contribution of the delayed neutrons, therefore in conclusion, for large positive reactivity insertion, the reactor response is dictated entirely by the prompt neutron life time  $l$ .

C)  $\rho_0 = \beta$

This latter case represents the "break-point" between the kinetic response governed by the delayed neutrons and that one strictly controlled by the ready neutrons. In reference to the point reactor kinetics equations (3.23), for the reactor to be critical, relying only on ready neutrons, it is necessary that  $\rho_0 = \beta$ . For  $\rho_0 < \beta$ , in order for the reactor criticality to be reached, the contribution of the delayed neutrons is necessary and therefore the response of the reactor over time depends on the delay in the decay that characterizes the precursors. The range  $0 < \rho < \beta$  is commonly referred to as *delayed critical*, while for  $\rho > \beta$  it is referred to as *critical prompt* or *supercritical prompt*.

## 4-THE ONE GROUP DIFFUSION EQUATION FOR THE REFLECTED REACTOR WITH DELAYED NEUTRONS

### 4.1-INTRODUCTION TO THE PROBLEM

I approached to the problem starting from the time depending equations, which I applied to a multiplying system in one dimensional slab geometry with the approximation of the one group diffusion model and in the presence of delayed emissions. In particular I consider a two-zoned reactor with a total thickness of  $2H = 2(H_C + H_R)$  consisting in a homogenous core of thickness  $2H_C$  reflected by two slabs of thickness  $H_R$ , both having constant material properties, and I treated the whole system as symmetrical with respect the plane  $x = 0$  due to the symmetry properties assumed for the excitation caused by the initial state and the sources, if there are.

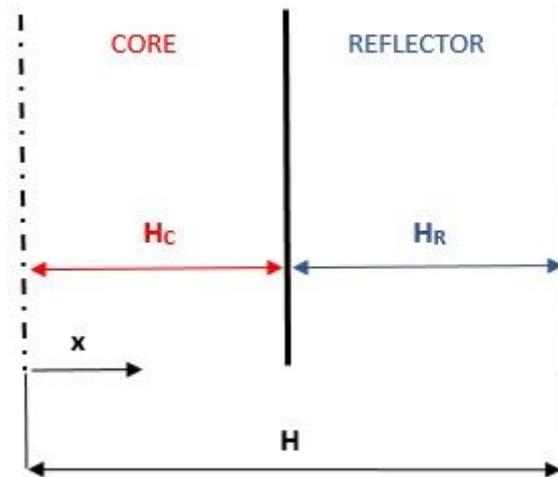


Fig. 9. Qualitative representation of the geometry of the reflected system.

Recalling the system (3.17), and adapting it to the one-dimensional case with the addition of the presence of an external source, the starting equations of neutron kinetics with the diffusion approximation, for the reactor zone can be written as:

$$\begin{cases} \frac{1}{vD} \frac{d\Phi(x,t)}{dt} = \left[ \frac{\partial^2}{\partial x^2} + \frac{k_\infty(1-\beta) - 1}{L^2} \right] \Phi(x,t) + \frac{1}{D} \sum_{i=1}^6 \lambda_i C_i(x,t) + \frac{S(x,t)}{D} \\ \frac{dC_i(x,t)}{dt} = -\lambda_i C_i(x,t) + \beta_i k_\infty \Sigma_a \Phi(x,t) \quad ; \quad i = 1, \dots, 6 \end{cases} \quad (4.1)$$

About the reflector zone I can say that there is not the second equation, but just the first one, without the presence of the delayed neutron precursors and the multiplicative term of course, as the equation (3.96').

$$\frac{1}{vD^{(R)}} \frac{\partial \Phi^{(R)}(x, t)}{\partial t} = \left[ \frac{\partial^2}{\partial x^2} - \frac{1}{L^{(R)2}} \right] \Phi(x, t) + \frac{S^{(R)}(x, t)}{D^{(R)}} \quad (4.2)$$

Thanks to the symmetrical properties I can observe for example, about the source term, that:

$$S(-x, t \geq 0) = S(x, t \geq 0) \quad ; \quad S^{(R)}(-x, t \geq 0) = S^{(R)}(x, t \geq 0) \quad (4.3)$$

I have four conditions that the solution of my system has to overcome:

- a) Symmetry with respect to the plane  $x = 0$  that I can express, when no source is located in the center of my system, like:

$$\left. \frac{\partial \Phi(x, t > 0)}{\partial x} \right|_{x=0} = 0 \quad ; \quad t > 0 \quad (4.4)$$

- b) Absence of the neutron flux outside the boundary of the reflector:

$$\Phi^{(R)}(H, t > 0) = 0 \quad (4.5)$$

- c) Continuity of the neutron flux at the core-reflector interface:

$$\Phi(H_C^-, t) = \Phi^{(R)}(H_C^+, t) \quad ; \quad t > 0 \quad (4.6)$$

- d) Continuity of the neutron current at the core-reflector interface:

$$D \left. \frac{\partial \Phi(x, t)}{\partial x} \right|_{x=H_C^-} = D^{(R)} \left. \frac{\partial \Phi^{(R)}(x, t)}{\partial x} \right|_{x=H_C^+} \quad ; \quad t > 0 \quad (4.7)$$

Regarding the initial conditions for the fluxes and the concentration of the precursors, I will use even functions for the starting situations at time  $t = 0$ :

$$\begin{aligned} \Phi(x, t = 0) &= \Phi_0(|x|) \quad ; \\ \Phi^{(R)}(x, t = 0) &= \Phi^{(R)}_0(|x|) \quad ; \\ C_i(x, 0) &= \begin{cases} C_{i,0}(|x|) & , \quad |x| \leq H_C \\ 0 & |x| > H_C \end{cases} \quad ; \end{aligned} \quad (4.8)$$

The first step is the solution of the equation of the delayed neutrons precursors with the aim to obtain the unknown concentration directly in term of flux. From the physical point of view, as seen above, this is possible thanks the irrelevant diffusion in space of the fission products since I have not a fuel circulating reactor.

Then if I integrate in time the second equation of the system (4.1), it is possible to write the concentration of neutrons precursors like the expression (4.9).

$$C_i(x, t) = C_{i,0}(x)e^{-\lambda_i t} + \beta_i k_\infty \Sigma_a \int_0^t \Phi(x, t') e^{-\lambda_i(t-t')} dt' \quad (4.9)$$

Now I substitute the above expression in the first equation of the reactor system (4.1), in order to obtain a single equation for the core zone. In this way, regarding the core region, I obtain a unique integro-differential equation instead of the system, (4.10).

$$\begin{aligned} \frac{1}{vD} \frac{\partial \Phi(x, t)}{\partial t} = & \left[ \frac{\partial^2}{\partial x^2} + \frac{k_\infty(1-\beta) - 1}{L^2} \right] \Phi(x, t) \\ & + \frac{1}{D} \sum_{i=1}^6 \lambda_i \left[ C_{i,0}(x)e^{-\lambda_i t} + \beta_i k_\infty \Sigma_a \int_0^t \Phi(x, t') e^{-\lambda_i(t-t')} dt' \right] + \frac{S(x, t)}{D} \end{aligned} \quad (4.10)$$

## 4.2-DETERMINATION OF THE SOLUTION BY USING THE LAPLACE TRANSFORM TECHNIQUE

Now the analytical solution of the core and reflector equations, related by the interface conditions, constitutes a unique problem. In order to find the appropriate eigenstates for the reflected reactor problem we can refer to the use of Laplace transform. I have to remember that I can obtain the Laplace transform of a function  $f(x_i, t)$  through the follow integral relation that allows to correlate the real variable  $t$  to the complex one  $p$ :

$$\mathcal{L}_{t \rightarrow p}[f(x_i, t)] = \int_0^{\infty} f(x_i, t) e^{-pt} dt \equiv f_L(x_i, p) \quad (4.11)$$

Thus by applying the Laplace transforms to each term of time dependent equations (4.2) and (4.10), and also to the continuity and boundary conditions, we obtain a system of virtually steady-state equations:

$$\frac{p}{vD} \Phi_L(x, p) - \frac{1}{vD} \Phi_0(x) = \frac{\partial^2 \Phi_L(x, p)}{\partial x^2} + \frac{k_\infty(1-\beta) - 1}{L^2} \Phi_L(x, p) + \frac{1}{D} \sum_{i=1}^6 \frac{\lambda_i C_{i,0}(x)}{p + \lambda_i} + \Phi_L(x, p) \frac{k_\infty}{L^2} \sum_{i=1}^6 \frac{\lambda_i \beta_i}{p + \lambda_i} + \frac{S_L(x, p)}{D} \quad (4.12)$$

$$\frac{p}{vD^{(R)}} \Phi_L^{(R)}(x, p) - \frac{1}{vD^{(R)}} \Phi_0^{(R)}(x) = \frac{\partial^2 \Phi_L^{(R)}(x, p)}{\partial x^2} - \frac{1}{L^{(R)2}} \Phi_L^{(R)}(x, p) + \frac{S_L^{(R)}(x, p)}{D^{(R)}} \quad (4.12')$$

Now I put together the similar terms with the purpose to have a simpler expressions:

$$\frac{\partial^2 \Phi_L(x, p)}{\partial x^2} + \left[ \frac{k_\infty(1-\beta) - 1}{L^2} - \frac{p}{vD} + \frac{k_\infty}{L^2} \sum_{i=1}^6 \frac{\beta_i}{p + \lambda_i} \right] \Phi_L(x, p) = -\frac{1}{D} \left[ \frac{1}{v} \Phi_0(x) + S_L(x, p) + \sum_{i=1}^6 \frac{\lambda_i C_{i,0}(x)}{p + \lambda_i} \right] \quad (4.13)$$

$$\frac{\partial^2 \Phi_L^{(R)}(x, p)}{\partial x^2} - \left[ \frac{1}{L^{(R)2}} + \frac{p}{vD^{(R)}} \right] \Phi_L^{(R)}(x, p) = -\frac{1}{D^{(R)}} \left[ \frac{1}{v} \Phi_0^{(R)}(x) + S_L^{(R)}(x, p) \right] \quad (4.13')$$

I have to remember that the previous equations represent two different zones: then I can use the first ones in the core region while the second one in the reflector region. In the end, to further simplify the relations, I could write (*Corno and Ravetto, 1976*):

$$\frac{\partial^2 \Phi_L(x, p)}{\partial x^2} + \alpha^2(p) \Phi_L(x, p) = m(x, p) \quad , \quad |x| \leq H_C \quad (4.14)$$

$$\frac{\partial^2 \Phi_L^{(R)}(x, p)}{\partial x^2} - \eta^2(p) \Phi_L^{(R)}(x, p) = m^{(R)}(x, p) \quad , \quad H_C \leq |x| \leq H \quad (4.14')$$

Therefore, in the above equalities there are the following functions:

$$\alpha^2(p) = \frac{k_\infty(1 - \beta) - 1}{L^2} - \frac{p}{vD} + \frac{k_\infty}{L^2} \sum_{i=1}^6 \frac{\beta_i}{p + \lambda_i}$$

$$\eta^2(p) = \left[ \frac{1}{L^{(R)^2} + \frac{p}{vD^{(R)}}} \right] \quad (4.15)$$

$$m(x, p) = -\frac{1}{D} \left[ \frac{1}{v} \Phi_0(x) + \frac{S_L(x, p)}{D} + \sum_{i=1}^6 \frac{\lambda_i C_{i,0}(x)}{p + \lambda_i} \right]$$

$$m^{(R)}(x, p) = -\frac{1}{D^{(R)}} \left[ \frac{1}{v} \Phi_0^{(R)}(x) + S_L^{(R)}(x, p) \right]$$

I can notice that I obtain two differential equation, easier respect equations (4.2) and (4.10) of the previous paragraph, for which the solutions are obtained with the usual methods of the calculus:

$$\Phi_L(x, p) = A(p) \cos(\alpha x) + C(p) \sin(\alpha x) + f(x, p) \quad (4.16)$$

$$\Phi_L^{(R)}(x, p) = A^{(R)}(p) e^{\eta x} + C^{(R)}(p) e^{-\eta x} + g(x, p) \quad (4.16')$$

In the formulas above  $A(p)$ ,  $C(p)$ ,  $A^{(R)}(p)$  and  $C^{(R)}(p)$  are determined by the boundary, symmetry and interface conditions. The formulas of the  $f(x, p)$  and  $g(x, p)$  functions are defined as (*Corno and Ravetto, 1976*):

$$f(x, p) = \frac{\sin(\alpha x)}{\alpha} \int_0^x m(x', p) \cos(\alpha x') dx' + \frac{\cos(\alpha x)}{\alpha} \int_x^{H_C} m(x', p) \sin(\alpha x') dx' \quad (4.17)$$

$$g(x, p) = -\frac{e^{\eta x}}{2\eta} \int_{H_C}^x m^{(R)}(x', p) e^{-\eta x'} dx' - \frac{e^{-\eta x}}{2\eta} \int_x^H m^{(R)}(x', p) e^{\eta x'} dx' \quad (4.17')$$

Now, to imposing the conditions of continuity on the interface, the next solutions, (4.18) and (4.18'), are to be found for the L-transform (*Corno and Ravetto, 1976*).

$$\Phi_L(x, p) = A(p) \cos(\alpha x) + f(x, p) \quad (4.18)$$

$$\Phi_L^{(R)}(x, p) = A^{(R)}(p) 2e^{\eta H} \sinh[\eta(H - x)] + f^{(R)}(x, p) \quad (4.18')$$

Where:

$$f^{(R)}(x, p) = -e^{\eta(H-x)} g(H, p) + g(x, p) \quad (4.19)$$

$$A(p) = \frac{a_{22}(p)b_1(p) - a_{12}(p)b_2(p)}{\Delta(p)} \quad (4.20)$$

$$A^{(R)}(p) = \frac{a_{11}(p)b_2(p) - a_{21}(p)b_1(p)}{\Delta(p)} \quad (4.20')$$

$$a_{11}(p) = \cos(\alpha H_C)$$

$$a_{12}(p) = 2e^{\eta H} \sinh(\eta H_R)$$

$$a_{21}(p) = D\alpha \sin(\alpha H_C) \quad (4.21)$$

$$a_{22}(p) = 2D^{(R)}\eta e^{\eta H} \cosh(\eta H_R)$$

$$b_1(p) = f^{(R)}(H_C, p) - f(H_C, p) \quad (4.21')$$

$$b_2(p) = \frac{\partial}{\partial x} [D^{(R)} f^{(R)}(H_C, p) - Df(H_C, p)]_{x=H_C}$$

Expressions such as (4.20') ensure the continuity of the flow and current at the interface for all values of  $p$  and for each time  $t > 0$ . Returning to the formulas of the coefficients  $A(p)$  and  $A(p)^{(R)}$ , the term  $\Delta(p)$  is defined as:

$$\begin{aligned} \Delta(p) &= a_{11}(p) \cdot a_{22}(p) - a_{12}(p) \cdot a_{21}(p) = \\ &= 2 \cos(\alpha H_C) D^{(R)} \eta e^{\eta H} \cosh(\eta H_R) \left[ 1 - \frac{D\alpha}{D^{(R)}\eta} \tan(\alpha H_C) \tanh(\eta H_R) \right] \end{aligned} \quad (4.22)$$

### 4.3-CHARACTERISTIC EQUATION AND DETERMINATION OF THE POLES OF THE LAPLACE TRANSFORM

The equation (4.22) seen above can also be written in this way (Corno et al., 2008):

$$\Delta(p) = D^{(R)}\eta(p) \cos[\alpha(p)H_C] \cosh[\eta(p)H_R] - D\alpha(p) \sin[\alpha(p)H_C] \sinh[\eta(p)H_R] \quad (4.23)$$

This very particular function is important cause depends only on geometry and on the physical characteristics of the material that constitutes the system and not on the

external excitation or initial state. Concerning the research of the poles, apart the contribution originating from the external source terms, the only contributions to the sum of the residues, for the two the integrand equations, are the roots of the transcendental equation  $\Delta(p) = 0$ . Therefore it is important to analyze the fundamental role of the equation  $\Delta(p) = 0$  in the description of the time behavior of the reactor: it is called the “characteristic equation” of the system, on which both the time eigenvalues and the space shape of their associated eigenfunctions result to be dependent.

Analysing the form of the characteristic equation for the case of not reflected reactor, meaning the limit case  $H_R \rightarrow 0$ , the equation (4.23) turns in the form:

$$\cos[\alpha(p)H_C] = 0 \quad (4.24)$$

The equation (4.24) is true for all values of  $p$  such that:

$$\alpha^2(p) = \left[ \frac{(2n-1)\pi}{2H_C} \right]^2 \equiv B_n^2 \quad ; \quad n = 1, 2, \dots, \infty, \quad n \in \mathbb{N} \quad (4.25)$$

The last formula, in which  $B_n^2$  identifies the geometrical buckling, is the generalized Inhour equation for the neutrons shared in accordance with the n-th order mode of the Helmholtz problem in the case of an not reflected, symmetrical system that I will use it to find the interval in which search the poles of the characteristic equation.

The inhour equation can also be written in the form given in the previous paragraph (3.37). In fact re-issuing the definition of  $k_\infty = v\Sigma_f/\Sigma_a$  and inserting the definition of  $\alpha^2(p)$  in (4.25) (Corno et al., 2008):

$$\rho_n = \frac{pl_n}{pl_n + 1} + \frac{pl_n}{pl_n + 1} \sum_{i=1}^6 \frac{\beta_i}{p + \lambda_i} \quad (4.26)$$

In which the reactivity and the neutron lifetime associated with the n-th spatial harmonic are defined as follows respectively:

$$\rho_n = \frac{k-1}{k} = 1 - \frac{1 + L^2 B_n^2}{k_\infty} \quad ; \quad l_n = \frac{1}{v\Sigma_a(1 + L^2 B_n^2)} \quad (4.27)$$

Now I begin the research of the poles with the solution of the Inhour equation by substituting the known expression of  $\alpha^2(p)$  from (4.15) in the (4.25) and solving the equation in the variable  $p$ , but using the case in which I have just one family of neutron precursor, obtaining the equation (4.28).

$$\left[ \frac{k_{\infty}(1-\beta) - 1}{L^2} - \frac{p}{vD} + \frac{k_{\infty}}{L^2} \frac{\beta}{p + \lambda} \right] - \left[ \frac{(2n-1)\pi}{2H_c} \right]^2 = 0 \quad (4.28)$$

It is clear that the previous equation is a simply second degree equation with the follow solutions:

$$p = \frac{-\left(\frac{\lambda}{vD} + \left[\frac{(2n-1)\pi}{2H_c}\right]^2 + \frac{k_{\infty}(1-\beta) - 1}{L^2}\right) \pm \sqrt{\left(\frac{\lambda}{vD} + \left[\frac{(2n-1)\pi}{2H_c}\right]^2 - \frac{k_{\infty}(1-\beta) - 1}{L^2}\right)^2 - \frac{4}{vD} \left(\left[\frac{(2n-1)\pi}{2H_c}\right]^2 \lambda - \frac{k_{\infty}(1-\beta) - 1}{L^2} \lambda - \frac{k_{\infty}\lambda\beta}{L^2}\right)}}{\frac{2}{vD}}$$

After I found the interval in which search the poles, I can talk about the solution of the characteristic equation (4.23). First of all I organize this one in a more useful form for the discussion about the localization of the pole on the real axis, then the equation (4.23) becomes (4.29), in which, as seen previously,  $\alpha$  and  $\eta$  are both functions in the complex variable  $p$ :

$$\tan(\alpha H_C) \tanh(\eta H_R) = \frac{D\alpha}{D^{(R)}\eta} \quad (4.29)$$

Proceeding with the localization of the poles on the real axis, I consider two main cases:

$$a) \quad p > -\frac{vD^{(R)}}{L^{(R)2}} \rightarrow \tan(\alpha H_C) \tanh(\eta H_R) - \frac{D\alpha}{D^{(R)}\eta} = 0 \quad (4.30)$$

$$b) \quad p < -\frac{vD^{(R)}}{L^{(R)2}} \rightarrow \begin{cases} \tan(\alpha H_C) \tan(\eta' H_R) - \frac{D\alpha}{D^{(R)}\eta'} = 0 \\ \eta' = \sqrt{-\frac{p}{vD^{(R)}} - \frac{1}{L^{(R)2}}} \end{cases} \quad (4.30')$$

Regarding case a)  $\eta(p)$  is a real number then in each interval previously found the function  $\tan(\alpha H_C)$  oscillates continuously from  $-\infty$  to  $+\infty$  intersecting for sure the function  $D\alpha/D^{(R)}\eta$  and then individuating a single pole. On the other hand in the case b)  $\eta(p)$  becomes imaginary then in the equation (4.30') there is the function  $\tan(\eta' H_R)$  instead of the hyperbolic one,  $\tanh(\eta H_R)$ , and the oscillations caused by the term  $\tan(\eta' H_R)$  interfere with the ones of  $\tan(\alpha H_C)$ .

As a consequence of this interference, between two consecutive solutions of the Inhour equation, I find more than one solution to the characteristic equation. In order to explain better the above concept, the graphs of the two cases presented will be plotted in the following section where a numerical case will be addressed.

#### 4.4-ABOUT THE ORTHOGONALITY OF THE EIGENFUNCTIONS: REGULAR AND ADJOINT EIGENVALUE PROBLEM

In this section, with the purpose to verify the orthogonality of the eigenfunctions and therefore the accuracy of the poles found, it is mentioned the method of the dynamic eigenstates, also called omega-d modes, and in particular the regular and adjoint eigenvalue problem for the reflected reactor. I analyze the problem proposed at the beginning but starting from the operator formulation (*Corno and Ravetto, 1976*):

$$\frac{\partial}{\partial t} |\mathcal{E}(x, t)\rangle = \{\widehat{K}\} |\mathcal{E}(x, t)\rangle + |S(x, t)\rangle \quad ; \quad x \in D, \quad 0 \leq t < \infty \quad (4.31)$$

Where the state and the source vector are respectively:

$$|\mathcal{E}(x, t)\rangle = \begin{pmatrix} \Phi(x, t) \\ C_1(x, t) \\ \cdot \\ \cdot \\ C_R(x, t) \end{pmatrix} \quad ; \quad |S(x, t)\rangle = \begin{pmatrix} vS(x, t) \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad (4.32)$$

The components of these dimensional vectors are functions defined on the reactor domain and moreover the flux presents in the state vector responds to boundary, symmetry and interface conditions seen before. The operator  $\{\widehat{K}\}$  is called dynamic operator of the reactor and I will define it later for the specific case under study. A few general overviews are necessary to proceed with the discussion of the problem's solution. For this kind of problem there are the related eigenvalue problem (4.33) and the associated problem, the adjoint to this one (4.34):

$$\{\widehat{K}\} \cdot |\xi_n(x, p)\rangle = p_n \cdot |\xi_n(x, p)\rangle \quad ; \quad n = 1, 2, \dots, \infty \quad (4.33)$$

$$\langle \xi_n(x, p) | \cdot \{\widehat{K}^+\} = \overline{p_n} \cdot \langle \xi_n(x, p) | \quad ; \quad n = 1, 2, \dots, \infty \quad (4.34)$$

If we presume the dynamic operator  $\{\widehat{K}\}$  to be linear and time independent, to approach the solution I take in account the following conditions:

- The eigenvalue problem has a solution such that the components of  $|\xi_n(x, p)\rangle$ , all belonging to different discrete eigenvalues and such that they satisfy the linearity independence, respect the same boundary and interface conditions as the unknown solution  $|\mathcal{E}(x, t)\rangle$  ;
- The adjoint problem has a solution such that the components of  $\langle \xi_n(x, p) |$  respect the same boundary and interface conditions as the unknown solution  $\langle \mathcal{E}(x, t) |$  ;

- The eigenstates  $|\xi_n(x)\rangle$  form a discrete, close set and the follow orthogonality relation is true:

$$\langle \xi_n | \xi_m \rangle = \int_0^H \sum_{i=1}^{R+1} [\xi(x, p_n)^{+(i)}(x) \cdot \xi^{(i)}(x, p_m)] dx = \delta_{nm} ; \quad m, n = 1, 2, \dots, \infty \quad (4.35)$$

Then, with the satisfaction of the three above conditions, it is possible to write the solution of the problem (4.31) in the form:

$$|\mathcal{E}(x, t)\rangle = \sum_n A_n(t) \cdot |\xi_n(x, p)\rangle \quad (4.36)$$

The form of the solution (4.36) is a series expansion of eigenstates of  $\hat{K}$ , however in this section I am interested in the demonstration of the accuracy of the poles obtained from the characteristic equation, then I have to talk about the orthogonality relation (4.35). First of all I write the kind of problem treated, that it is the one group diffusion approximation case and with one family of delayed precursors, one equation for the core and one for the reflector region:

$$\begin{cases} \frac{1}{vD} \frac{\partial \Phi(x, t)}{\partial t} = \left[ \frac{\partial^2}{\partial x^2} + \frac{k_\infty(1-\beta) - 1}{L^2} \right] \Phi(x, t) + \frac{\lambda C(x, t)}{D} + \frac{S(x, t)}{D} \\ \beta k_\infty \Sigma_a \Phi(x, t) - \lambda C(x, t) = \frac{\partial C(x, t)}{\partial t} \end{cases} \quad (4.37)$$

$$\frac{1}{vD^{(R)}} \frac{\partial \Phi^{(R)}(x, t)}{\partial t} = \left[ \frac{\partial^2}{\partial x^2} - \frac{1}{L^{(R)2}} \right] \Phi(x, t) + \frac{S^{(R)}(x, t)}{D^{(R)}} \quad (4.38)$$

Now I define two important functions useful to write in contracted form the next relations:

$$I_C = 1 \Leftrightarrow x \in \mathcal{D}_C \quad (4.39)$$

$$I_R = 1 \Leftrightarrow x \in \mathcal{D}_R$$

Where  $\mathcal{D}_C$  and  $\mathcal{D}_R$  are respectively the core's and the reflector's domain. I write below the dynamic operator for this kind of problem (*Corno and Ravetto, 1976*):

$$\hat{K} = \begin{Bmatrix} k_{11} & I_C v \lambda \\ I_C \beta k_\infty \Sigma_a & -I_C v \lambda \end{Bmatrix} \quad (4.40)$$

This is a matrix in which I have the only one element that incorporates a second order differential operator:

$$k_{11} = I_C v D \left[ \frac{\partial^2}{\partial x^2} * + \frac{k_\infty(1-\beta) - 1}{L^2} \right] + I_R v D^{(R)} \left[ \frac{\partial^2}{\partial x^2} * - \frac{1}{L^{(R)2}} \right] \quad (4.41)$$

Then I write the operators  $|\xi_n(x, p)\rangle$  and  $\langle \xi_n(x, p)|$  for the case under study, with  $\alpha(p_n) = \alpha_n$  and  $\eta(p_n) = \eta_n$ , both the formulas from (Corno and Ravetto, 1976):

$$|\xi_m(x, p)\rangle = I_C \cos(\alpha_m x) \left| \frac{1}{p_m + \lambda} \right\rangle + I_R \frac{\cos(\alpha_m H_C)}{\sinh(\eta_m H_R)} \sinh[\eta_m(H - x)] \left| \frac{1}{0} \right\rangle \quad (4.42)$$

$$\langle \xi_n(x, p)| = \left\{ I_C \left\langle 1, \frac{v\lambda}{p_n + \lambda} \right| \cos(\alpha_n x) + I_R \langle 1, 0| \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \sinh[\eta_n(H - x)] \right\} \frac{1}{N_n^2} \quad (4.43)$$

With the term  $N_n^2$ , called normalization constant, equal to the expression:

$$N_n^2 = \left( 1 + \frac{vk_{\infty}\Sigma_a\beta\lambda}{(p_n + \lambda)^2} \right) \left( \frac{\sin(\alpha_n H_C) \cos(\alpha_n H_C) + \alpha_n H_C}{2\alpha_n} \right) + \left( \frac{\cos[\alpha_n H_C]}{\sinh[\eta_n H_R]} \right)^2 \left( \frac{\sinh[\eta_n H_R] \cosh[\eta_n H_R] - \eta_n H_R}{2\eta_n} \right) \quad (4.44)$$

Finally I can return to the orthogonality relation (4.35) to apply it at this case, remembering that the product between the two specific functions  $I_C \cdot I_R$  is equal to zero:

$$\begin{aligned} \langle \xi_n | \xi_m \rangle &= \int_0^H \sum_{i=1}^2 \left[ \xi(x, p_n)^{+(i)}(x) \cdot \xi^{(i)}(x, p_m) \right] dx = \\ &= \frac{1}{N_n^2} \int_0^H \left[ I_C \cos(\alpha_n x) + I_R \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \sinh[\eta_n(H - x)] \right] \left[ I_C \cos(\alpha_m x) + I_R \frac{\cos(\alpha_m H_C)}{\sinh(\eta_m H_R)} \sinh[\eta_m(H - x)] \right] dx \quad (i = 1) \\ &+ \frac{1}{N_n^2} \int_0^H \left[ I_C \frac{v\lambda}{p_n + \lambda} \cos(\alpha_n x) + I_R \cdot 0 \right] \left[ I_C \frac{k_{\infty}\Sigma_a\beta}{p_m + \lambda} \cos(\alpha_m x) + I_R \cdot 0 \right] dx = \quad (i = 2) \\ &= \frac{1}{N_n^2} \int_0^H \left\{ \left[ I_C \cos(\alpha_n x) \cos(\alpha_m x) \right] + \left[ I_R \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \sinh[\eta_n(H - x)] \frac{\cos(\alpha_m H_C)}{\sinh(\eta_m H_R)} \sinh[\eta_m(H - x)] \right] \right\} dx + \quad (i = 1) \\ &+ \frac{1}{N_n^2} \int_0^H \left[ I_C \frac{v\lambda}{p_n + \lambda} \cos(\alpha_n x) \frac{k_{\infty}\Sigma_a\beta}{p_m + \lambda} \cos(\alpha_m x) \right] dx = \quad (i = 2) \\ &= \frac{1}{N_n^2} \left\{ \int_0^{H_C} \cos(\alpha_n x) \cos(\alpha_m x) dx + \frac{\cos(\alpha_n H_C) \cos(\alpha_m H_C)}{\sinh(\eta_n H_R) \sinh(\eta_m H_R)} \int_{H_C}^H \sinh[\eta_n(H - x)] \sinh[\eta_m(H - x)] dx \right\} + \quad (i = 1) \\ &+ \frac{1}{N_n^2} \left\{ \frac{v\lambda}{p_n + \lambda} \frac{k_{\infty}\Sigma_a\beta}{p_m + \lambda} \int_0^{H_C} \cos(\alpha_n x) \cos(\alpha_m x) dx \right\} \quad (i = 2) \end{aligned}$$

Arrived at this point is important to distinguish the case in which the index  $n$  is equal to the index  $m$ , the case that represents the principal diagonal in the matrix

represented by the operator  $\delta_{nm}$ , from the one in which the two index are different, the other elements present in the matrix. The matrix that is obtained if all the eigenfunctions are orthogonal each other is obviously of the type:

$$\delta_{nm} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad (4.45)$$

The dimensions of this matrix depend on the total number of poles taken in account.

I write the integral (4.35) using the index  $n = m$ :

$$\begin{aligned} \langle \xi_n(x) | \xi_n(x) \rangle &= \quad (4.46) \\ &= \frac{1}{N_n^2} \left\{ \int_0^{H_C} \cos^2(\alpha_n x) dx + \left[ \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \right]^2 \int_{H_C}^H \sinh^2[\eta_n(H-x)] dx \right\} + \quad (i=1) \\ &+ \frac{1}{N_n^2} \left\{ \frac{v\lambda}{p_n + \lambda} \frac{k_\infty \Sigma_a \beta}{p_n + \lambda} \int_0^{H_C} \cos^2(\alpha_n x) dx \right\} = \quad (i=2) \\ &= \frac{1}{N_n^2} \left\{ \left[ 1 + \frac{v\lambda k_\infty \Sigma_a \beta}{(p_n + \lambda)^2} \right] \int_0^{H_C} \cos^2(\alpha_n x) dx + \left[ \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \right]^2 \int_{H_C}^H \sinh^2[\eta_n(H-x)] dx \right\} \end{aligned}$$

I solve separately the two definite integrals. For the first one I have to remember the trigonometric formula  $[\cos(ax)]^2 = 1/2 \cdot [1 + \cos(2ax)]$ :

$$\int_0^{H_C} \cos^2(\alpha_n x) dx = \frac{1}{2} \left[ x + \frac{\sin(2\alpha_n x)}{4\alpha_n} \right]_0^{H_C} = \frac{1}{2} H_C + \frac{\sin(2\alpha_n H_C)}{4\alpha_n} \quad (4.47)$$

I solved the second one using the integration by part and the hyperbolic formula  $\sinh(2x) = 2 \sinh(x) \cosh(x)$ :

$$\int_{H_C}^H \sinh^2[\eta_n(H-x)] dx = \left[ \frac{H-x}{2} - \frac{\sinh[2\eta_n(H-x)]}{4\eta_n} \right]_{H_C}^H = \frac{\sinh[2\eta_n(H-H_C)]}{4\eta_n} - \frac{H_R}{2} \quad (4.48)$$

I return to the orthogonality relation, then I put (4.47) and (4.48) in (4.46):

$$\langle \xi_n(x) | \xi_n(x) \rangle = \frac{1}{N_n^2} \left\{ \left[ 1 + \frac{v\lambda k_\infty \Sigma_a \beta}{(p_n + \lambda)^2} \right] \left( \frac{1}{2} H_C + \frac{\sin(2\alpha_n H_C)}{4\alpha_n} \right) + \left[ \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \right]^2 \left( \frac{\sinh[2\eta_n(H-H_C)]}{4\eta_n} - \frac{H_R}{2} \right) \right\} \quad (4.49)$$

Now I examine the case in which  $n \neq m$ :

$$\begin{aligned} \langle \xi_n(x) | \xi_m(x) \rangle &= \frac{1}{N_n^2} \left\{ \left[ 1 + \frac{v\lambda}{p_n + \lambda} \frac{k_\infty \Sigma_a \beta}{p_m + \lambda} \right] \int_0^{H_C} [\cos(\alpha_n x) \cos(\alpha_m x)] dx + \right. \\ &\quad \left. + \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \frac{\cos(\alpha_m H_C)}{\sinh(\eta_m H_R)} \int_{H_C}^H [\sinh[\eta_n(H-x)] \sinh[\eta_m(H-x)]] dx \right\} \quad (4.50) \end{aligned}$$

As before I solve the two integral separately:

$$\begin{aligned}
& \int_0^{H_C} [\cos(\alpha_n x) \cos(\alpha_m x)] dx = \\
& = \int_0^{H_C} \frac{\cos[x(\alpha_n + \alpha_m)] + \cos[x(\alpha_m - \alpha_n)]}{2} dx = \\
& = \frac{1}{2} \left\{ \left[ \frac{\sin[x(\alpha_n + \alpha_m)]}{\alpha_n + \alpha_m} \right]_0^{H_C} + \left[ \frac{\sin[x(\alpha_m - \alpha_n)]}{\alpha_m - \alpha_n} \right]_0^{H_C} \right\} = \\
& = \frac{1}{2} \left\{ \frac{\sin[H_C(\alpha_n + \alpha_m)]}{\alpha_n + \alpha_m} + \frac{\sin[H_C(\alpha_m - \alpha_n)]}{\alpha_m - \alpha_n} \right\}
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
& \int_{H_C}^H \{ \sinh[\eta_n(H-x)] \sinh[\eta_m(H-x)] \} dx = \\
& = \int_{H_C}^H \frac{\cosh[(\eta_n + \eta_m)(x-H)] + \cosh[(\eta_n - \eta_m)(x-H)]}{2} dx = \\
& = \frac{1}{2} \left\{ \left[ \frac{\sinh[(\eta_n + \eta_m)(x-H)]}{\eta_n + \eta_m} \right]_{H_C}^H + \left[ \frac{\sinh[(\eta_n - \eta_m)(x-H)]}{\eta_n - \eta_m} \right]_{H_C}^H \right\} = \\
& = \frac{1}{2} \left\{ \frac{\sinh[H_R(\eta_n - \eta_m)]}{\eta_n - \eta_m} - \frac{\sinh[H_R(\eta_n + \eta_m)]}{\eta_n + \eta_m} \right\} =
\end{aligned} \tag{4.52}$$

I return to the original integral putting together (4.51) and (4.52) in (4.50):

$$\begin{aligned}
\langle \xi_n(x) | \xi_m(x) \rangle = & \frac{1}{2 N_n^2} \left\{ \left[ 1 + \frac{v\lambda}{p_n + \lambda} \frac{k_\infty \Sigma_a \beta}{p_m + \lambda} \right] \left[ \frac{\sin[H_C(\alpha_n + \alpha_m)]}{\alpha_n + \alpha_m} + \frac{\sin[H_C(\alpha_m - \alpha_n)]}{\alpha_m - \alpha_n} \right] + \right. \\
& \left. + \frac{\cos(\alpha_n H_C)}{\sinh(\eta_n H_R)} \frac{\cos(\alpha_m H_C)}{\sinh(\eta_m H_R)} \left[ \frac{\sinh[H_R(\eta_n - \eta_m)]}{\eta_n - \eta_m} - \frac{\sinh[H_R(\eta_n + \eta_m)]}{\eta_n + \eta_m} \right] \right\}
\end{aligned} \tag{4.53}$$

In the end, after the computation of the formulas (4.49) and (4.53) the accuracy of the poles found and the orthogonality of the eigenfunctions can be verified.

In the progress of the calculations it is important to distinguish between the poles before and after the value  $p_{rif} < -vD^{(R)}/L^{(R)2}$  : for the poles with a value smaller than  $p_{rif}$  in fact I have to use the function  $\eta'$  and the circular functions instead the hyperbolic ones.

## 4.5- INVERSE TRANSFORMATION, SOLUTION ADOPTED AND EIGENFUNCTIONS OF THE PROBLEM

Now it is the moment to return on the solution of the problem with the purpose to study the space and time evolution of the flux in a reflected reactor. The next step is to perform the inverse Laplace transformation of the function in the complex variable  $\Phi_{\mathcal{L}}(x, p)$  and  $\Phi_{\mathcal{L}}^{(R)}(x, p)$ , in order to return to the time variable by using the classical formula (Corno and Ravetto, 1976):

$$\Phi(x, t) \equiv \mathcal{L}^{-1}[\Phi_{\mathcal{L}}(x, p)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi_{\mathcal{L}}(x, p) e^{pt} dp \quad (4.54)$$

$$\Phi^{(R)}(x, t) \equiv \mathcal{L}^{-1}[\Phi_{\mathcal{L}}^{(R)}(x, p)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi_{\mathcal{L}}^{(R)}(x, p) e^{pt} dp \quad (4.54')$$

In which  $c$  is the convergence abscissa. About the system being examined, for the reason that a reflected reactor is certainly less reactive respect a uniform reactor with similar dimensions and the same material configuration as the core studied right now, I can say that a finite upper bound for  $c$  exists for sure: then is possible to solve the previous integrals using the residual theorem as is done in the mentioned works, (Corno and Ravetto, 1976) and (Corno et al., 2008). Therefore, once the poles have been found, it is possible to apply the theorem of the residues on the inverse transformation in order to find the expression of the flux.

The flux is written in form of “series of clusters of eigenfunctions”, and the fact that it appears on a form of infinite series of space dependent eigenfunctions is due to the infinite number of roots of the characteristic equation (4.23), which play the role of time constant for the system. Finally the neutron flux expression for a reflected reactor, in the general case presented in the beginning of this section by equations (4.1) and (4.2), both for the core and the reflector (Corno et al., 2008) is:

$$\Phi(x, t) = \sum_{n=1}^{\infty} \sum_{i=1}^6 \frac{A_{n,j}(t)}{N_n^2} e^{-p_{n,j}t} \psi_{n,j}(x) \quad (4.55)$$

With the normalization constant  $N_n$  equal to expression (4.44) and the term  $A_{n,j}(t)$  equal to the formula (4.56). About the space dependence of the solution there is the term  $\psi_{n,j}(x)$  that identifies the eigenfunctions of the problem and it represents the spatial shape of my problem.

$$\begin{aligned}
A_{n,j}(t) = & \int_0^{H_C} \left[ \Phi_0(x') + v \sum_{i=1}^6 \frac{\lambda_i C_{i,0}(x')}{p_{n,j} + \lambda_i} + v \int_0^t S(x', t') e^{-p_{n,j} t'} dt' \right] \cos(\alpha_{n,j} x') dx' \\
& + \frac{\cos(\alpha_{n,j} H_C)}{\sinh(\eta_{n,j} H_R)} \int_{H_C}^H \left[ \Phi_0^{(R)}(x') + v \int_0^t S^{(R)}(x', t') e^{-p_{n,j} t'} dt' \right] \sinh[\eta_{n,j} (H - x')] dx'
\end{aligned} \tag{4.56}$$

The eigenfunctions of the problem are explicitly (Corno et al., 2008):

$$\psi_{n,j}(x) = \begin{cases} \cos[\alpha(p_{n,j})x] & , \quad |x| \leq H_C \\ \frac{\cos[\alpha(p_{n,j})H_C]}{\sinh[\eta(p_{n,j})H_R]} \sinh[\eta(p_{n,j})(H - x)] & , \quad H_C < |x| \leq H \end{cases} \tag{4.57}$$

About the form of the solution, from the physical point of view, it is important to notice how a given excitation restricted to one specific region alone is able to produce a transient in the neutron population within the whole system. Furthermore, at all instant of the transient, the satisfaction of the boundary, symmetric and continuity conditions is guaranteed through the fulfillment of such conditions by each eigenfunction that evolves as a single exponential. About a source-free system, it is possible to say that the time dependent function  $A_{n,j}(t)$  has a constant value: it can happen just in the case in which the external source injected has steadily fallen to zero.

About the double index that I use for the notations of the poles and the eigenfunctions:  $n$  is called cluster index,  $j$  the inhour index. Furthermore, due to fact that  $\eta$  is a purely imaginary number for  $p_{rif} < -vD^{(R)}/L^{(R)2}$ , I used the function  $\eta'$  and the circular functions instead the hyperbolic ones when I done the calculations, as previously mentioned.

## 5-SOME SIGNIFICANT NUMERICAL RESULTS

In this section numerical examples have been made using a computer code, in particular using the environment for the numerical calculation MATLAB. The results obtained range from the representation of the eigenfunctions to the calculation of the flux and power inside the reactor in different specific cases.

### 5.1-LOCALIZATION OF THE POLES AND EIGENFUNCTIONS

In order to explain better the above concept regarding the localization of the poles on the real axis, I plot a qualitative representation for the two main cases mentioned in the paragraph 4.3. I'm starting from a reflected reactor with one family of delayed precursors, without the presence of an external source and with the following data reported in the below table, Tab. 2, in which the dimensions and composition of the materials making up the reactor are shown.

$k_{\infty}$	1.009
$D$	0.89 cm
$\Sigma_a$	$0.1695 \cdot 10^{-1} \text{ cm}^{-1}$
$v$	$0.22 \cdot 10^6 \text{ cm/s}$
$D^{(R)}$	2 cm
$\Sigma_a^{(R)}$	$0.692 \cdot 10^{-4} \text{ cm}^{-1}$
$\beta$	650 pcm
$\lambda$	$0.1 \text{ s}^{-1}$
$H_C$	100 cm
$H_R$	100 cm

**Tab. 2.** Material composition and dimensions of the reactor under exam.

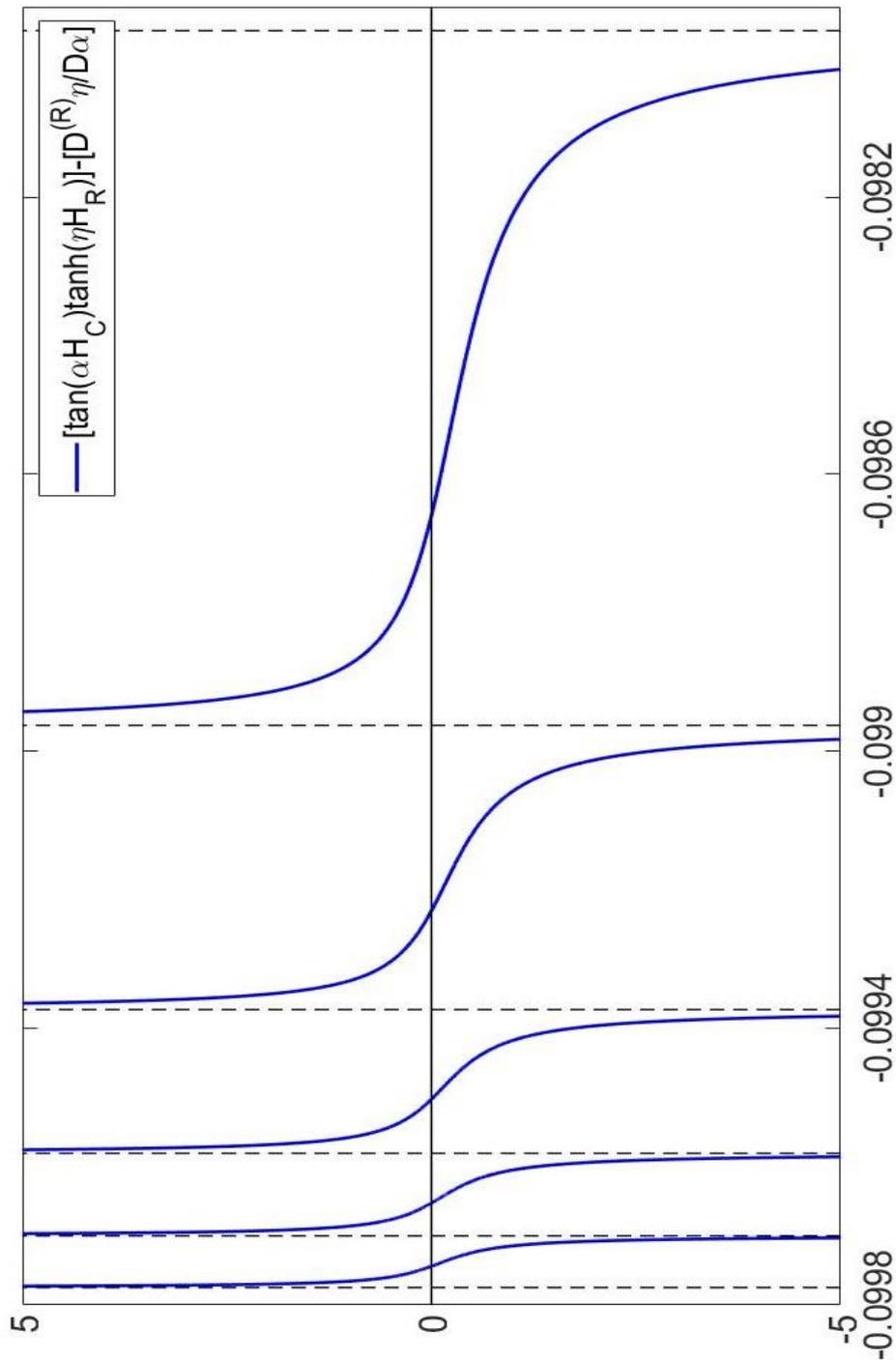
We recall that the two main cases mentioned above are expressed with the formulas (4.30) and (4.30'):

$$a) \quad p > -\frac{vD^{(R)}}{L^{(R)2}} \rightarrow \tan(\alpha H_C) \tanh(\eta H_R) - \frac{D\alpha}{D^{(R)}\eta} = 0 \quad (5.1)$$

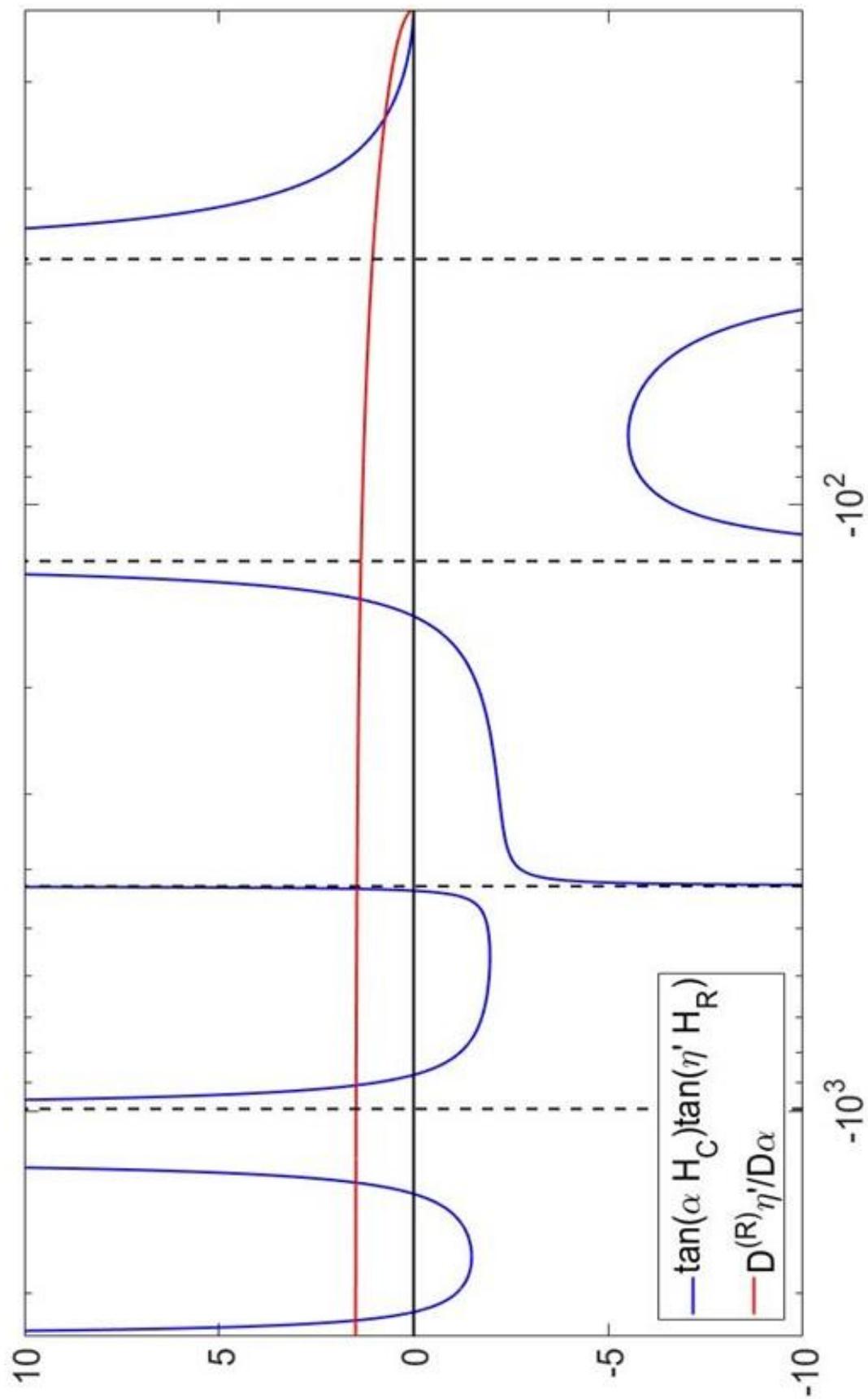
$$b) \quad p < -\frac{vD^{(R)}}{L^{(R)2}} \rightarrow \begin{cases} \tan(\alpha H_C) \tan(\eta' H_R) - \frac{D\alpha}{D^{(R)}\eta'} = 0 \\ \eta' = \sqrt{-\frac{p}{vD^{(R)}} - \frac{1}{L^{(R)2}}} \end{cases} \quad (5.1')$$

The first plot, Fig.10, represents the case a) while the second one, Fig.11, the case b): in both graphics I computed the corresponding relation in function of the poles. How it

is possible to observe from the plots, in the first one is evident the behavior of the characteristic equation that oscillates between  $-\infty$  and  $+\infty$  with the consequence, mentioned before, that there is one solution of the characteristic equation for two consecutive solutions of the Inhour equation, whereas in the second one it is possible to notice the presence of more than one pole for each interval taken into account, as previously mentioned.



**Fig. 10.** Qualitative representation of the characteristic eq. for  $p > -\frac{vD^{(R)}}{L^{(R)^2}}$



**Fig. 11.** Qualitative representation of the characteristic eq. for  $p < -\frac{vD^{(R)}}{L^{(R)^2}}$

Let us now proceed to examine the trend of the eigenfunctions remembering the expression (4.57) seen above:

$$\psi_{n,j}(x) = \begin{cases} \cos[\alpha(p_{n,j})x] & , \quad |x| \leq H_C \\ \frac{\cos[\alpha(p_{n,j})H_C]}{\sinh[\eta(p_{n,j})H_R]} \sinh[\eta(p_{n,j})(H-x)] & , \quad H_C < |x| \leq H \end{cases} \quad (5.2)$$

Moreover, as previously mentioned, for poles for  $p_{rif} < -vD^{(R)}/L^{(R)2}$  it is necessary to use the  $\eta'$  function and the circular functions instead the hyperbolic ones.

Then in this section it is done the analysis of the eigenfunctions of my problem for four different system configuration, for which I used the data in Tab. 3 to do the simulations: I utilized the letter *C* to indicate the core and the letter *R* for the reflector.

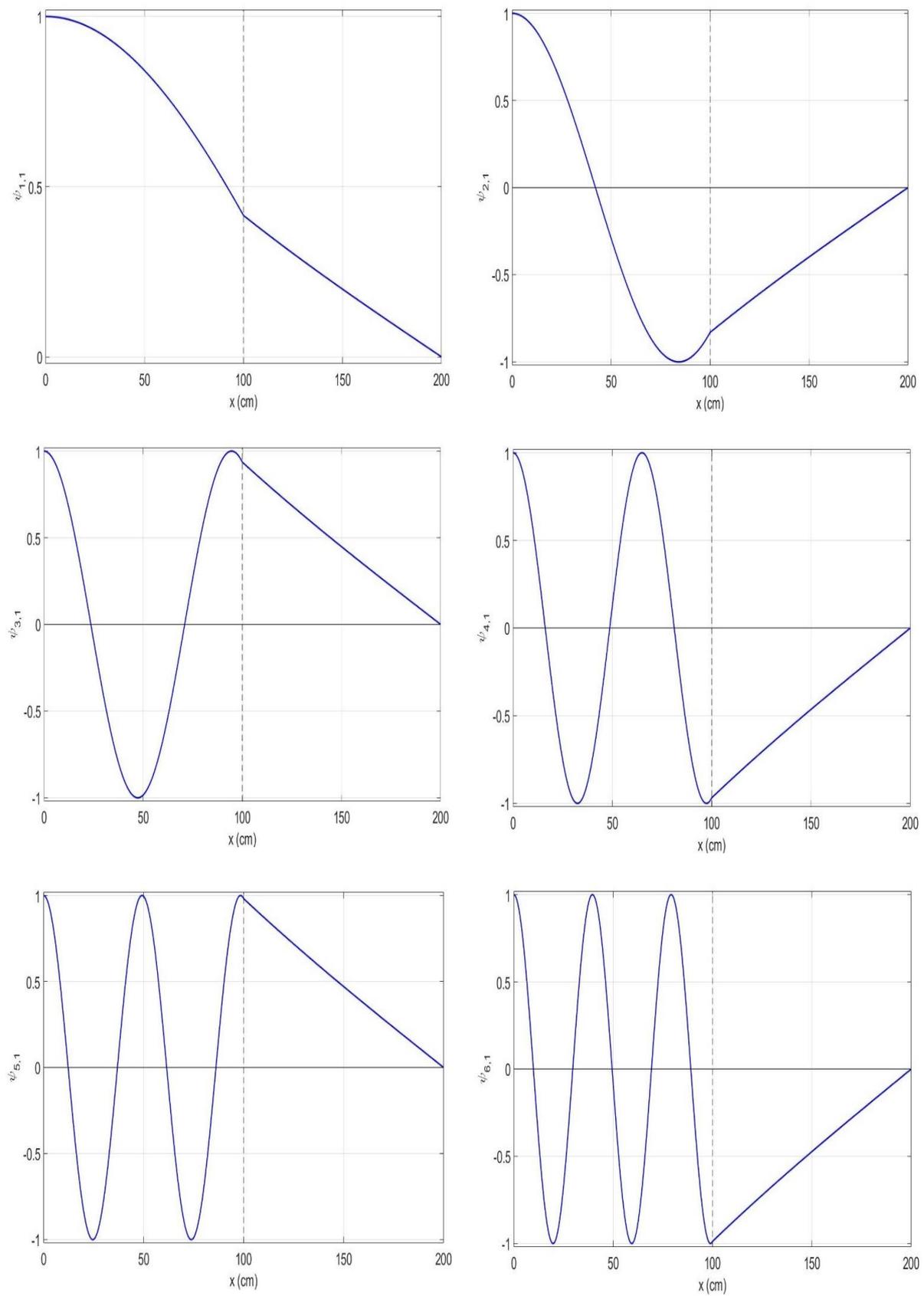
Case	Region	Width [cm]	$k_\infty$	$\Sigma_a$ [cm]	D [cm]
A	C	100	1.009	$1.695 * 10^{-2}$	0.89
	R	100	0	$0.692 * 10^{-4}$	2
B	C	100	1.001	$1.695 * 10^{-2}$	0.89
	R	100	0	$0.692 * 10^{-2}$	2
C	C	100	1.009	$1.695 * 10^{-2}$	0.89
	R	900	0	$0.692 * 10^{-4}$	2
D	C	100	1.009	$1.695 * 10^{-2}$	0.89
	R	10	0	$0.692 * 10^{-4}$	2

**Tab. 3.** Geometric and nuclear properties used during the various simulations.

I assumed only one family of delayed precursors with a decay constant  $\lambda = 0,1 s^{-1}$  and  $\beta = 650 pcm$ , and a system without any external source. All the eigenfunctions are plotted in arbitrary units because are solution of a homogeneous problem.

The first six graphs shown in Fig. 12 constitute the case A for poles larger than  $-\lambda$ . It is possible to see that the fundamental eigenfunction  $\psi_{1,1}$  does not change the sign while there are more and more nodes for eigenfunctions with higher cluster index: they have a negative buckling in the reflector.

For eigenfunctions corresponding to the poles smaller than  $-\lambda$ , for the same case, I obtained the plots in Fig. 13. In this case it is possible to observe that  $\psi_{1,2}$  is very close to  $\psi_{1,1}$  while  $\psi_{2,2}$  is different respect  $\psi_{2,1}$  and this is due to the fact that there is a positive buckling in the reflector, in general I have the same number of nodes in the core and a growing number of nodes in the reflector.



**Fig. 12.** Spatial eigenfunctions for case A, poles larger than  $-\lambda$ .

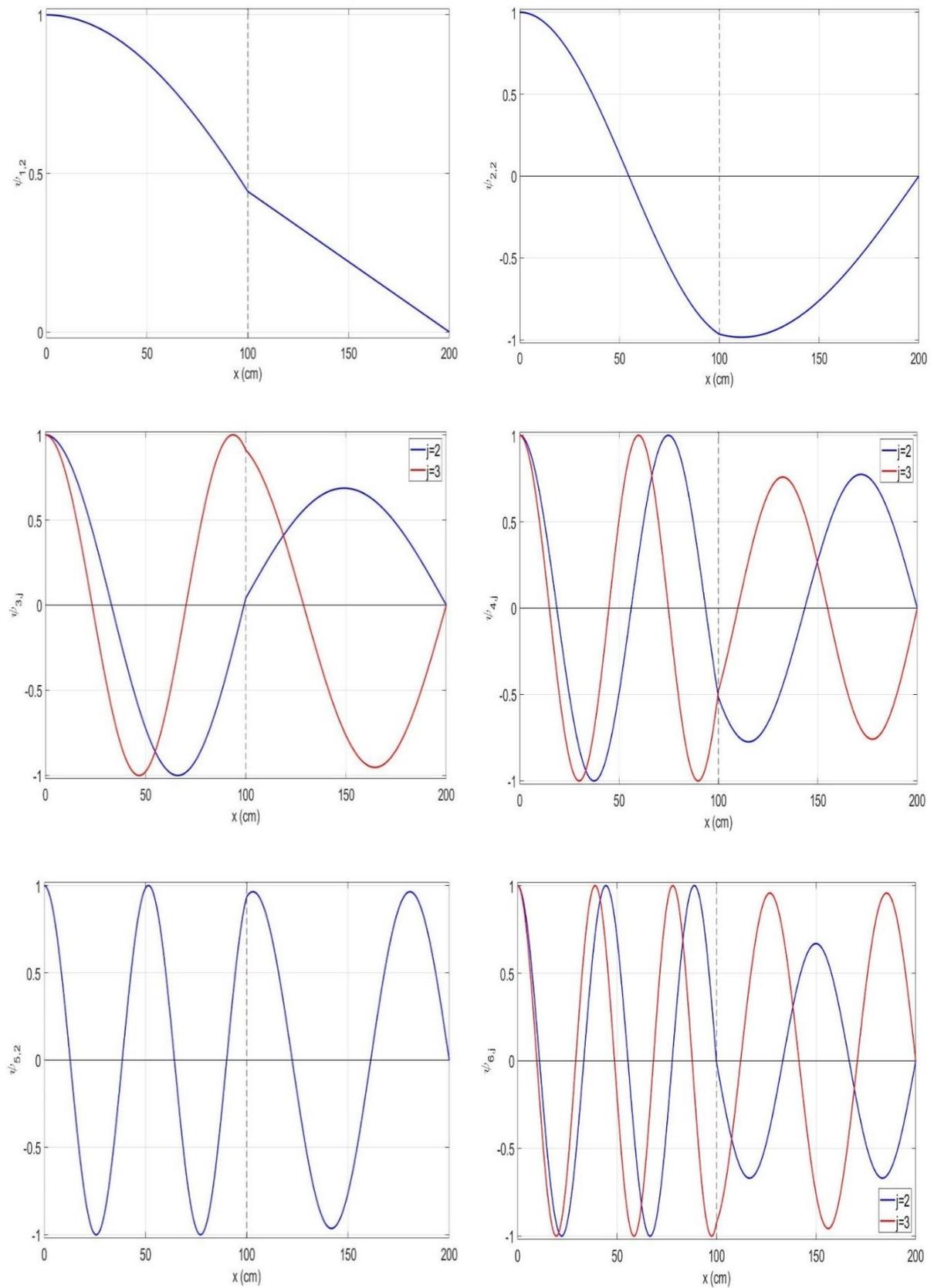
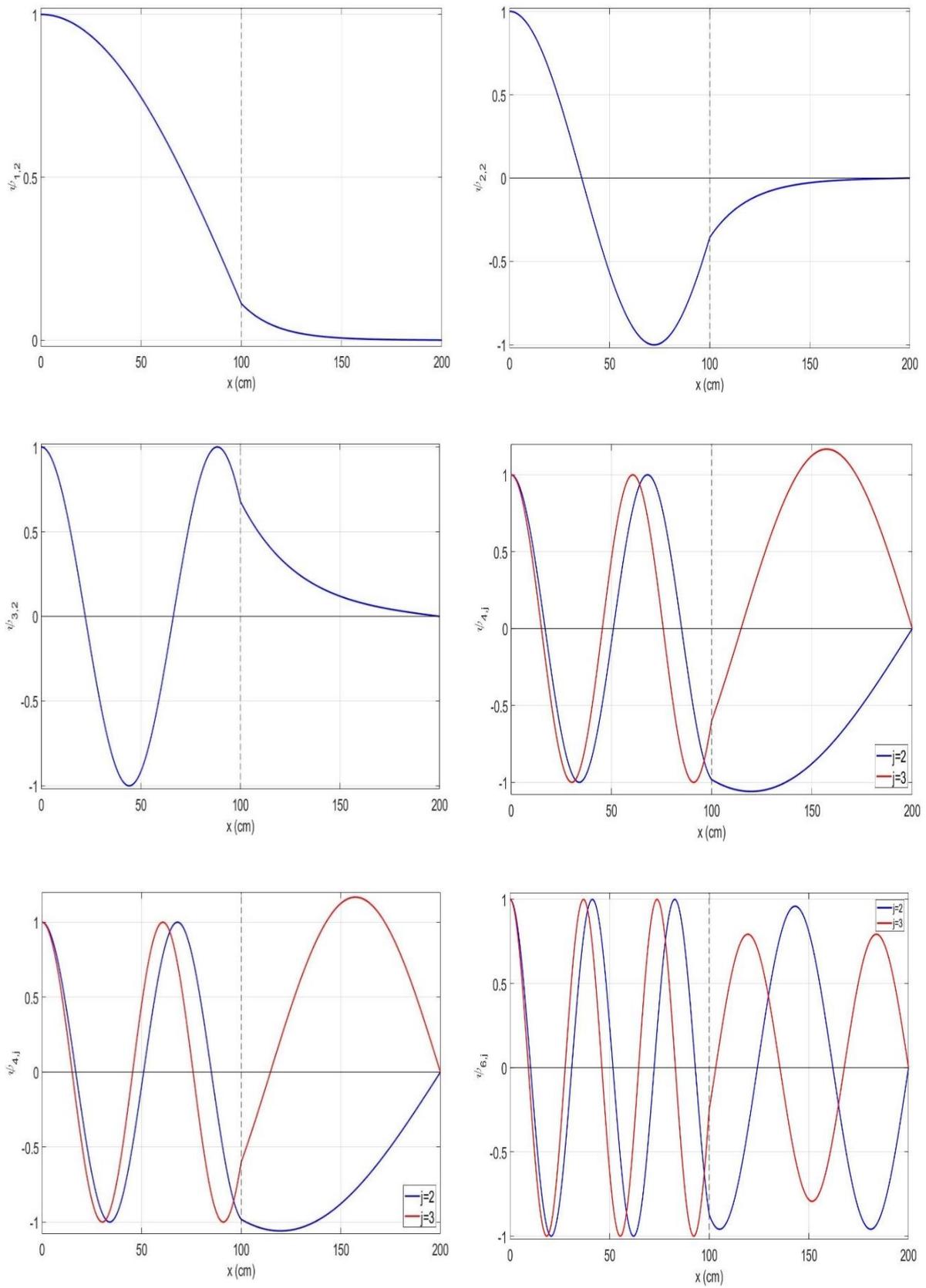


Fig. 13. Spatial eigenfunctions for case A, poles smaller than  $-\lambda$ .



**Fig. 14.** Spatial eigenfunctions for case B.

The charts in Fig. 14 shows the behavior of the system in the case B: due the increase of absorption cross section in the reflector zone and the decrease of  $k_{\infty}$  respect the previous case, it is possible to notice from the physical point of view a stronger spatial decay of the shape in the reflector whereas the transition to oscillating behavior appears for higher-order clusters.

In the plots in Fig. 15 I analyzed the case C, in which I increased the dimensions of the reflector, we see that the behavior in the core is not so different respect the case A, while in the reflector I have a stronger oscillating behavior than the cases seen before. This is due to an increase of the inhour index and consequently in the number of poles: if in case A the index " $j$ " was equal to three, in this case the inhour index is equal to eight.

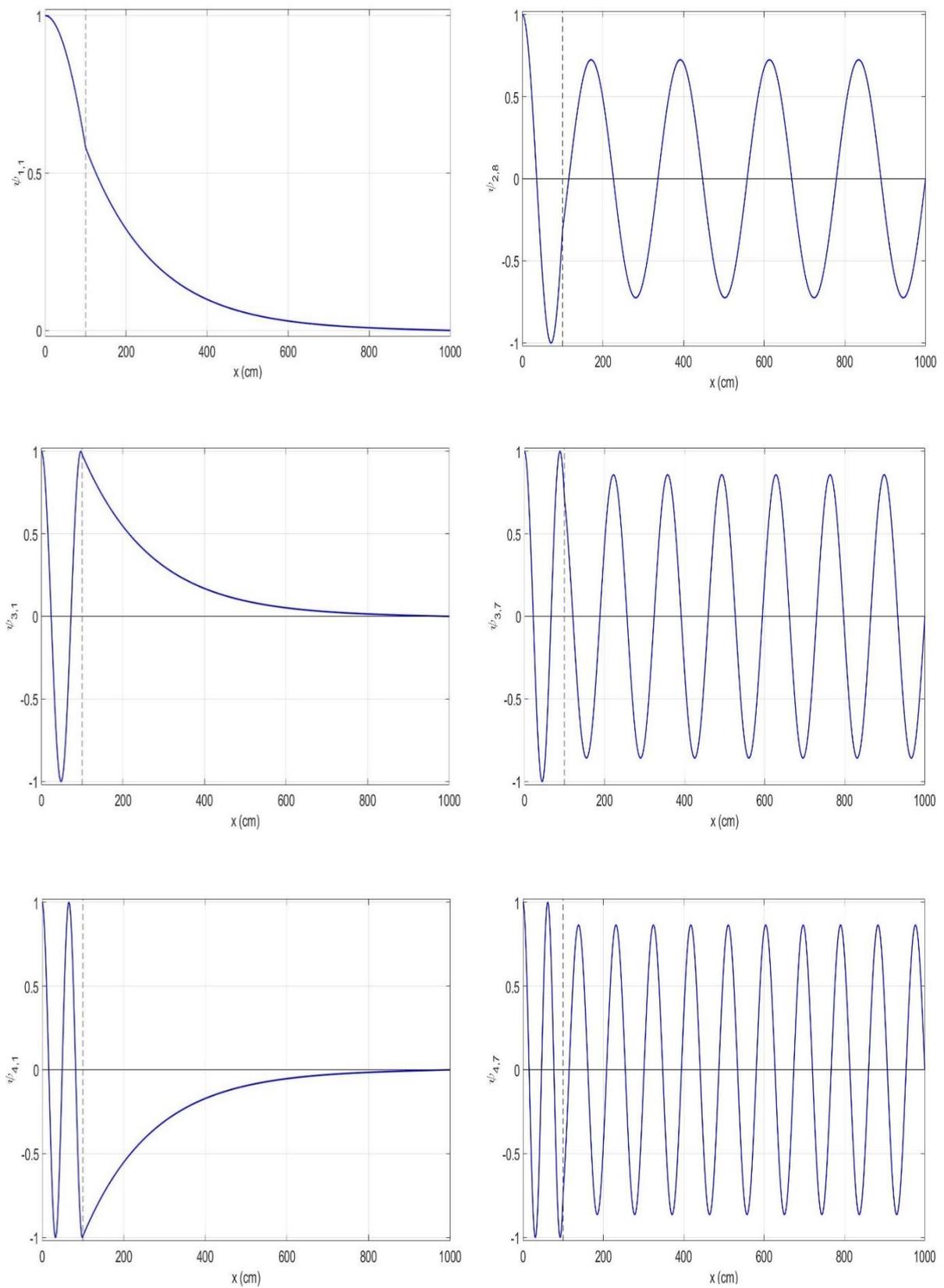
In the last case, the case D, I studied what happen if the dimensions of the reflector decreases. It is evident that in the reflector a big change in the behavior happens, indeed due to a strong decrease in the number of poles regarding the inhour index the oscillations disappear. About the core, the shape is similar respect the other case, however it is possible to observe a faster decrease/increase in the shape of the eigenfunctions approaching the reflector region. It is possible to see the results of this case in Fig. 16.

The reason why I studied cases in which I change the dimensions of the reflector is to talk about the cases for reactor dimensions approaching their possible limits. In a reflected reactor these limit cases could be obviously  $H_R \rightarrow 0$  and  $H_R \rightarrow \infty$ .

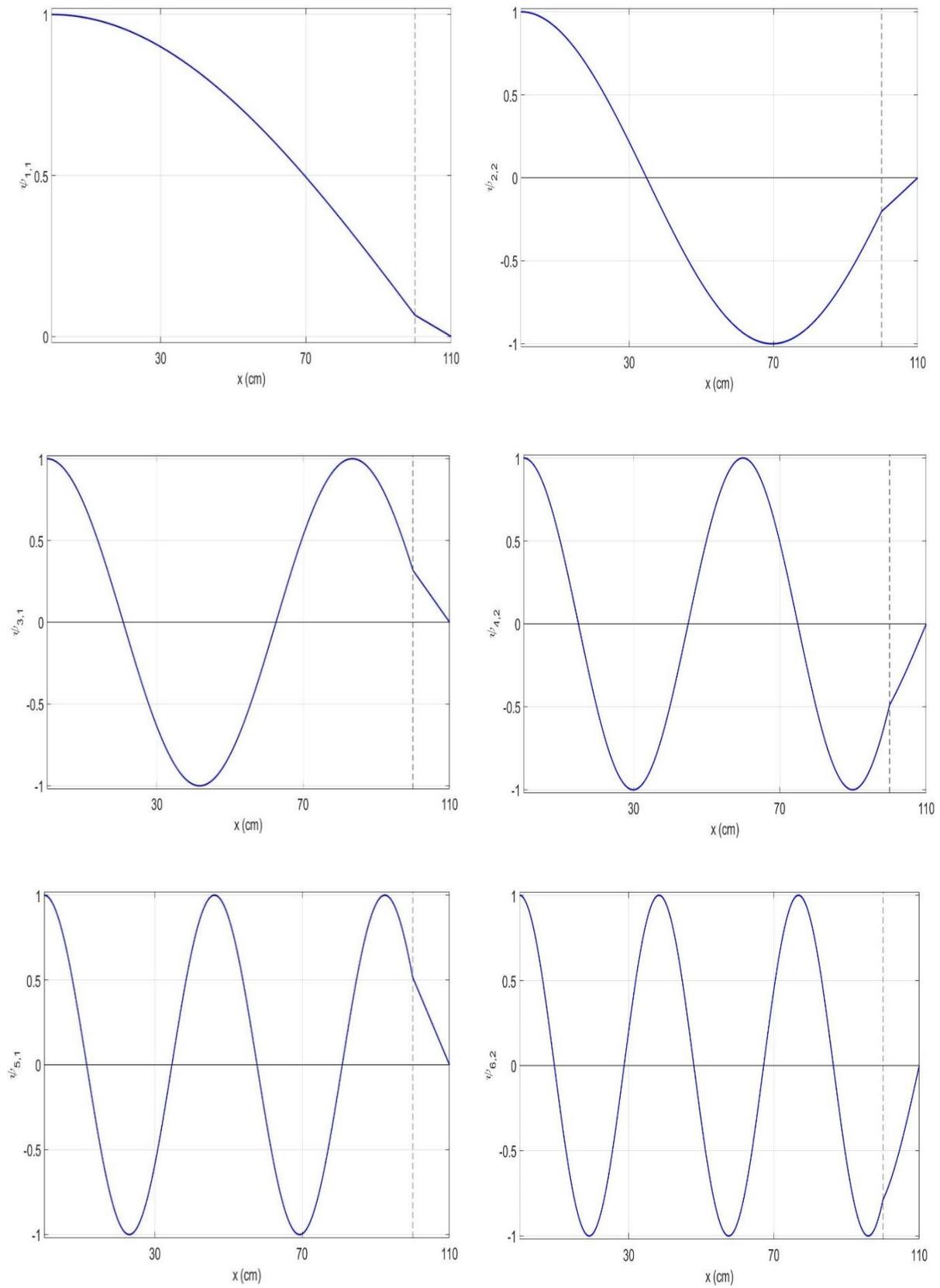
Therefore, if the thickness of the reflector tends to zero, the solution degenerates to the well-known solution for a bare reactor and in particular, from the mathematical point of view, it becomes the classical Helmholtz expansion of the transient solution.

On the other hand, when the thickness of the reflector is increased, the distance between two consecutive solutions of the Inhour equation decreases, then asymptotes getting closer and closer are the consequence in the introduction of a larger number of poles. Consequently, this growth in the number of poles is associated with an oscillating behavior in the reflector, as it is possible to observe in the case in which the dimensions of the reflector increase as seen above. Furthermore, for  $H_R \rightarrow \infty$  happens that  $\tanh[(\eta(p) H_R)] \rightarrow 1$  and then the characteristic equation of the problem becomes:

$$\tan(\alpha H_C) = \frac{D\alpha}{D^{(R)}\eta} \quad (5.3)$$



**Fig. 15.** Spatial eigenfunctions for case C.

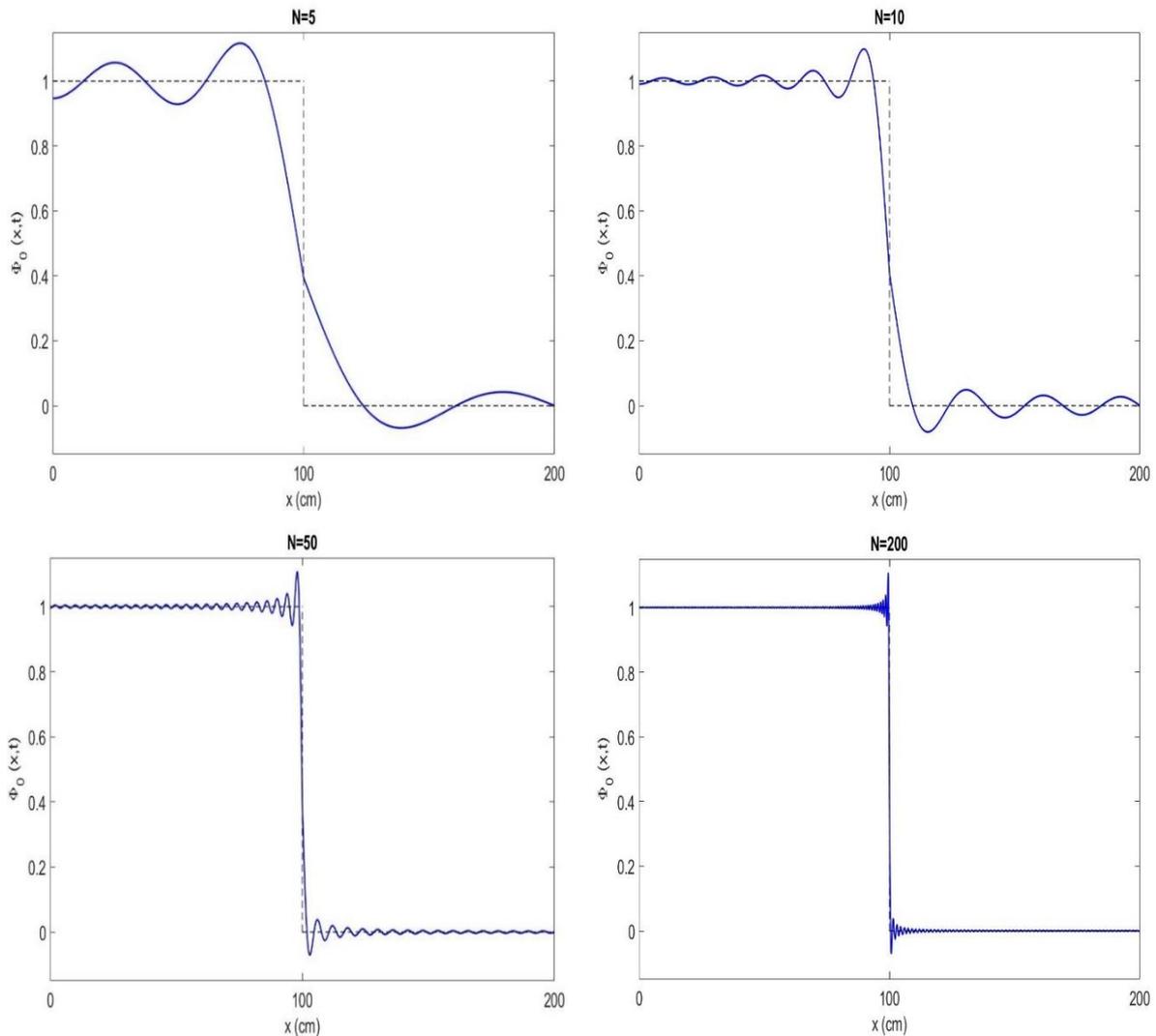


**Fig. 16.** Spatial eigenfunctions for case D.

Therefore, before to talk about the time evolution of the flux for different cases, I want to show how the number of clusters of eigenfunctions used to perform the calculations affects the solution. I took in to account two simple configuration to study this effect, a constant unitary initial condition in the core and in the reflector, without an external source and with one family of delayed precursors, with the data used in case A, for which the expressions of the time dependent function of the solution  $A_{n,j}(t)$  are:

$$\Phi_0(x) = \begin{cases} 1, & |x| \leq H_C \\ 0, & H_C < |x| \leq H \end{cases} \rightarrow A_{n,j}(t) = \frac{\sin(\alpha_{n,j}H_C)}{H_C} \quad (5.4)$$

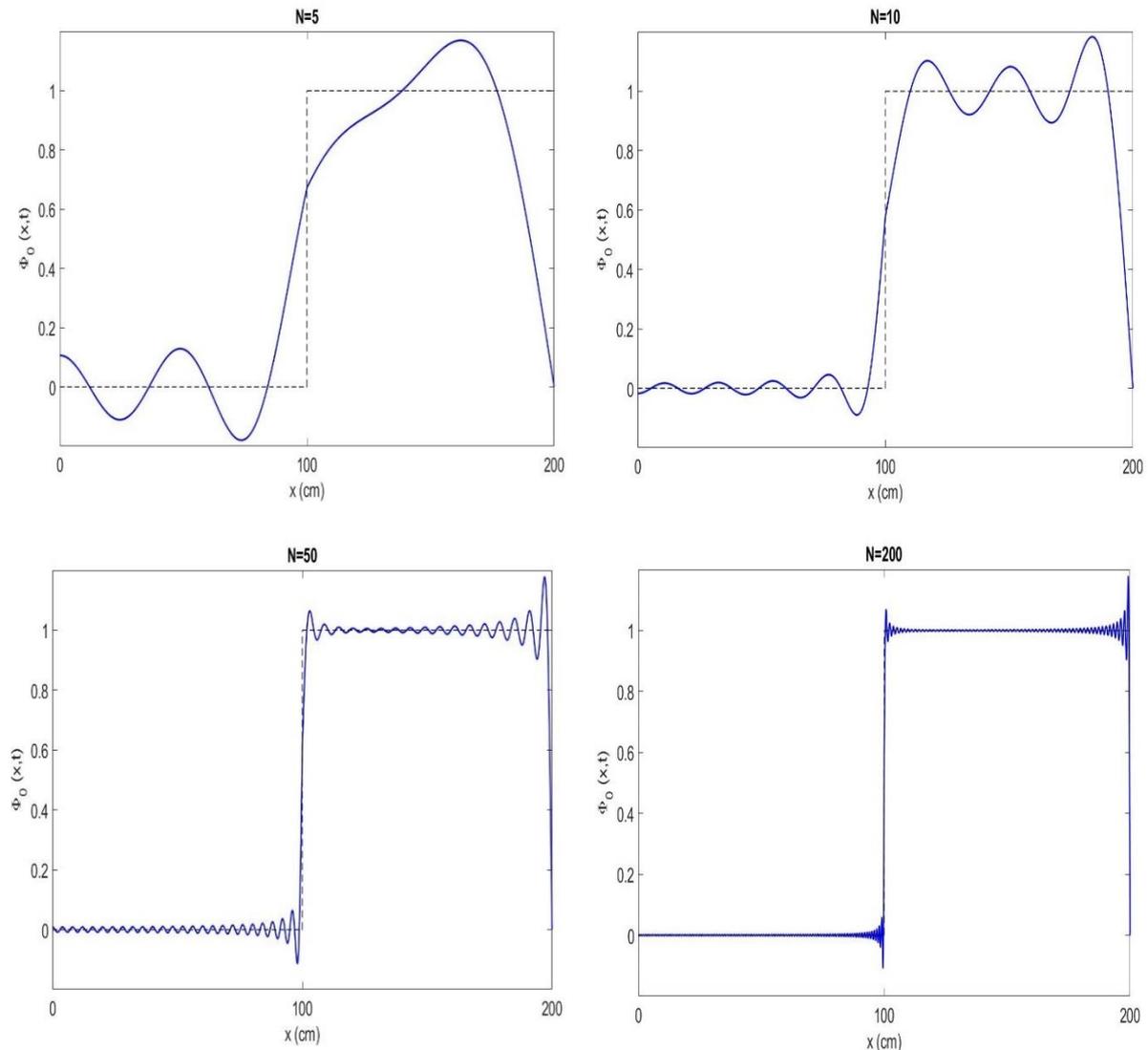
$$\Phi_0(x) = \begin{cases} 0, & |x| \leq H_C \\ 1, & H_C < |x| \leq H \end{cases} \rightarrow A_{n,j}(t) = \frac{1}{\eta_{n,j}} \frac{\cos(\alpha_{n,j}H_C)}{\sinh(\eta_{n,j}H)} [\cosh(\eta_{n,j}H_R) - 1] \quad (5.4')$$



**Fig. 17.** Convergence trend of the spatial eigenfunctions, uniform initial flux in the core and case A.

It is evident for the both case represented in Fig. 17 and Fig. 18 that with the increase of the number of eigenfunctions used the signal is more precise and stable. However

when there is an initial uniform distribution in the reflector the representation of the flux results to be worse than the other case, with the same number of poles used, for the lower number of oscillations that appear in the reflector than the ones appearing in the core zone.



**Fig. 18.** Convergence trend of the spatial eigenfunctions, uniform initial flux in the reflector and case A.

## 5.2-TRANSIENT ANALYSIS: SOURCE FREE SYSTEM

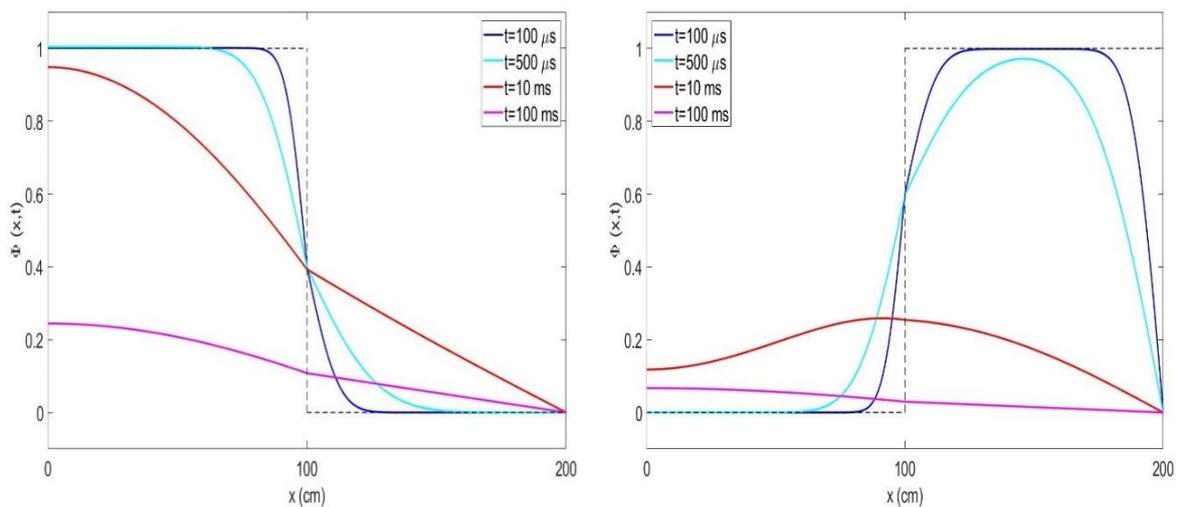
In this section I will analyze the evolution in time and space of my system without the presence of any kind of source in particular studying different initial state, the influence of the delayed precursors in the evolution of my system and the modification of system properties. The data used for the calculations correspond to the system configuration A seen before, unless otherwise specified.

In particular, the cases studied are:

- Source-free system without an initial concentration of precursors.
- Source-free system with the initial concentration of precursors in equilibrium with the initial flux.
- Source-free system without an initial concentration of precursors and source pulse in different positions.
- Modification of system properties.

### 5.2.1-SOURCE FREE SYSTEM WITHOUT AN INITIAL CONCENTRATION OF DELAYED PRECURSORS

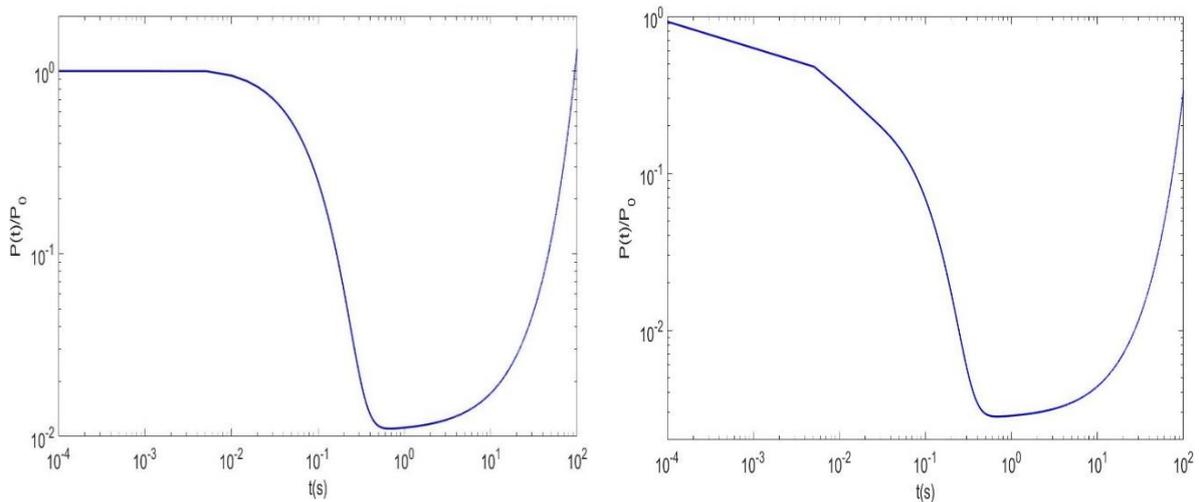
First of all I will analyze the transient calculations for the two initial states seen before, uniform neutron flux in the core and in the reflector, using the initial state and the time dependent function  $A_{n,j}(t)$  mentioned in the formulas (5.4) and (5.4'). The evolution of the flux inside the reactor for the two different initial conditions, at different times at the beginning of the transient, is reported in Fig. 19.



**Fig. 19.** Evolution of the neutron flux with uniform condition in the core (left) and in the reflector (right).

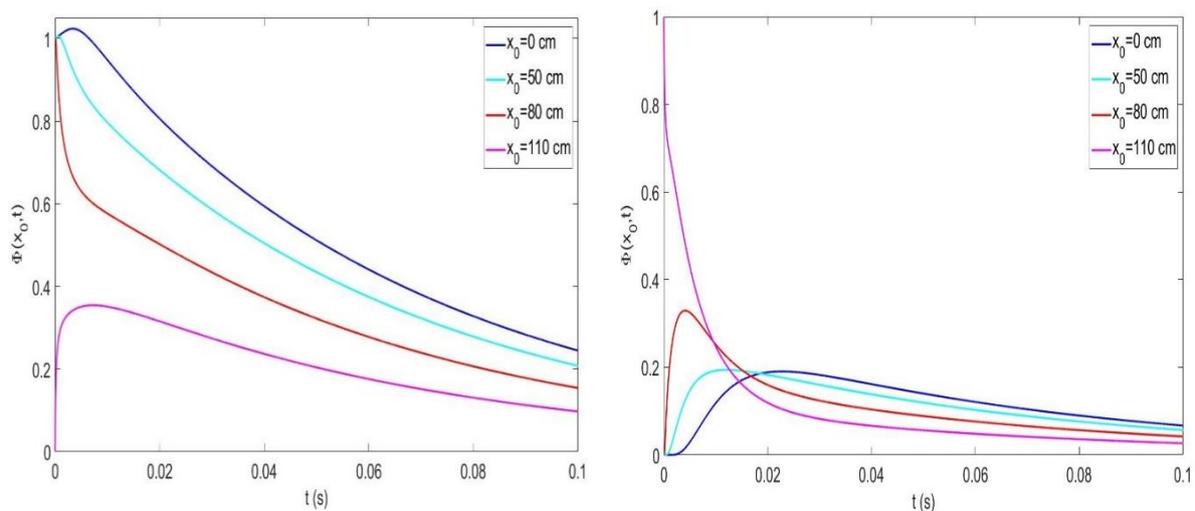
Regarding the power evolution in the reactor, in the case of non-zero initial condition in the core, it is evident a decreasing of the flux after the beginning of the transient and just when it is reached the equilibrium of the neutron precursors population the power in the system will begin to increase and subsequently to diverge due to the super criticality as it shown in the Fig. 20. For the other case, the non-zero initial condition in the reflector, the flux start to decreasing after the beginning of the transient as before then the decrease in the plot of the power is due to the non-equilibrium of the delayed neutron precursors and finally there is the growth in power

and divergence always due to the super criticality of the system. About the evolution of the power in time, despite the two different kind of the initial state, the answer of the system is similar in the two cases analyzed both from the point of view of the shape and the reaction time for the reason that the system is supercritical and it is possible just to observe that there is a difference in the power's level between the two cases due to the fact that the core is a multiplicative layer and the reflector not.



**Fig. 20.** Evolution of the power in the system with uniform condition in the core (left) and in the reflector (right).

I also analyzed the signal of detectors put in different positions for the two initial states taken into consideration, how it is possible to observe in Fig 21.

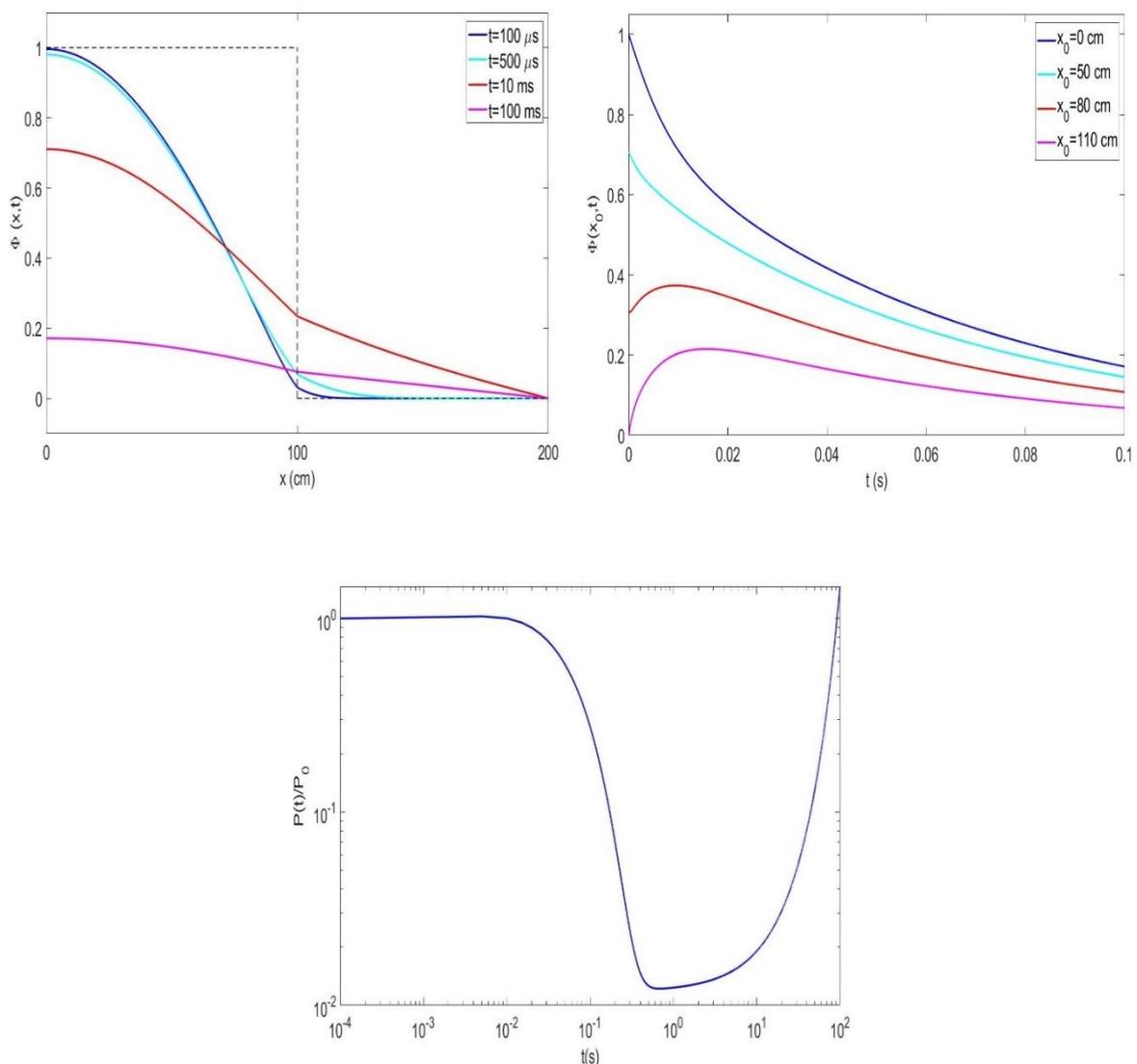


**Fig. 21.** Detector signals with uniform condition in the core (left) and in the reflector (right).

When I applied the excitation in the core region, as noted during the transient, before the effect due to the equilibrium achieved by the precursors, the flux decrease in the

time and just the detector located in the center show a different trend: in this location I observe firstly a growth due to the fact that the multiplication process takes place in a high-importance space position. Regarding the case in which the excitation is applied to the reflector, I notice that detectors in the core have a non-monotonic response due to the delay interested to the motion of the neutrons.

In the end I considering a case very closed to the non-zero initial condition in the core, that it will be useful in the next section where I study the one in which I have an initial concentration of delayed precursors other than zero: the case in which I have an initial cosine distribution of neutrons restricted to the core region.



**Fig. 22.** Initial cosine distribution: flux transient (top left), detector signal (top right), power evolution (low center).

The mathematic expression of the flux at  $t = 0$  in this condition is:

$$\Phi_0(x) = \begin{cases} \cos\left(\frac{\pi x}{H_R}\right) & 0 \leq x \leq H_C \\ 0 & H_C < x \leq H \end{cases} \quad (5.5)$$

Initial state to which correspond the follow expression for the term  $A_{n,j}(t)$ :

$$A_{n,j}(t) = \frac{H_R}{(\pi - \alpha_{n,j})(\pi + \alpha_{n,j})} \left[ \pi \cos(\alpha_{n,j}H_C) \sin\left(\frac{\pi x}{H_R}\right) - \alpha_{n,j}H_R \sin(\alpha_{n,j}H_C) \cos\left(\frac{\pi x}{H_R}\right) \right] \quad (5.6)$$

The plots, representing the space evolution of the flux in specific moments, the time evolution of the flux in specific locations and the time evolution of the power, are very similar to the plots given to the non-zero initial condition in the core region, as it is possible to observe in Fig. 22.

### 5.2.2- SOURCE FREE SYSTEM WITH AN INITIAL CONCENTRATION OF DELAYED PRECURSORS

In this paragraph I studied the condition in which I have an initial concentration of neutrons precursors in equilibrium with the initial flux, then I analyzed the non-zero initial condition in the core and the initial cosine distribution of neutrons restricted to the core region. The initial concentration of precursors is easily obtained from the second equation of the initial system (4.1):

$$C_0(x) = \frac{\beta k_{\infty} \Sigma_a}{\lambda} \Phi_0(x) \quad (5.7)$$

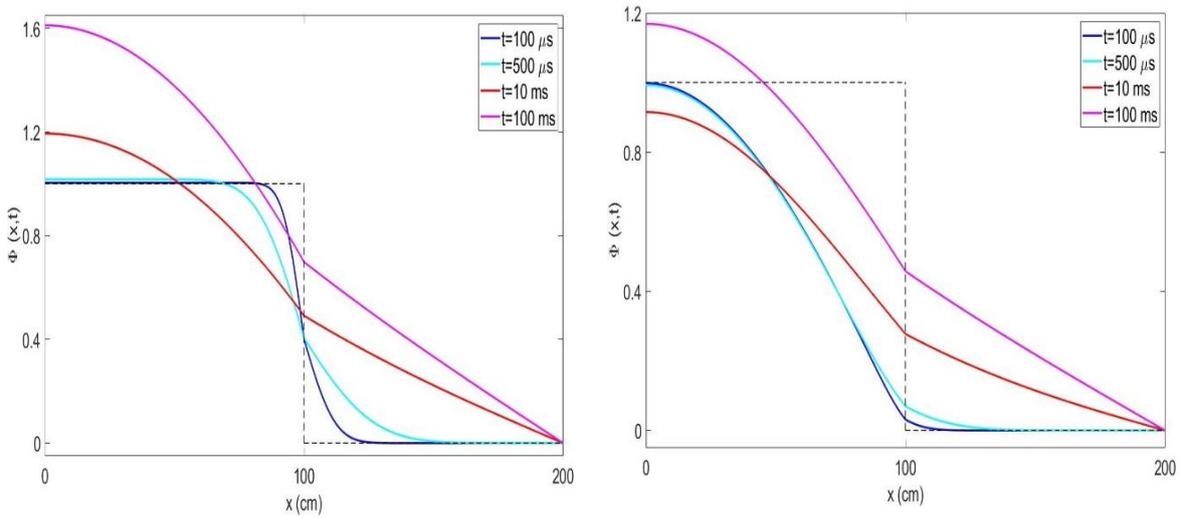
From the mathematic point on view the expression of the time function  $A_{n,j}(t)$  changes and becomes for the two different initial states examined:

$$A_{n,j}(t) = \left[ 1 + \left( \frac{v\lambda}{p_{n,j} + \lambda} \right) \left( \frac{\beta k_{\infty} \Sigma_a}{\lambda} \right) \right] \left( \frac{\sin(\alpha_{n,j}H_C)}{H_C} \right) \quad (5.8)$$

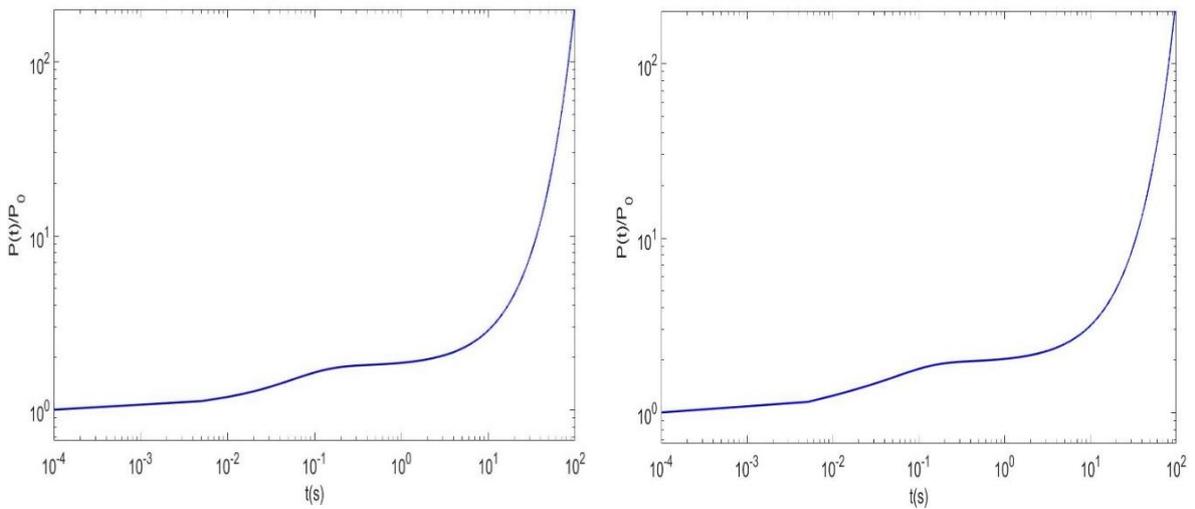
$$A_{n,j}(t) = \frac{H_R}{(\pi - \alpha_{n,j})(\pi + \alpha_{n,j})} \left[ 1 + \left( \frac{v\lambda}{p_{n,j} + \lambda} \right) \left( \frac{\beta k_{\infty} \Sigma_a}{\lambda} \right) \right] \left[ \pi \cos(\alpha_{n,j}H_C) \sin\left(\frac{\pi x}{H_R}\right) - \alpha_{n,j}H_R \sin(\alpha_{n,j}H_C) \cos\left(\frac{\pi x}{H_R}\right) \right] \quad (5.9)$$

First of all, I analyzed the space evolution of the flux at determined moments: in Fig. 23 it is shown that the increase of the flux in time is due to the presence of an initial concentration of neutron precursors though the time interval considered is very short. The increase of the flux in time previously observed is more evident for the graphs representing the evolution of power in time like in Fig. 24, in fact thanks to the delayed precursors, respect the cases without any initial concentration of them, the power tends to be constant at the beginning of the transient and then it tends to diverge

because of the super criticality of the system. Regarding the time evolution in different space points, I obtained plots in Fig. 25 that confirm the previous observation.



**Fig. 23.** Evolution of neutron flux with uniform (left) and cosine (right) initial condition in the core.

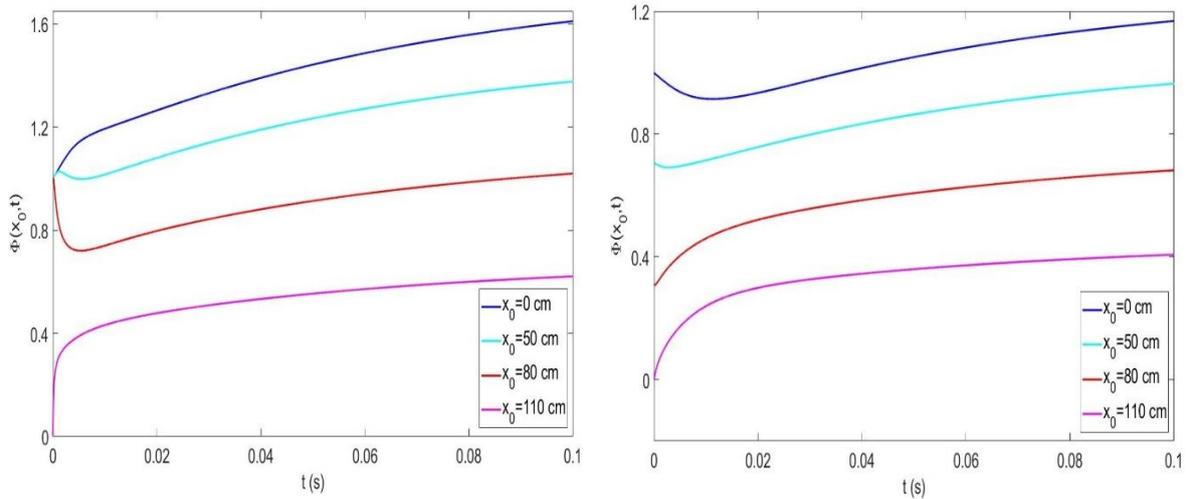


**Fig. 24.** Power evolution with uniform (left) and cosine (right) initial condition in the core.

### 5.2.3-SYSTEM WITH NEUTRON PULSE INJECTED IN DIFFERENT POSITION

In this part I will treat three cases without an initial concentration of neutron precursors and with an initial state corresponding to a neutron pulse injected at certain positions. I begin with the case in which the neutron pulse is injected at  $x = 0$ , the center of the whole system:

$$\Phi_0(x) = \delta(x) \rightarrow A_{n,j}(t) = \int_0^{H_C} \delta(x) \cos(\alpha_{n,j} x) dx = 1 \quad (5.10)$$



**Fig. 25.** Detector signals with uniform (left) and cosine (right) initial condition in the core.

Regarding the power evolution in time the system responds as the previous cases both in time and in space: an initial decrease in time of the flux due to non-equilibrium of the neutron precursors followed by a growth and the resultant divergence due to the super criticality like it is possible to observe in the time evolution of the power plot in Fig. 26; about the detectors, the signal is stronger how much the detectors are near to the neutron pulse.

In the second case the neutron pulse is injected in the symmetric middle of the reflector, exactly in the position  $x = 150 \text{ cm}$ , then the initial flux is  $\Phi_0(x) = \delta(x - 150)$  and  $A_{n,j}(t)$  becomes:

$$A_{n,j}(t) = \frac{\cos(\alpha_{n,j}H_C)}{\sinh(\eta_{n,j}H)} \int_{H_C}^{H_R} \delta(x - 150) \sinh[\eta_{n,j}(H - x)] dx = \frac{\cos(\alpha_{n,j}H_C)}{\sinh(\eta_{n,j}H)} \sinh[\eta_{n,j}(H - 150)] \quad (5.11)$$

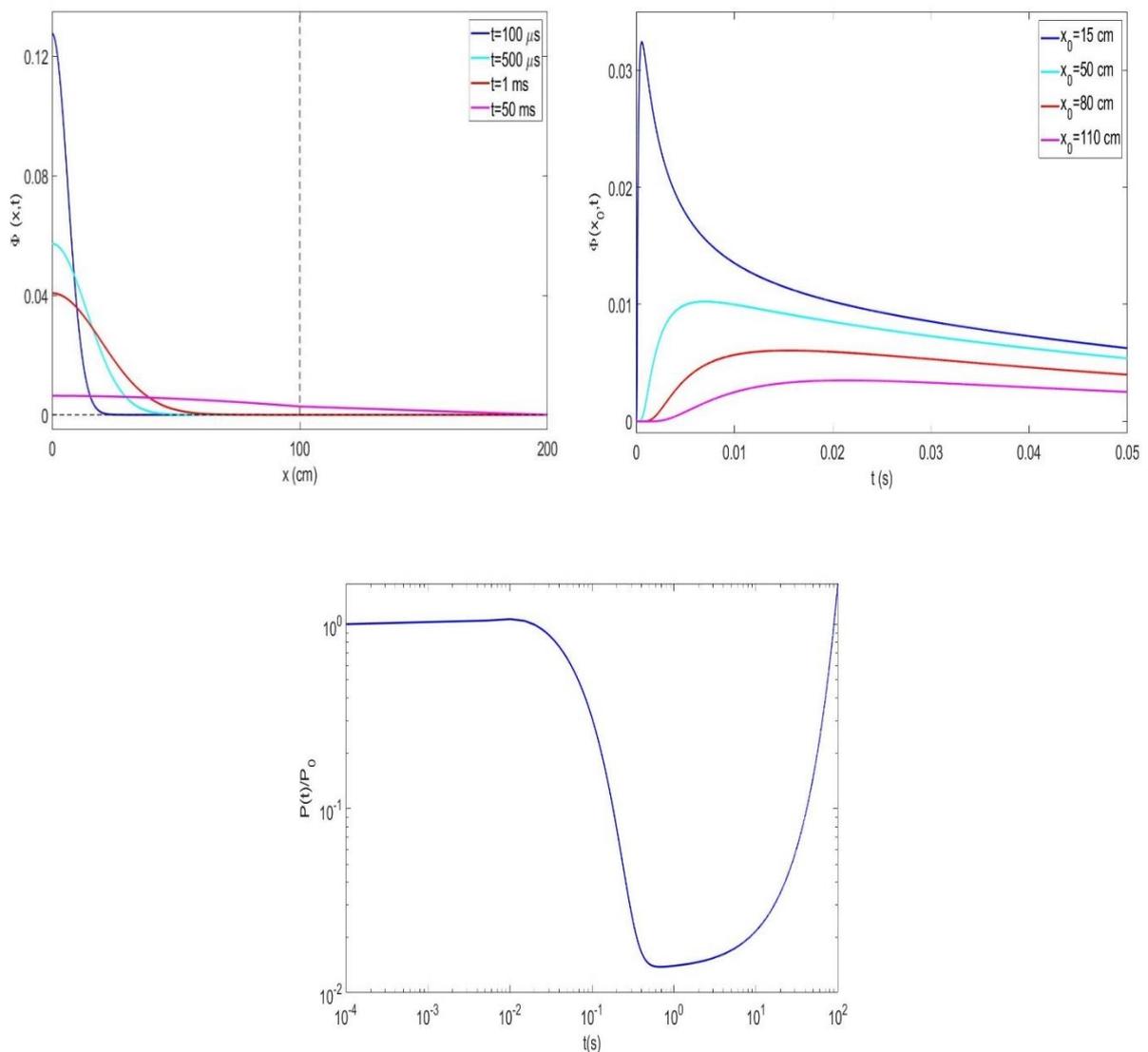
Also in this case the system responds like before, but in particular I can observe in Fig. 27, left row, that the signal of detectors at the beginning is higher in the zones near to the pulse but with the passage of time it become higher in the center of the system in which I have higher importance space position.

In the third case the neutron pulse is injected at the interface between the core and the reflector, then  $\Phi_0(x) = \delta(x - H_C)$ . In this case the integral of the term  $A_{n,j}(t)$  involves both in the core and in the rector region:

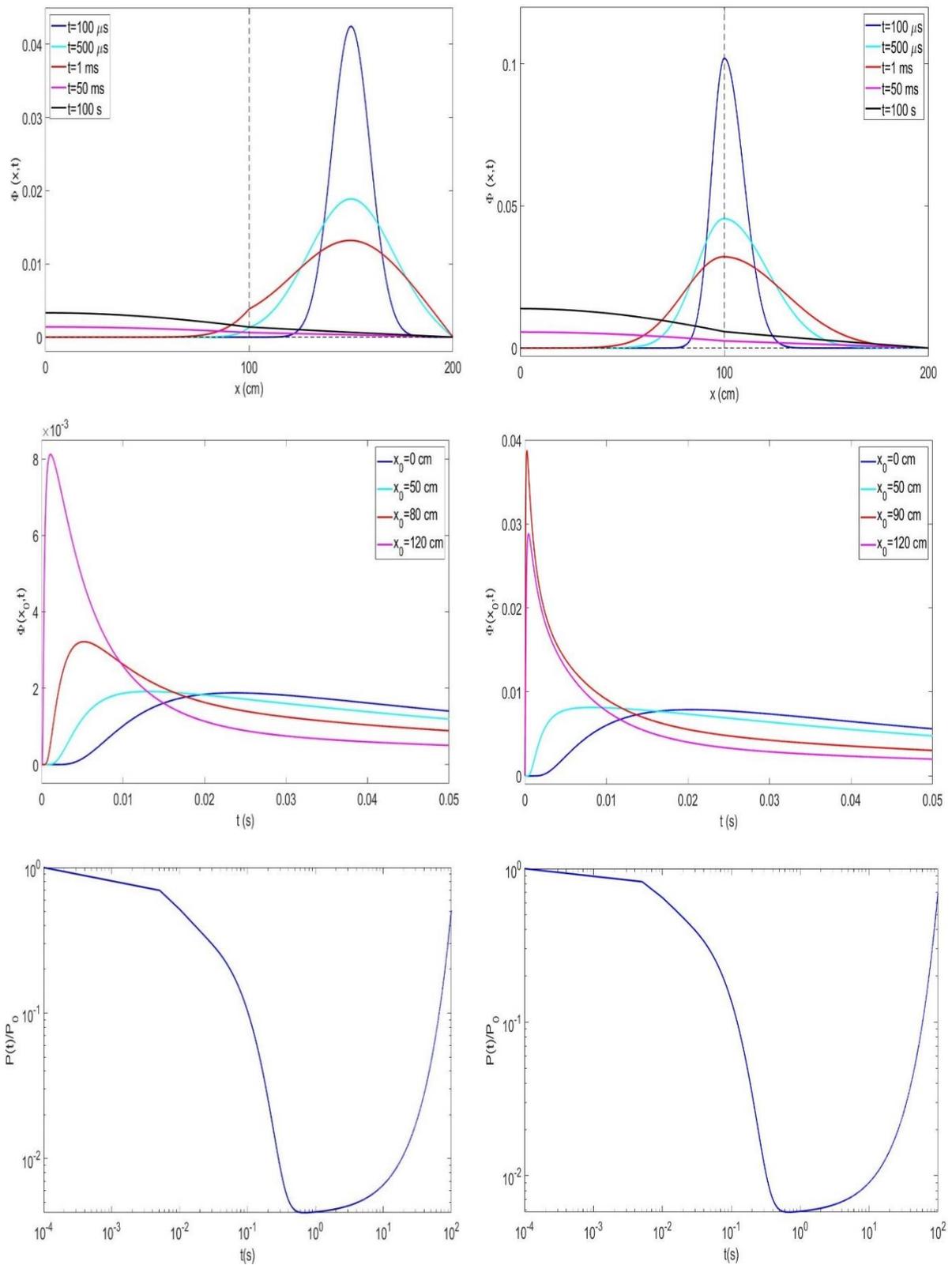
$$\begin{aligned} A_{n,j}(t) &= \left\{ \int_0^{H_C} \delta(x - H_C) \cos(\alpha_{n,j}x) dx \right\} + \left\{ \frac{\cos(\alpha_{n,j}H_C)}{\sinh(\eta_{n,j}H)} \int_{H_C}^{H_R} \delta(x - H_C) \sinh[\eta_{n,j}(H - x)] dx \right\} = \\ &= 2 \cos(\alpha_{n,j}H_C) \end{aligned} \quad (5.12)$$

The results of this case are always shown in Fig. 27, right row. Watching the three cases I can observe that when I injected the neutron pulse in a position with a high importance, as the center of the core, the power in the system is higher respect the other cases, in which the neutron pulse is injected in the reflector and at the interface between the two layers.

Furthermore, thanks to the plot about the time evolution of the flux regarding the last two cases, it is possible to understand in which way an excitation restricted to a certain area contributes to generate a reaction in the whole system.



**Fig. 26.** Neutron pulse in  $x=0$  cm: flux transient (top left), detector signal (top right), power evolution (low center).



**Fig. 27.** Flux transient (top), detector signal (middle) and power evolution (down) for neutron pulse in  $x=150$  cm (left figures) and neutron pulse in  $x=H_c$  (right figures).

### 5.2.4-VARIATION OF THE MULTIPLICATIVITY IN THE SYSTEM

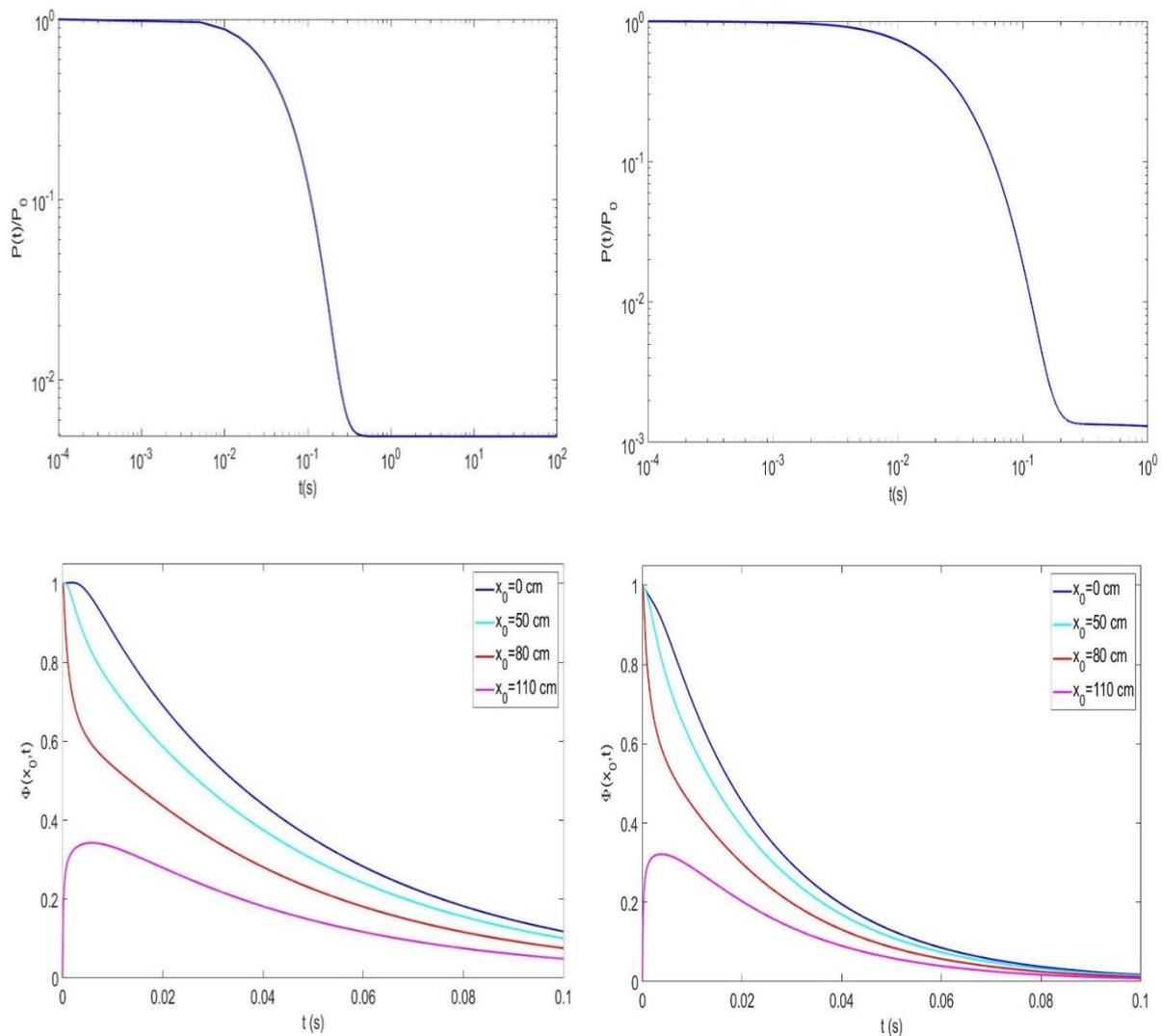
In the case seen before, the system studied corresponds to the system configuration A in which the system is supercritical, here I tried to solve the problem using two different value for  $k_\infty$ :

$$k_\infty = k_{\infty,1} = 1.006851 \rightarrow \text{critical system}$$

$$k_\infty = k_{\infty,2} = 1.001 \rightarrow \text{subcritical system}$$

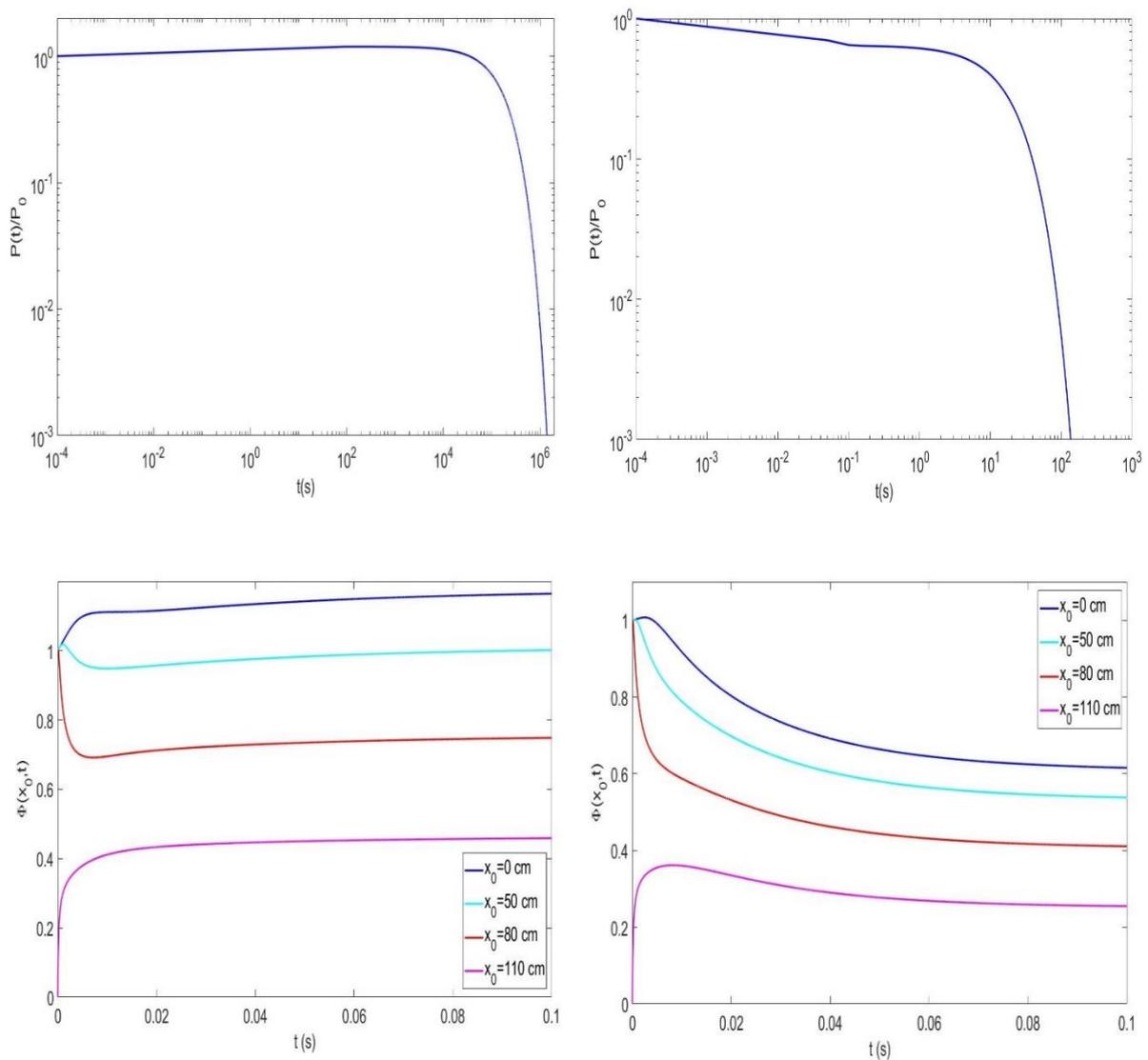
(5.13)

I start to analyze the cases in which I have a uniform initial distribution in the core without an initial concentration of delayed precursors for the two different values of multiplicativity.



**Fig. 28.** Power evolution (top) and signal detector (down) for  $k_{\infty,1}$ (left) and  $k_{\infty,2}$ (right).

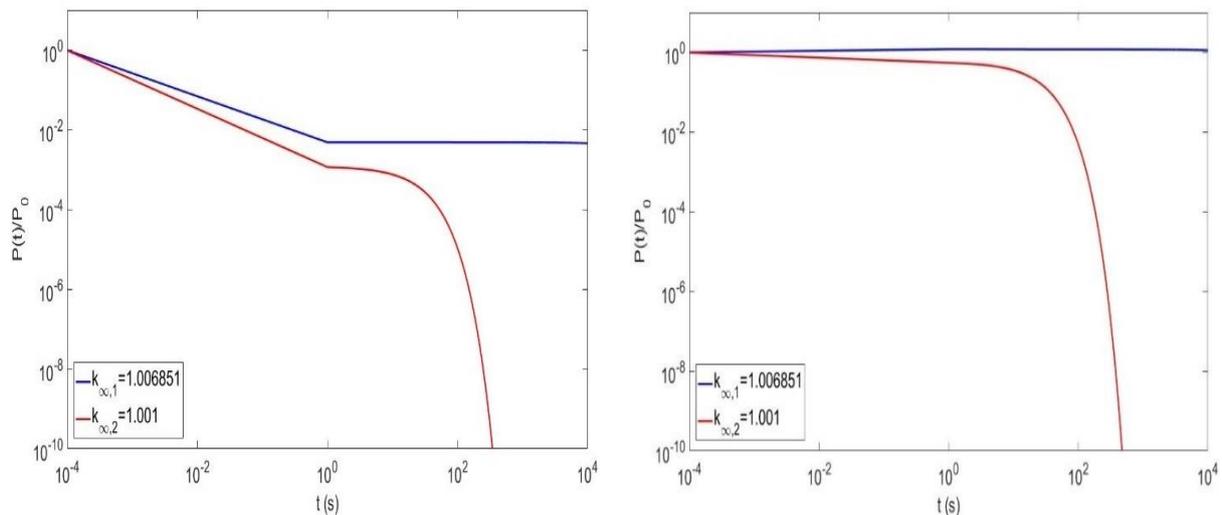
In Fig. 28 is evident that the system reacts in similar way if I use  $k_{\infty,1}$  or  $k_{\infty,2}$ , indeed in both cases the system tends to turn off as fast as the value of the multiplicativity of the system is little. This trend changes for both if I suppose an initial concentration of the delayed precursors in equilibrium with the initial configuration, as it is shown in Fig. 29. In this case, both for  $k_{\infty} = k_{\infty,1}$  and  $k_{\infty} = k_{\infty,2}$ , the power in the system is constant in time for a longer period respect the previous case, and in the end it decrease until zero: this behavior is confirmed both by the plot of the power and by the plot of the detectors. In the end, in Fig.30, it is present a comparison between the critical and subcritical case without and with an initial concentration of delayed precursors: the graphs show the previous conclusions more clearly.



**Fig. 29.** Power evolution (top) and signal detector (down) for  $k_{\infty,1}$ (left) and  $k_{\infty,2}$ (right) with an initial concentration of precursors.

### 5.2.5-VARIATION OF THE ABSORPTION CROSS SECTION IN THE REFLECTOR

In this part of my work I will study what happens to my system if I change the absorption cross section. I analyze the case in which I have a uniform initial distribution in the core without an initial concentration of delayed precursors for the case A but with  $\Sigma_a^{(R)} = 0.692 \cdot 10^{-2} \text{ 1/cm}$  and the case B.



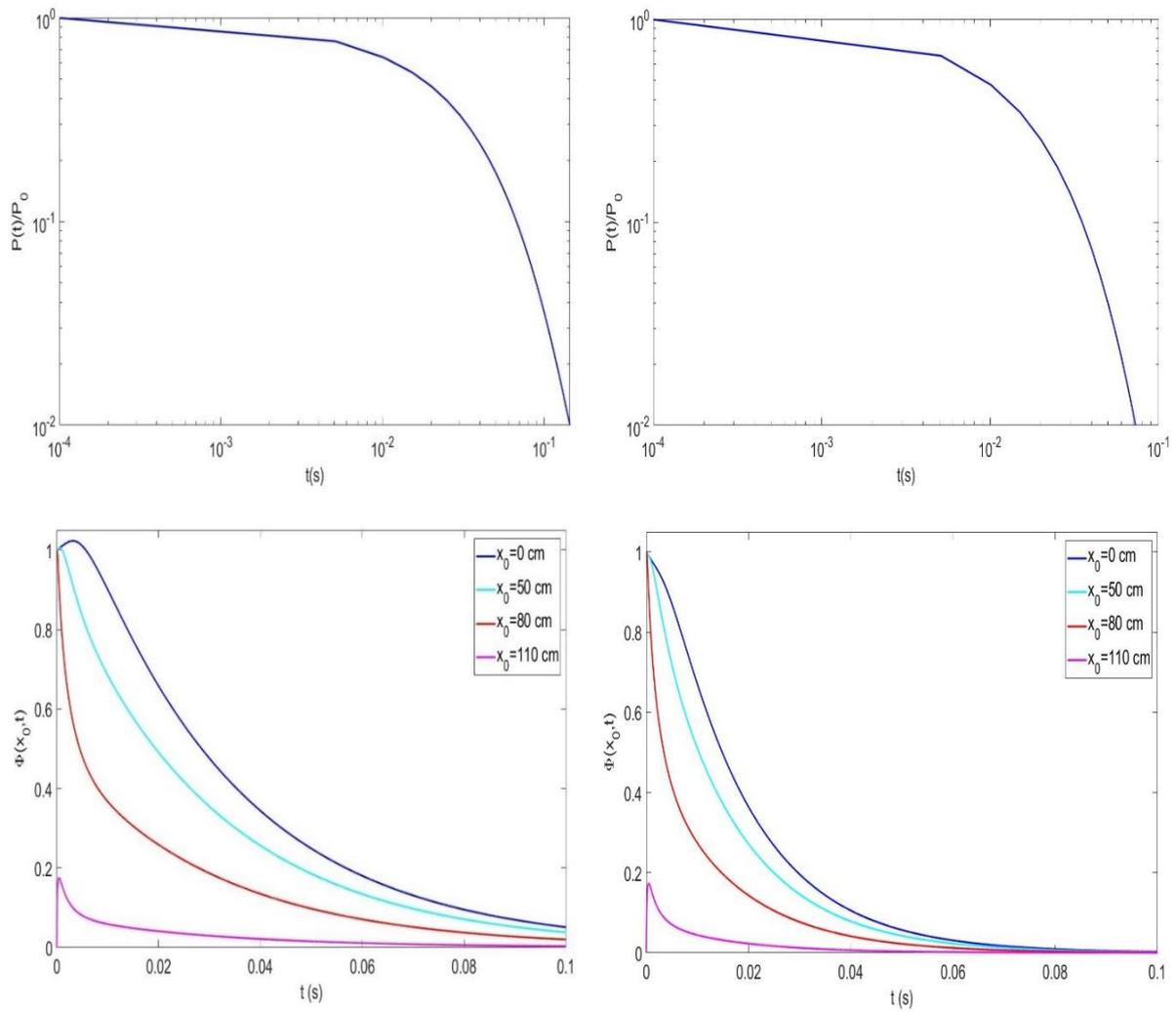
**Fig. 30.** Constant unitary initial condition in the core without (left) and with (right) an initial concentration of delayed precursors.

In Fig.31, in the left graphs, it is important to observe that also if the system has a supercritical multiplicative term, the reactor tends to switch off due to the greater loss of neutrons caused by a major absorption cross section in the reflector region. Then the plots regarding the case B, the right graphs in Fig.31, confirm the trend seen in the previous section: more the multiplicativity of the system is small, faster the system goes out.

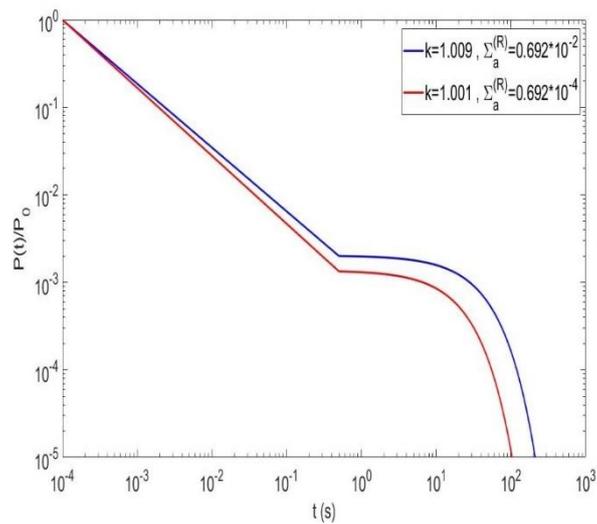
In the end, in Fig.32 is shown how much it is similar the shape of the power evolution for a supercritical system with  $\Sigma_a^{(R)} = 0.692 \cdot 10^{-2} \text{ 1/cm}$  and a subcritical system with  $\Sigma_a^{(R)} = 0.692 \cdot 10^{-4} \text{ 1/cm}$ .

### 5.3-TRANSIENT ANALYSIS: SOURCE TRANSIENTS

In this part of the work I analyzed the transients with an external source acting on system configuration B, the subcritical one. In particular I studied two main cases: a source consisting in a pulse of a certain duration and a constant source in time. For the both cases under examination the initial neutron flux  $\Phi_0(x)$  and the initial precursor concentration  $C_0(x)$  are set to zero.



**Fig. 31.** Uniform initial distribution in the core: power evolution (top) and signal detectors (down) with  $\Sigma_a^{(R)} = 0.692 \cdot 10^{-2} \text{ 1/cm}$  for case A (left) and case B (right).



**Fig. 32.** Comparison between sub-critical case A and super-critical case B.

### 5.3.1- PULSED SOURCE

In this section the kind of source under examination is the follow, a function depending both in time and space:

$$S(x, t) = \begin{cases} S(x), & 0 < t \leq T \\ 0, & T < t < \infty \end{cases} \quad (5.14)$$

$$S^{(R)}(x, t) = \begin{cases} S^{(R)}(x), & 0 < t \leq T \\ 0, & T < t < \infty \end{cases} \quad (5.15)$$

In the expressions above T represents the time during the source is on while  $S(x)$  and  $S^{(R)}(x)$  are the space distribution of the source in the space for the core and the reflector region, respectively. Regarding the time function  $A_{n,j}(t)$ , the source depends also on time, then the convolution term inside it have to be solved and for this case we have the follow general expression:

$$A(t) = \begin{cases} \left\{ \int_0^{H_C} S(x') \cos(\alpha_{n,j} x') dx' + \frac{\cos(\alpha_{n,j} H_C)}{\sinh(\eta_{n,j} H_R)} \int_{H_C}^H S^{(R)}(x') \sinh[\eta_{n,j}(H - x')] dx' \right\} \frac{1 - e^{-p_n t}}{p_n}, & 0 < t \leq T \\ \left\{ \int_0^{H_C} S(x') \cos(\alpha_{n,j} x') dx' + \frac{\cos(\alpha_{n,j} H_C)}{\sinh(\eta_{n,j} H_R)} \int_{H_C}^H S^{(R)}(x') \sinh[\eta_{n,j}(H - x')] dx' \right\} \frac{1 - e^{-p_n T}}{p_n}, & T < t < \infty \end{cases} \quad (5.16)$$

In the two first cases considered I used a unitary pulse of the duration of  $T = 10 \mu s$  applied symmetrically along 1/4 of the core thickness and along 1/4 of the reflector thickness, in the middle precisely. In Fig. 33 it is possible to observe the flux evolution in time and the flux evolution in time of detectors in different positions: in both case, with the configuration system B, the system tends to decay quickly.

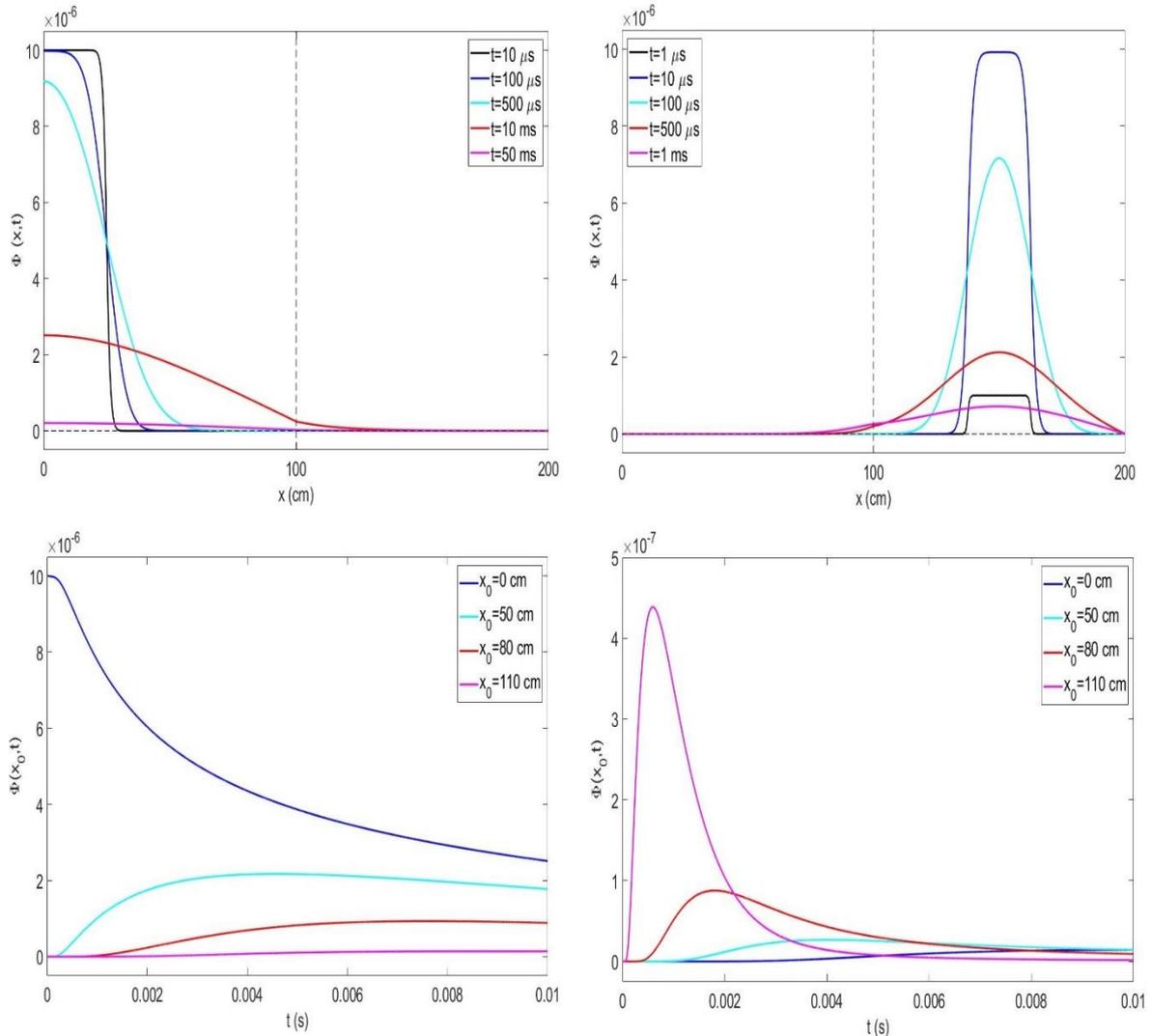
I obtained the same trends also if I used pulsed source located in different positions: in Fig. 34 I examined the cases in which I used a pulse in the middle of the core and a pulse at the interface between the core and the reflector.

### 5.3.2- CONSTANT SOURCE

The next case studied is the one in which the source is constant, for which the time function  $A(t)$  is:

$$A(t) = \left\{ \int_0^{H_C} S(x') \cos(\alpha_{n,j} x') dx' + \frac{\cos(\alpha_{n,j} H_C)}{\sinh(\eta_{n,j} H_R)} \int_{H_C}^H S^{(R)}(x') \sinh[\eta_{n,j}(H - x')] dx' \right\} \frac{1 - e^{-p_n t}}{p_n} \quad (5.17)$$

Concerning the spatial distribution of the source, I examined the same events seen in the previous section, starting from the uniform constant sources acting on a certain portions of the core and of the reflector.



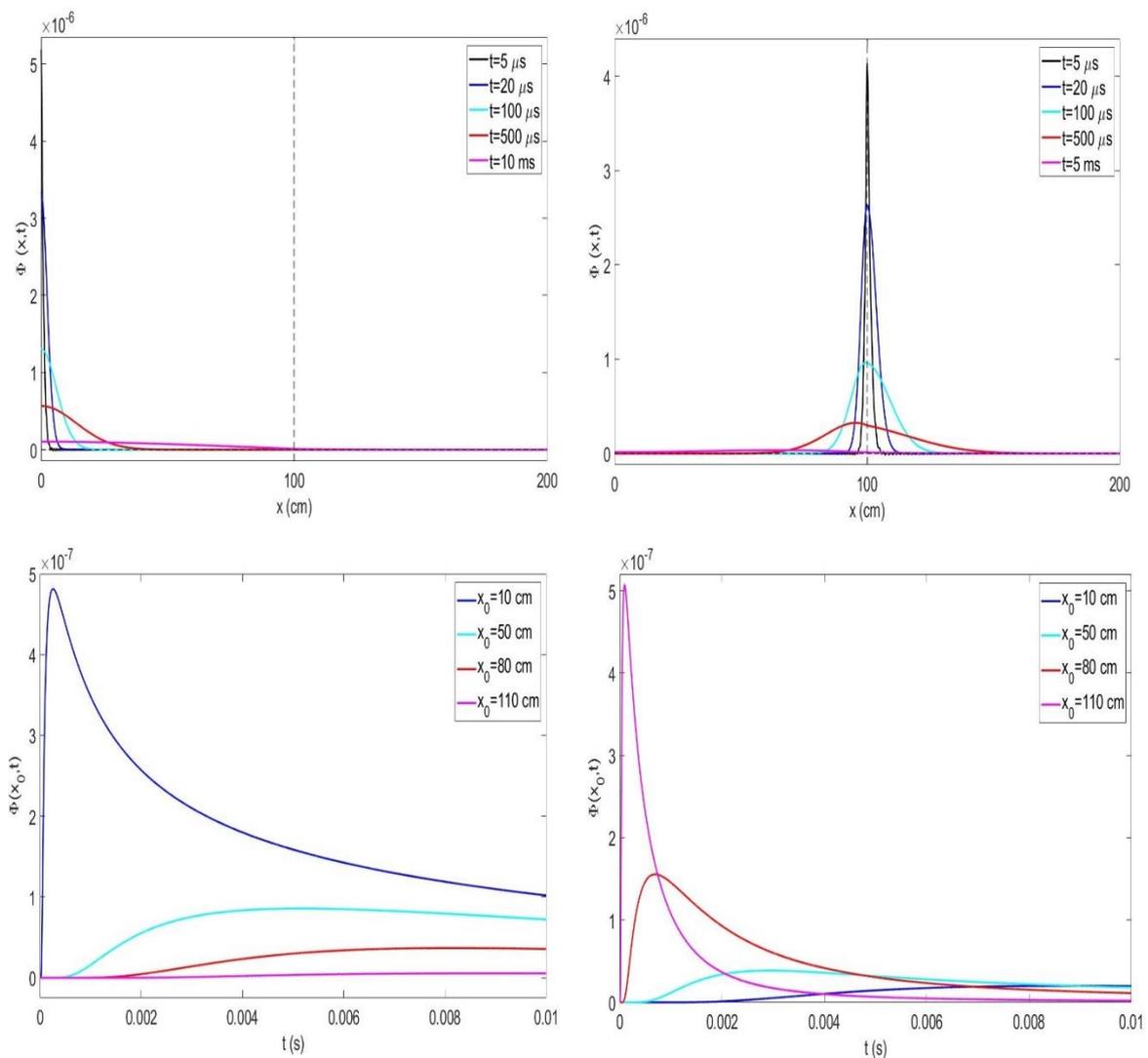
**Fig. 33.** Evolution of neutron flux (top) and detector signals (bottom) after a source pulse  $T$  in core (left) and in reflector (right).

It is evident, if I use a constant source in time, a very big difference respect the previous case, in which the source was constant just for a limited time period  $T$ : in the first row of the Fig. 35 we can observe that the flux grows in time, in addition to being greater than before.

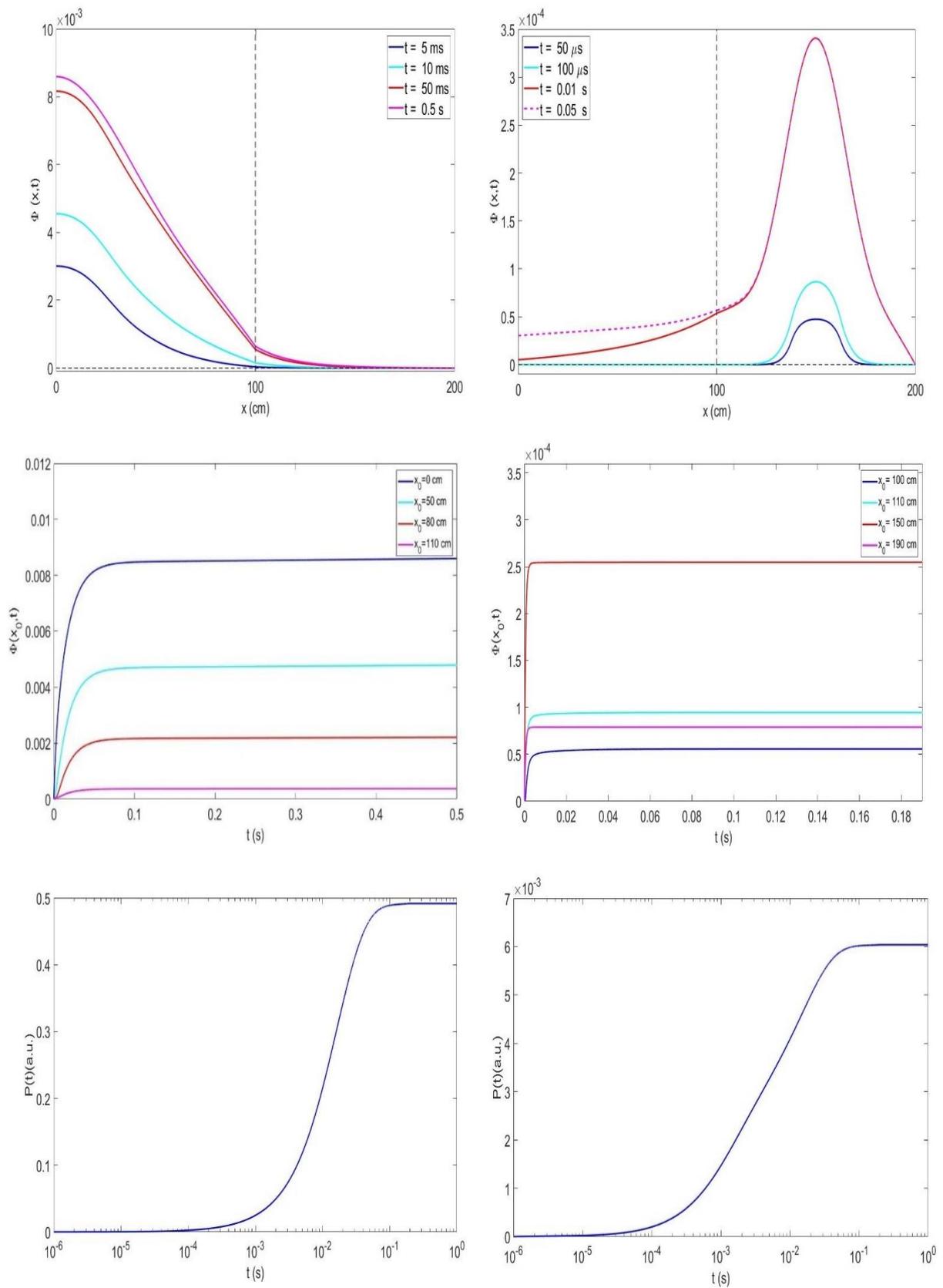
Moreover, always respect the flux evolution plots, both the distributions do not approximate the fundamental distribution, especially the reflector one. Regarding the signals from different detectors, second row in Fig. 35, from a certain moment onwards the system reaches a point-like evolution, so also the ratio of the flux

detected at different positions is constant. In addition in the graphs of the power evolution, third row in Fig.35, the value reached approaches a constant behavior in both the cases analyzed, as expected if a constant source is injected: the ration between the two asymptotic values obtained for the two different distributions clarifies how significant it is to choice a proper position to inject a source, then it explains the different importance of neutrons in the two cases examined.

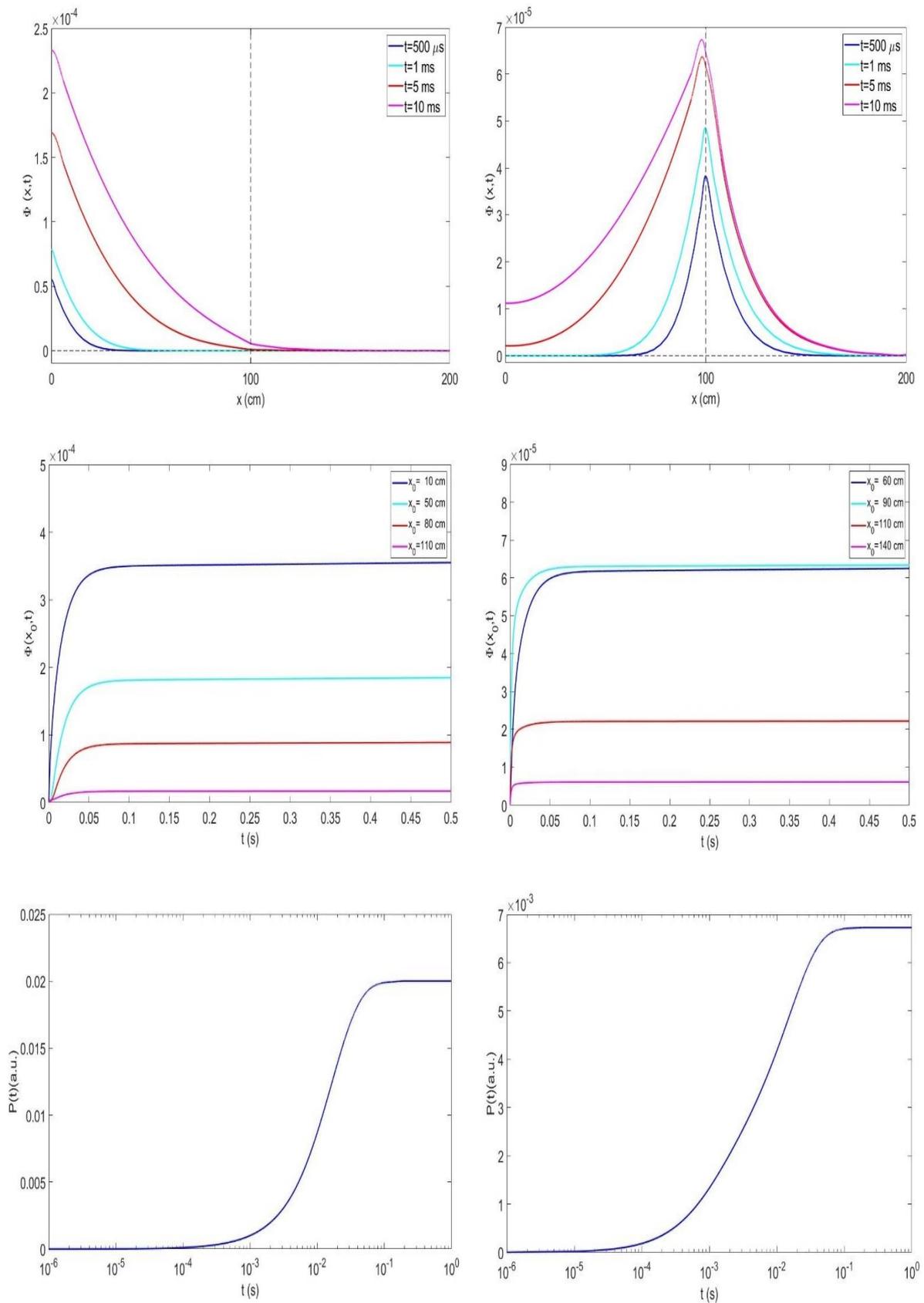
In the end, the same results just commented are exposed in Fig. 36, in which a pulsed source is positioned symmetrically in the center of the core and at the interface between the core and the reflector.



**Fig. 34.** Evolution of neutron flux (top) and detector signals (bottom) after a source pulsed in space and in time in the middle of the core (left) and at the interface (right).



**Fig. 35.** Evolution of neutron flux (top), detector signals (middle) and evolution of power (bottom) after a constant source injected in the core (left) and in the reflector (right).



**Fig. 36.** Evolution of neutron flux (top), detector signals (middle) and evolution of power (bottom) after a constant source pulse injected in the middle of core (left) and at the interface (right).

## 6-CONCLUSIONS

In this work the first part is entirely dedicated to introducing diffusion theory and the reactor kinetic theory with the aim of better understanding how the diffusion equation is constructed in the presence of delayed neutrons in the specific case of the reflected reactor. Therefore also the solutions, presented without numerical examples, have the purpose of constructing the heterogeneous reactor problem in the kinetics of the reactor and understanding the information at the basis of a solution, such as the boundary conditions and the initial conditions.

Then the neutron dynamics problem is explained in generic form for a reactor reflected in one-group diffusion theory in the simple geometric configuration of the one-dimensional slab. We proceeded to obtain the analytical solution using Laplace transforms and using the residue theorem for the inverse transformation.

Furthermore, as regards the poles that are obtained using this resolution technique, their orthonormality has been verified by mentioning another resolution method, the  $\omega$ -d modes. The numerical cases examined, in which the types of initial conditions or the characteristics of the materials that make up the reactor have changed, have helped us to better understand which parameters most influence the functioning of a reactor and what types of reactions take place in its various operational phases. To give a practical example, the hypothesis of the presence of an initial concentration of delayed precursors could be assimilated to the condition in which the reactor is turned off and then turned on again. Furthermore, the study of cases in which there is an external source that emits a constant amount of neutrons over time is interesting to understand how to obtain a constant power level inside a reactor.

Finally, the type of analytical solution found is useful for understanding how the dynamics of a reactor works and can also be compared to the results obtained by using numerical algorithms in solving the same kind of problem.

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