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Partially Hinged Rectangular Plates: Theoretical Analysis and Applications

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Introduction

The aim of the thesis is to present a rigorous theoretical analysis of the dynamics of partially hinged rectangular plates and their possible role in mathematical models of footbridges or suspension bridges expanding on the recent paper by Ferrero and Gazzola [8]. It is known that simplified models can prove useful since finer analytical methods can be applied to provide valuable insights into the physical properties of the structures, furthermore if the simple model exhibits a behaviour that can be considered dangerous for the structure, we can expect that the more precise model will behave similarly. On the other hand, a reliable model of bridges —even a very simple one— should have enough degrees of freedom to express the manifestation of torsional oscillations, which are considered by engineers a crucial factor in many collapse events. Motivated by these considerations, in [8] a simplified mathematical model of bridge is proposed where the deck is indeed seen as a long narrow thin plate hinged at short edges and free on the remaining two; this reflects the fact that the deck is supported by the ground at short edges.

The dissertation is opened by a compendium of suitable theoretical background material on higher order Sobolev spaces and elliptic boundary value problems reported here for the convenience of the reader, furthermore proper references are given to the update literature. Then we proceed by recalling the bending elastic energy of a deflected plate originating from the classical Kirchhoff-Love theory of elasticity, then the fourth order partial differential equation satisfied by the equilibrium position of the plate is derived together with the associated boundary conditions, which are of Navier type on short edges and of Neumann type on the remaining edges. This is followed by a rigorous theoretical study of existence, uniqueness and regularity of solutions of the boundary value problem obtained. Additionally, the explicit form of the solution is written by applying a suitable technique of separation of variables, refined to reflect the presence of peculiar boundary conditions. We improve on the results given in [8] by relaxing the assumptions on the dependence of the load from the position, thus extending the result to arbitrary forcing terms. As shown in [8], the same separation of variables technique may be exploited to characterize completely the oscillating modes of the plate, i.e., the spectrum and the corresponding eigenfunctions of the biharmonic operator under partially hinged boundary conditions.

The thesis is organized as follows: Chapter 2 is devoted to review some classical definitions and results on subjects such as higher order Sobolev spaces, polyharmonic operators and spectral theory, which will be used extensively in the following chapters. In Chapter 3 we lay out the physical motivations which lead to the fourth order problems we tackle in the thesis, specifically we derive a linear stationary model as well as nonlinear stationary and dynamic models, where the nonlinearity is introduced by the sustaining action of the bridge's hangers. In Chapter 4 we focus our attention on the linear stationary problem and we prove existence and uniqueness of its solutions. Moreover we provide the explicit Fourier expansion of solutions generalizing the results of [8] to more arbitrary load functions. The chapter finishes by providing qualitative results on the solutions' behaviour, in particular by illustrating the relation between the two-dimensional plate model and a one-dimensional beam corresponding to a plate of infinitesimal width. Finally, Chapter 5 analyses the eigenvalue problem related to the modes of oscillations of the plate and a characterization of the relative eigenvalues and eigenfunctions is given.

Chapter 2

Mathematical background

In this chapter we present some mathematical background that will be useful in the following.

We start by introducing Sobolev spaces, which will serve as a natural setting for our problem, and we formulate the Sobolev embeddings, which are useful to study the relation between classical and weak solutions of the problem. Then we review some crucial theoretical facts about polyharmonic differential equations in a general setting, presenting in particular the complementing conditions, which are algebraic constraints on the boundary conditions that guarantee the well posedness of our equation. We end the chapter by stating some results on spectral theory of compact and symmetrical operators, which we apply to obtain a theorem characterizing the solutions of a generalized polyharmonic eigenvalue problem. In this way we build the setting for the core chapters of this work.

2.1 Higher order Sobolev spaces

In this section we briefly recall the definition and basic properties of higher order Sobolev spaces and of their embedding into L^q spaces. In particular, we need to define the trace operators in order to give some meaning to the boundary conditions. Throughout the chapter Ω denotes an open and connected domain of $\mathbb{R}^n$ ($n \geq 2$). The smoothness assumptions on the boundary $\partial\Omega$ will be made precise in each situation considered. For this section we refer the reader to Section 2.2 of the book by Gazzola, Grunau and Sweers [9], which we follow in our description.

2.1.1 Definitions and basic properties

Given a domain $\Omega \subset \mathbb{R}^n$, $\|\cdot\|_{L^p}$ denotes the standard $L^p(\Omega)$-norm for $1 \leq p \leq \infty$. For all $m \in \mathbb{N}^+$ let us define the norm

$$u \mapsto N(u) := \left(\sum_{k=0}^{m} \left\| |D^k u| \right\|_{L^p}^p \right)^{1/p} \tag{2.1}$$

where $D^0u = u$,

$$D^ku \cdot D^k\nu = \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k u}{\partial x_{i_i}\dots\partial x_{i_k}} \frac{\partial^k \nu}{\partial x_{i_i}\dots\partial x_{i_k}} \text{ and } |D^ku| = \left(D^ku \cdot D^ku\right)^{1/2} .$$

Note that we will specify the domain Ω in $\|\cdot\|_{L^p}$ only when it is not clear from the context. Next, we define the space

$$W^{m,p}(\Omega) := \overline{\{u \in C^m(\Omega); N(u) < \infty\}}^N ,$$

that is, the completion with respect to the norm (2.1). Alternatively, $W^{m,p}(\Omega)$ may be defined as the subspace of $L^p(\Omega)$ of functions having generalised derivatives up to order m in $L^p(\Omega)$. If $\Omega \neq \mathbb{R}^n$ and its boundary $\partial\Omega$ is smooth, then a function $u \in W^{m,p}(\Omega)$ admits some traces on $\partial\Omega$ where, for our purposes, it is enough to restrict the attention to the case $p \in (1,\infty)$. More precisely, if ν denotes the unit outer normal to $\partial\Omega$, then for any $u \in C^m(\Omega)$ and any $j = 0,\dots,m$ we define the traces

$$\gamma_j u := \frac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega} . \tag{2.2}$$

By [11, Théorème 8.3], these linear operators may be extended continuously to the larger space $W^{m,p}(\Omega)$. We set

$$W^{m-j-1/p,p}(\partial\Omega) := \gamma_j[W^{m,p}(\Omega)] \text{ for } j = 0,\dots,m-1. \tag{2.3}$$

In particular, $W^{1/p',p}(\partial\Omega) = \gamma_{m-1}[W^{m,p}(\Omega)]$, where p' is the conjugate of p (that is, $p + p' = pp'$). We also put

$$\begin{aligned}\gamma_m[W^{m,p}(\Omega)] = W^{-1/p,p}(\partial\Omega) &:= [W^{1/p,p'}(\partial\Omega)]' \\ &= \text{the dual space of } W(\partial\Omega),\end{aligned} \tag{2.4}$$

so that (2.3) makes sense for all $j = 0,\dots,m$. With an abuse of notation, in the sequel we simply write u (respectively $\frac{\partial^j u}{\partial \nu^j}$) instead of $\gamma_0 u$ (respectively $\gamma_j u$ for $j = 1,\dots,m$).

When $p = 2$, we put $H^m(\Omega) := W^{m,2}(\Omega)$. Moreover, when $p = 2$ and $m \geq 1$ we write $H^{m-1/2}(\partial\Omega) = W^{m-1/2,2}(\partial\Omega)$ and

$$H^{-m+\frac{1}{2}}(\partial\Omega) = [H^{m-\frac{1}{2}}(\partial\Omega)]' = \text{the dual space of } H^{m-\frac{1}{2}}(\partial\Omega) . \tag{2.5}$$

The space $H^m(\Omega)$ becomes a Hilbert space when endowed with the scalar product

$$(u,v) \mapsto \sum_{k=0}^{m} \int_\Omega D^ku \cdot D^kv dx \qquad \text{for all } u,v \in H^m(\Omega) .$$

In some cases one may simplify the just defined norms and scalar products. As a first step, we mention that thanks to interpolation theory, see [1, Theorem 4.14], one can neglect intermediate derivatives in (2.1). More precisely, $W^{m,p}(\Omega)$ is a Banach space also when endowed with the following norm, which is equivalent to (2.1):

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \||D^mu|\|_{L^p} \qquad \text{for all } u \in W^{m,p}(\Omega) , \tag{2.6}$$

whereas $H^m(\Omega)$ is a Hilbert space also with the scalar product

$$(u,v)_{H^m} := \int_\Omega (uv + D^m u \cdot D^m v)dx \qquad \text{for all } u,v \in H^m(\Omega)\,. \tag{2.7}$$

Of particular interest is the closed subspace of $W^{m,p}$ defined as the intersection of the kernels of the trace operators in (2.2), that is for any bounded domain Ω we consider

$$W_0^{m,p}(\Omega) := \bigcap_{j=0}^{m-1} \ker \gamma_j\,.$$

Moreover, for bounded domains Ω and for $1 < p < \infty$, if p' is the conjugate of p we write

$$W^{-m,p'}(\Omega) := [W_0^{m,p}(\Omega)]' = \text{ the dual space of } W_0^{m,p}(\Omega) \tag{2.8}$$

and, for $p = 2$,

$$H^{-m}(\Omega) := [H_0^m(\Omega)]'\,.$$

Consider the bilinear form

$$(u,v)_{H_0^m} := \begin{cases} \int_\Omega \Delta^k u \Delta^k v dx & \text{if } m = 2k \\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v)dx & \text{if } m = 2k+1\,, \end{cases} \tag{2.9}$$

and the corresponding norm

$$\|u\|_{H_0^m} := \begin{cases} \left\|\Delta^k u\right\|_{L^2} & \text{if } m = 2k \\ \left\|\nabla(\Delta^k u)\right\|_{L^2} & \text{if } m = 2k+1\,. \end{cases} \tag{2.10}$$

For general $p \in (1,\infty)$, one has the choice of taking the L^p-version of (2.10) or the equivalent norm

$$\|u\|_{W_0^{m,p}} := \| |D^m u| \|_{L^p}\,.$$

Thanks to these norms, in the case of a bounded domain Ω, one may define the above spaces in a different way, namely

$$\begin{aligned} W_0^{m,p}(\Omega) &= \text{the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\cdot\|_{W^{m,p}} \\ &= \text{the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\cdot\|_{W_0^{m,p}}\,. \end{aligned} \tag{2.11}$$

The above definitions follow by combining interpolation inequalities (see [1, Theorem 4.14]) with the classical Poincaré inequality $\|\nabla u\|_{L^p} \geq c\,\|u\|_{L^p}$ for all $u \in W_0^{1,p}(\Omega)$. If Ω is unbounded, including the case where $\Omega = \mathbb{R}^n$, we define

$$\begin{aligned} \|u\|_{\mathcal{D}^{m,p}} &:= \| |D^m u| \|_{L^p}\,, \\ \mathcal{D}^{m,p} &:= \text{the closure of } C_c^\infty(\Omega) \text{with respect to the norm } \|\cdot\|_{\mathcal{D}^{m,p}}\,, \end{aligned}$$

and, again, let $W_0^{m,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}}$. In this unbounded case, the alternative definition (2.11) is no longer valid since although $W_0^{m,p}(\Omega) \subset \mathcal{D}^{m,p}(\Omega)$, the converse inclusion fails. For instance, if $\Omega = \mathbb{R}^n$, then

$W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$, whereas the function $u(x) = (1+|x|^2)^{(1-n)/4}$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^n)$ but not to $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

As a consequence of (2.11) we have that, when Ω is bounded, the space $H_0^m(\Omega)$ is a Hilbert space when endowed with the scalar product (2.9). The striking fact is that not only all lower order derivatives are neglected but also that some of the highest order derivatives are dropped. This fact has a simple explanation since

$$(u,v)_{H_0^m} = \int_\Omega D^m u \cdot D^m v dx \qquad \text{for all } u,v \in H_0^m(\Omega)\,. \tag{2.12}$$

One can verify (2.12) by using a density argument, namely for all $u, v \in C_c^\infty(\Omega)$. And with this restriction, one can integrate by parts several times in order to obtain (2.12). The bilinear form (2.9) also defines a scalar product on the space $\mathcal{D}^{m,2}(\Omega)$ whenever Ω is an unbounded domain. We summarise all these facts in

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain. Then the bilinear form*

$$(u,v) \mapsto \begin{cases} \int_\Omega \Delta^k u \Delta^k v dx & \text{if } m = 2k\,, \\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v) dx & \text{if } m = 2k+1\,, \end{cases} \tag{2.13}$$

defines a scalar product on $H_0^m(\Omega)$ (respectively $\mathcal{D}^{m,2}(\Omega)$) if Ω is bounded (respectively unbounded). If Ω is bounded, then this scalar product induces a norm equivalent to (2.1).

2.1.2 Embedding theorems

Consider first the case of unbounded domains.

Theorem 2.1.2. *Let $m \in \mathbb{N}^+$, $1 \le p < \infty$, with $n > mp$. Assume that $\Omega \subset \mathbb{R}^n$ is an unbounded domain with uniformly Lipschitzian boundary $\partial\Omega$, then:*

1. *$\mathcal{D}^{m,p}(\Omega) \subset L^{np/(n-mp)}(\Omega)$;*
2. *$W^{m,p}(\Omega) \subset L^q(\Omega)$ for all $p \le q \le \frac{np}{n-mp}$.*

On the other hand, in bounded domains subcritical embeddings become compact.

Theorem 2.1.3. *Let $m \in \mathbb{N}^+$, $1 \le p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitzian domain, then for any $1 \le q < \frac{np}{n-mp}$ there exists a compact embedding $W^{m,p}(\Omega) \subset L^q(\Omega)$. Here we make the convention that $\frac{np}{n-mp} = \infty$ if $n \le mp$.*

Remark 2.1.4. The optimal constants of the compact embeddings in Theorem 2.1.3 are attained on functions solving corresponding Euler–Lagrange equations.

In fact, if $n < mp$, Theorem 2.1.3 may be improved by the following statement.

Theorem 2.1.5. *Let $m \in \mathbb{N}^+$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitzian boundary. Assume that there exists $k \in \mathbb{N}$ such that $n < (m-k)p$. Then*

$$W^{m,p}(\Omega) \subset C^{k,\gamma}(\Omega) \qquad \textit{for all } \gamma \in \left(0, m-k-\frac{n}{p}\right] \cap (0,1)$$

with compact embedding if $\gamma < m-k-\frac{n}{p}$.

The statements of Theorems 2.1.3 and 2.1.5 also hold if we replace $W^{m,p}(\Omega)$ with its proper subspace $W_0^{m,p}(\Omega)$. In this case, no regularity assumption on the boundary $\partial\Omega$ is needed.

2.2 Polyharmonic operators and complementing conditions

In this section we present the more general framework of polyharmonic operators among which the biharmonic operator Δ^2 appearing in the classical equation (3.8) is a special case. In this setting we describe the complementing conditions and their role in ensuring existence and uniqueness of the solutions of a polyharmonic differential equation. For this part the exposition is based on Chapter 2 of the book by Gazzola, Grunau and Sweers [9].

2.2.1 Polyharmonic operators

Throughout this section we assume of the domain $\Omega \subset \mathbb{R}^n$ not only that it is open an connected but also that it is bounded. Moreover, we shall always assume that $\partial\Omega$ is Lipschitzian so that the tangent hyperplane and the unit outward normal $\nu = \nu(x)$ are well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(n-1)$-dimensional Hausdorff measure. When it is clear from the context, in the sequel we omit writing "a.e."

The Laplacian Δu of a smooth function $u : \Omega \to \mathbb{R}$ is the trace of its Hessian matrix, namely

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} .$$

We are interested in iterations of the Laplace operator, namely polyharmonic operators defined inductively by

$$\Delta^m u = \Delta(\Delta^{m-1} u) \qquad \text{for } m = 2, 3, \dots .$$

Arguing by induction on m, it is straightforward to verify that

$$\Delta^m u = \sum_{\ell_1+\cdots+\ell_n=m} \frac{m!}{\ell_1! \dots \ell_n!} \frac{\partial^{2m} u}{\partial x_1^{2\ell_1} \dots \partial x_n^{2\ell_n}} .$$

The polyharmonic operator Δ^m may also be seen in an abstract way through the polynomial $L_m : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L_m(\xi) = \sum_{\ell_1+\cdots+\ell_n=m} \frac{m!}{\ell_1! \dots \ell_n!} \left(\prod_{i=1}^{n} \xi_i^{2\ell_i} \right) = |\xi|^{2m} \text{ for } \xi \in \mathbb{R}^n .$$

Formally, $\Delta^m = L_m(\nabla)$. In particular, this shows that $L_m(\xi) > 0$ for all $\xi \neq 0$ so that Δ^m is an elliptic operator, see [3, p. 625] or [11, p. 121]. Ellipticity is a property of the principal part (containing the highest order partial derivatives) of the differential operator.

In this section, we study linear differential elliptic operators of the kind

$$u \mapsto Au = (-\Delta)^m u + \mathcal{A}(x; D)u \,, \tag{2.14}$$

where

$$\mathcal{A} : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^{2m-1}} \to R$$

is a linear operator containing all the lower order partial derivatives of the function u. The coefficients of the derivatives are measurable functions of x in Ω . For elliptic differential operators A of the form (2.14) and under suitable assumptions on f, we shall consider solutions $u = u(x)$ to the equation

$$(-\Delta)^m u + \mathcal{A}(x; D)u = f \qquad \text{in } \Omega \,, \tag{2.15}$$

which satisfy some boundary conditions on $\partial\Omega$. We discuss the class of "admissible" boundary conditions in Section 2.2.2. What we mean by solution to (2.15) will be made clear in each situation considered.

2.2.2 Boundary conditions

We assume the domain Ω to be bounded. Under suitable assumptions on $\partial\Omega$, to equation (2.15) we may associate m boundary conditions. These conditions will be expressed by linear differential operators $B_j(x; D)$, namely

$$B_j(x; D)u = 0 \text{ for } j = 1, \dots, m \text{ on } \partial\Omega \,. \tag{2.16}$$

Each B_j has a maximal order of derivatives $m_j \in \mathbb{N}$ and the coefficients of the derivatives are sufficiently smooth functions on $\partial\Omega$. For the problem considered in this thesis, it holds that

$$m_j \leq 2m - 1 \qquad \forall j = 1, \dots, m \,. \tag{2.17}$$

Indeed, $m_j \leq 3$ and $m = 2$ since we consider a biharmonic operator.

The choice of the B_j's is not completely free, we need to impose a certain algebraic constraint, the so-called complementing condition. For any j, let B'_j denote the highest order part of B_j which is precisely of order m_j. Then for equations (2.15), which have the polyharmonic operator as principal part, we have the following.

Definition 2.2.1. For every point $x \in \partial\Omega$, let $\nu(x)$ denote the normal unit vector. We say that the complementing condition holds for (2.16) if, for any non-trivial tangential vector $\tau(x)$, the polynomials in t $B'_j(x; \tau + t\nu)$ are linearly independent modulo the polynomial $(t - i|\tau|)^m$.

The complementing condition is crucial in order to obtain a priori estimates for solutions to (2.15)-(2.16) and, in turn, existence and uniqueness results.

Example. Consider the following polyharmonic problem over a sufficiently smooth domain Ω, e.g., $\Omega = B_1(0)$.

$$\begin{cases} \Delta^2 u(x) = 0 & \text{for } x \in \Omega \\ \Delta u(x) = 0 & \text{for } x \in \partial\Omega \\ (\Delta u)_\nu(x) = 0 & \text{for } x \in \partial\Omega \,, \end{cases} \tag{2.18}$$

where $(\cdot)_\nu$ denotes the directional derivative along the external unit normal vector ν to the border $\partial\Omega$ of the domain.

Observe that problem (2.18) does not have an unique solution. Indeed, if we take a function $\tilde{u}$ which is harmonic, i.e., $\tilde{u}$ is an infinitely differentiable function satisfying $\Delta\tilde{u} = 0$ over $\overline{\Omega}$, then $\tilde{u}$ satisfies both the equation and the boundary conditions of (2.18). There are infinitely many linearly independent harmonic functions, for example considering harmonic polynomials; we conclude that the problem, albeit looking reasonable as the boundary conditions are independent from each other, in not well-posed. Furthermore, if we take any point in $\mathbb{R}^n \setminus \overline{\Omega}$, the fundamental solution u_0 of $-\Delta$ having pole in x_0 (namely, $u_0(x) = \log|x - x_0|$ if $n = 2$ and $u_0(x) = |x - x_0|^{2-n}$ if $n \geq 3$) solves (2.18). This shows that it is not possible to obtain uniform a priori bounds in any norm. Indeed, as x_0 approaches the boundary $\partial\Omega$ the H^1-norm of the solution cannot be bounded uniformly in terms of its L^2-norm.

The reason for this —as we are about to explain— is that problem (2.18) does not satisfy the complementing conditions. In order to show this, we first rewrite the boundary conditions as in (2.16) by defining the boundary operators

$$\begin{aligned}
B_1(x; D) = \Delta &= \sum_{i=1}^{n} (D_i)^2 \\
B_1(x; \xi) &= \sum_{i=1}^{n} \xi_i^2 = |\xi|^2 \\
B_2(x; D) &= \frac{\partial}{\partial\nu}\Delta = \left(\sum_{i=1}^{n} \nu_i D_i\right)\left(\sum_{i=1}^{n} (D_i)^2\right) \\
B_2(x; \xi) &= \left(\sum_{i=1}^{n} \nu_i \xi_i\right)\left(\sum_{i=1}^{n} \xi_i^2\right) = (\nu \cdot \xi)|\xi|^2 \,.
\end{aligned}$$

These boundary operators do not contain lower order terms so that they coincide with their principal parts. We can then consider $B_j(x, \xi)$ instead of the principal part $B_j'(x, \xi)$. Fix a point $x \in \partial\Omega$ and a non-trivial tangent vector $\tau = (\tau_1, \tau_2)$; one can compute the polynomials (in t) $B_j(x, y; \tau + t\nu)$, $j = 1{,}2$ and their remainder modulo $(t - i\tau)^2$. Using the fact that ν is unit normal while τ is tangential we have that $|\nu| = 1$ and $\tau \cdot \nu = 0$, from which we obtain

$$\begin{aligned}
B_1(x; \tau + t\nu) &= |\tau + t\nu|^2 = |\tau|^2 + t^2|\nu|^2 + \tau \cdot t\nu = t^2 + |\tau|^2 \\
&\cong 2|\tau|^2(it + 1) \qquad \bmod (t - i\tau)^2 \\
B_2(x; \tau + t\nu) &= (\nu \cdot (\tau + t\nu))|\tau + t\nu|^2 = t^3 + t|\tau|^2 \\
&\cong 2|\tau|^2 \left[t(1 - 2|\tau|) + i|\tau|^2\right] \qquad \bmod (t - i\tau)^2 \,.
\end{aligned}$$

Finally, to know if the polynomials are linearly independent it suffices to check if the following determinant is non-zero

$$\det\begin{pmatrix} i & 1 - 2|\tau| \\ 1 & i|\tau|^2 \end{pmatrix} = -|\tau|^2 - 1 + 2|\tau| = -(|\tau| - 1)^2 \,.$$

The above determinant vanishes for some values of the tangent vector τ, namely those for which $|\tau| = 1$. We conclude that, as we claimed, the complementing conditions do not hold for problem (2.18) and indeed the problem is not well-posed.

Clearly, the solubility of (2.15)–(2.16) depends on the assumptions made on $\mathcal{A}$, f and B_j. We are here interested in *structural assumptions*, namely properties of the problem and not of its data.

Assumptions on the homogeneous problem. If we assume that $f = 0$ in Ω then (2.15)-(2.16) admits the trivial solution $u = 0$, in whatever sense this is intended. The natural question is then to find out whether this is the only solution. The answer depends on the structure of the problem. In fact, for any "reasonable" $\mathcal{A}$ and B_j's there exists a discrete set $\Sigma \subset \mathbb{R}$ such that, if $\lambda \notin \Sigma$, then the problem

$$\begin{cases} (-\Delta)^m u + \lambda \mathcal{A}(x; D)u = 0 \quad \in \Omega\,, \\ B_j(x; D)u = 0 \quad \text{with } j = 1, \ldots, m \text{ on } \partial\Omega\,, \end{cases} \tag{2.19}$$

only admits the trivial solution. If $\lambda \in \Sigma$, then the solutions of (2.19) form a non-trivial linear space; if $\mathcal{A}$ and the B_j's are well-behaved (in the sense specified below) this space has finite dimension. Therefore, we shall assume that

$$\text{the associated homogeneous problem only admits the trivial solution } u = 0. \tag{2.20}$$

Assumptions on $\mathcal{A}$. Assume that $\mathcal{A}$ has the following form

$$\mathcal{A}(x; D)u = \sum_{|\beta| \le 2m-1} a_\beta(x) D^\beta u \qquad a_\beta \in C^{|\beta|}(\bar{\Omega})\,. \tag{2.21}$$

Assumptions on the boundary conditions. Assume that, according to Definition 2.2.1,

$$\text{the linear boundary operators } B_j\text{'s satisfy the complementing condition.} \tag{2.22}$$

We are finally ready to state

Theorem 2.2.2. *Let $\Sigma_R := \{(x', x_n) \in \mathbb{R}^{n+1} : x_n > 0,\ |x'|^2 + x_n^2 < R\}$ be an half-sphere in $\mathbb{R}^{n+1}$ and let $\sigma_R := \{(x', 0) \in \mathbb{R}^{n+1} : |x'|^2 < R\}$ denote its planar boundary. Consider the problem*

$$\begin{cases} (-\Delta)^m u + \mathcal{A}(x; D)u = f & \text{in } \Sigma_R \\ B_j(x; D)u = 0 \text{ for } j = 1, \ldots, m & \text{on } \sigma_R\,, \end{cases} \tag{2.23}$$

with the boundary operators $B_j(x; \xi)$ satisfying the complementing conditions.

Then, for all $f \in H^{k-2m}(\Sigma_R+)$ with $k \ge 2m$ problem (2.23) admits a unique weak solution $u \in H^{2m}(\Omega)$ and there exist a positive constant C depending only on R, m, k, $\mathcal{A}$ and B_j, $j = 1, \ldots, m$ such that

$$\|u\|_{H^k(\Sigma_R)} \le C \|f\|_{H^{k-2m}(\Sigma_R)}\,.$$

The above statement can be extended to general bounded domains $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{2m}$, The basic idea is to decompose the domain into pieces using a partition of unity and «flattening out» any partition of the boundary. More precisely one has

Theorem 2.2.3. *Let $1 < p < \infty$ and take an integer $k \geq 2m$. Assume that $\partial\Omega \in C^k$ and also that (2.20) holds and that $\mathcal{A}$ and the B_j's satisfy (2.21)–(2.22).*

Then for all $f \in W^{k-2m,p}(\Omega)$, the problem (2.15)–(2.16) admits a unique strong solution $u \in W^{k,p}(\Omega)$. Moreover, there exists a positive constant $C = C(\Omega, k, m, \mathcal{A}, B_j)$ independent of f, such that the following a priori estimate holds

$$\|u\|_{W^{k,p}(\Omega)} \leq C \|f\|_{W^{k-2m,p}(\Omega)} .$$

The constant C depends on Ω only through the measure $|\Omega|$ and the C^k-norms of the local maps which define the boundary $\partial\Omega$. If $k > 2m + \frac{n}{p}$ then u is a classical solution.

Finally, if (2.20) is dropped, then for any solution u to (2.15)–(2.16) one has the following local variant of the estimate

$$\|u\|_{W^{k,p}(\Omega\cap B_R(x_0))} \leq C \|f\|_{W^{k-2m,p}(\Omega\cap B_{2R}(x_0))}$$

for any $R > 0$ and any $x_0 \in \Omega$. Here, C also depends on R.

2.3 Spectral theory

In this section we recall some classical results about spectral theory of compact and self-adjoint operators and proceed to expose a possible way in which one could analyse a general elliptic eigenvalue problem of which (5.2) studied in Chapter 5 is a special case.

2.3.1 Spectral theory of compact operators

This section presents some classical results on the spectrum of compact operators which are needed to tackle the general elliptic eigenvalue problem. We refer to the textbook by Evans [7] on which we base our exposition.

Definition 2.3.1. Let X and Y denote two real Banach spaces. A bounded linear operator $K : X \to Y$ is called compact provided for each bounded sequence $\{u_k\}_{k=1}^{\infty} \subset X$, the sequence $\{Ku_k\}_{k=1}^{\infty}$ is precompact in Y; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{u_{k_j}\}_{j=1}^{\infty}$ converges in Y.

Now let H denote a real Hilbert space, with inner product $(\cdot,\cdot)$. It is easy to see that if a linear operator $K : H \to H$ is compact and $u_k \rightharpoonup u$, then $Ku_k \to Ku$, where we have used $\rightharpoonup$ to denote convergence in the weak sense.

Theorem 2.3.2. *If $K : H \to H$ is compact, so is $K^* : H \to H$.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in H and extract a weakly convergent subsequence $u_{k_j} \rightharpoonup$ in H. We will prove $K^*u_{k_j} \to K^*u$. Indeed,

$$\begin{aligned}\left\|K^*u_{k_j} - K^*u\right\|^2 &= \left(K^*u_{k_j} - K^*u, K^*\left[u_{k_j} - u\right]\right)\\ &= \left(KK^*u_{k_j} - KK^*u, u_{k_j} - u\right) .\end{aligned}$$

Now, since K^* is linear, $K^*u_{k_j} \rightharpoonup K^*u$, and so $KK^*u_{k_j} \to KK^*u$. Thus $K^*u_{k_j} \to K^*u$. □

Theorem 2.3.3. *Let $K : H \to H$ be a compact linear operator. Then*

(i) $N(I-K)$ *is finite dimensional,*

(ii) $R(I-K)$ *is closed,*

(iii) $R(I-K) = N(I-K^*)^\perp$,

(iv) $N(I_K) = \{0\}$ *if and only if* $R(I-K) = H$,

and

(v) $\dim N(I-K) = \dim N(I-K^*)$.

Proof. 1. If $\dim N(I-K) = +\infty$, we can define an infinite orthonormal set $\{u_k\}_{k=1}^\infty \subset N(I-K)$. Then

$$Ku_k = u_k \quad (j = 1, \dots).$$

Now $\|u_k - u_l\|^2 = \|u_k\|^2 - 2(u_k, u_l) + \|u_l\|^2 = 2$ if $k \neq l$, and so $\|Ku_k - Ku_l\| = \sqrt{2}$ for $k \neq l$. This however contradicts the compactness of K, as $\{Ku_k\}_{k=1}^\infty$ would then contain no convergent subsequence. Assertion (i) is proved.

2. We next claim there exists a constant $\gamma > 0$ such that

$$\|u - Ku\| \geq \gamma \|u\| \qquad \text{for all } u \in N(I-K)^\perp. \tag{2.24}$$

Indeed, if not, there would exist for $k = 1, \dots$ elements $u_k \in N(I-K)^\perp$ with $\|u_k\| = 1$ and $\|u_k - Ku_k\| < \frac{1}{k}$. Consequently

$$u_k - Ku_k \to 0. \tag{2.25}$$

But since $\{u_k\}_{k=1}^\infty$ is bounded, there exists a weakly convergent subsequence $u_{k_j} \rightharpoonup u$. By compactness $Ku_{k_j} \to Ku$, and then (2.25) implies $u_{k_j} \to u$. We therefore have $u \in N(I-K)$ and so

$$(u_{k_j}, u) = 0 \quad (j = 1, \dots).$$

Let $k_j \to \infty$ to derive a contradiction to (2.24).

3. Next let $\{v_k\}_{k=1}^\infty \subset R(I-K)$, $v_k \to v$. We can find $u_k \in N(I-K)^\perp$ solving $u_k - Ku_k = v_k$. Using (2.24) we deduce

$$\|v_k - v_l\| \geq \gamma \|u_k - u_l\|.$$

Thus $u_k \to u$ and $u - Ku = v$. This proves (ii).

4. Assertion (iii) is now a consequence of (ii) and the general fact

$$\overline{R(A)} = N(A^*)^\perp \text{ for each bounded linear operator } A : H \to H.$$

5. To verify (iv), let us suppose to start with that $N(I-K)=\{0\}$, but $H_1=(I-K)(H)\subsetneq H$. According to (ii) H_1 is a closed subspace of H. Furthermore $H_k:=(I-K)(H_1)\subsetneq H_1$, since $I-K$ is one-to-one. Similarly if we write $H_k:=(I-K)^k(H)$ $(k=1,\ldots)$., we see that H_k is a closed subspace of H, $H_{k+1}\subsetneq H_k$ $(k=1,\ldots)$.

   Choose $u_k\in H_k$ with $\|u_k\|=1$, $u_k\in H_{k+1}^{\perp}$. Then $Ku_k-ku_l=-(u_k-Ku_k)+(u_l-Ku_l)+(u_k-u_l)$. Now if $k>l$, $H_{k+1}\subsetneq H_k\subset H_{l+1}\subsetneq H_l$. Thus u_k-Ku_k, u_l-Ku_l, $u_k\in H_{l+1}$. But this is impossible since K is compact.

6. Now conversely assume $R(I-K)=H$. Then owing to (iii) we see that $N(I-K^*)=\{0\}$. Since K^* is compact, we may utilize step 5 to conclude $R(I-K^*)=H$. But then $N(I-K)=R(I-K^*)^{\perp}=\{0\}$. This conclusion and step 5 complete the proof of assertion (iv).

7. Next we assert
$$\dim N(I-K)\geq \dim R(I-K)^{\perp}\,.$$
   To prove this, suppose instead $\dim N(I-K)<\dim R(I-K)^{\perp}$. Then there exists a bounded linear mapping $A:N(I-K)\to R(I-K)^{\perp}$ which is one-to-one, but *not* onto. Extend A to a linear mapping $A:H\to R(I-K)^{\perp}$ by setting $Au=0$ for $u\in N(I-K)^{\perp}$. Now A has a finite dimensional range and so A, and thus $K+A$, are compact. Furthermore $N(I-(K+A))=\{0\}$. Indeed, if $Ku+Au=u$, then $u-Ku=Au\in R(I-K)^{\perp}$; whence $u-Ku=Au=0$. Thus $u\in N(I-K)$ and so in fact $u=0$, since A is one-to-one on $N(I-K)$. Now apply assertion (iv) to $\tilde{K}=K+A$. We conclude $R(I-(K+A))=H$. But this is impossible: if $v\in R(I-K)^{\perp}$, but $v\notin R(A)$, the equation
$$u-(Ku+Au)=v$$
   has no solution.

8. Since $R(I-K^*)^{\perp}=N(I-K)$, we deduce from step 7
$$\dim N(I-K^*)\geq \dim R(I-K^*)^{\perp}= \qquad \dim N(I-K)\,.$$
   The opposite inequality comes from interchanging the roles of K and K^*. This establishes (v).

□

Remark 2.3.4. Theorem 2.3.3 asserts in particular either

$$\text{for each } f\in H,\text{ the equation } u-Ku=f \text{ has a unique solution} \tag{α}$$

or else

$$\text{the homogeneous equation } u-Ku=0 \text{ has solutions } u\neq 0. \tag{β}$$

This dichotomy is the *Fredholm alternative*. In addition, should (β) obtain, the space of solutions of the homogeneous problem is finite dimensional, and the homogeneous equation

$$u-Ku=f \tag{γ}$$

has a solution if and only if $f\in N(I-K)^{\perp}$.

Now we investigate the spectrum of a compact linear operator.

Theorem 2.3.5. *Assume* $\dim H = \infty$ *and* $K : H \to H$ *is compact. Then*

(i) $0 \in \sigma(K)$,

(ii) $\sigma(K) - \{0\} = \sigma_p(K) - \{0\}$,

and

(iii) $\begin{cases} \sigma(K) - \{0\} \text{ is finite, ore else} \\ \sigma(K) - \{0\} \text{ is a sequence tending to } 0. \end{cases}$

Proof. 1. Assume $0 \notin \sigma(K)$. Then $K : H \to H$ is bijective and so $I = K \circ K^{-1}$, being the composition of a compact and a bounded linear operator, is compact. This is impossible, since $\dim H = \infty$.

2. Assume $\eta \in \sigma(K)$, $\eta \neq 0$. Then if $N(K - \eta I) = \{0\}$, the Fredholm alternative would imply $R(K - \eta I) = H$. But then $\eta \in \rho(K)$, a contradiction.

3. Suppose now $\{\eta_k\}_{k=1}^{\infty}$ is a sequence of *distinct* elements of $\sigma(K) - \{0\}$, and $\eta_k \to \eta$. We will show $\eta = 0$.

   Indeed, since $\eta_k \in \sigma_p(K)$ there exists $w_k \neq 0$ such that $Kw_k = \eta_k w_k$. Let H_k denote the subspace of H spanned by $\{w_1, \dots, w_k\}$. Then $H_k \subsetneq H_{k+1}$ for each $k = 1,2,\dots$, since the $\{w_k\}_{k=1}^{\infty}$ are linearly independent.

   Observe also $(K - \eta_k I)H_k \subseteq H_{k-1}$ $(k = 2, \dots)$. Choose now for $k = 1, \dots$ an element $u_k \in H_k$, with $u_k \in H_{k-1}^{\perp}$ and $\|u_k\| = 1$. Now if $k > l$, $H_{l-1} \subsetneq H_l \subseteq H_{k-1} \subsetneq H_k$. Thus

$$\left\| \frac{Ku_k}{\eta_k} - \frac{Ku_l}{\eta_l} \right\| = \left\| \frac{(Ku_k - \eta_k u_k)}{\eta_k} - \frac{(Ku_l - \eta_l u_l)}{\eta_l} + u_k - u_l \right\| \geq 1,$$

   since $Ku_k - \eta_k u_k$, $Ku_l - \eta_l u_l$, $u_l \in H_{k-1}$. If $\eta_k \to \eta \neq 0$, we obtain a contradiction to the compactness of K.

□

2.3.2 Symmetric operators

Now let $S : H \to H$ be bounded, symmetric, and write

$$m := \inf_{\substack{u \in H \\ \|u\|=1}} (Su, u), \quad M := \sup_{\substack{u \in H \\ \|u\|=1}} (Su, u).$$

Lemma 2.3.6. *We have*

(i) $\sigma(S) \subset [m, M]$, *and*

(ii) $m, M \in \sigma(S)$.

Proof. 1. Let $\eta > M$. Then

$$(\eta u - Su, u) \geq (\eta - M)\|u\|^2 \quad (u \in H).$$

Hence the Lax Milgram Theorem asserts $\eta I - S$ is one-to-one and onto, and thus $\eta \in \rho(S)$. Similarly $\eta \in \rho(S)$ if $\rho < m$. This proves (i).

2. We will prove $M \in \sigma(S)$. Since the pairing $[u, v] := (Mu - Su, v)$ is symmetric, with $[u, u] > 0$ for all $u \in H$, the Cauchy–Schwarz inequality implies

$$|(Mu - Su, v)| \leq (Mu - Su, u)^{1/2}(Mv - Sv, v)^{1/2}$$

for all $u, v \in H$. In particular

$$\|Mu - Su\| \leq C(Mu - Su, u)^{1/2} \quad (u \in H) \tag{2.26}$$

for some constant C.

Now let $\{u_k\}_{k=1}^{\infty} \subset H$ satisfy $\|u_k\| = 1$ $(k = 1, \dots)$ and $(Su_k, u_k) \to M$. Then (2.26) implies $\|Mu_k - Su_k\| \to 0$. Now id $M \in \rho(S)$, then

$$u_k = (MI - S)^{-1}(Mu_k - Su_k) \to 0,$$

a contradiction, Thus $M \in \sigma(S)$, and likewise $M \in \sigma(S)$.

□

Theorem 2.3.7. *Let H be a separable Hilbert space, and suppose $S : H \to H$ is a compact and symmetric operator. Then there exists a countable orthonormal basis of H consisting of eigenvectors of S.*

Proof. 1. let $\{\eta_k\}$ comprise the sequence of distinct eigenvalues of S, excepting 0. Set $\eta_0 = 0$. Write $H_0 = N(S)$, $H_k = N(S - \eta_k I)$ $(k = 1, \dots)$. Then $0 \leq \dim H_0 \leq \infty$, and $0 < \dim H_k < \infty$, according to the Fredholm alternative.

2. Let $u \in H_k$, $v \in H_l$ for $k \neq l$. Then $Su = \eta_k u$, $Sv = \eta_l v$ and so

3. $$\eta_k(u, v) = (Su, v) = (u, Sv) = \eta_l(u, v).$$

As $\eta_k \neq \eta_l$, we deduce $(u, v) = 0$. Consequently we see subspaces H_k and H_l are orthogonal.

4. Now let $\tilde{H}$ be the smallest subspace of H containing H_0, H_1, … . Thus $\tilde{H} = \{\sum_{k=0}^{m} a_k u_k \mid m \in \{0, \dots\}, u_k \in H_k, a_k \in \mathbb{R}\}$. W next demonstrate $\tilde{H}$ is dense in H. Clearly $S(\tilde{H}) \subseteq \tilde{H}$. Furthermore $S(\tilde{H}^{\perp}) \subseteq \tilde{H}^{\perp}$: indeed if $u \in \tilde{H}^{\perp}$ and $v \in \tilde{H}$, then $(Su, v) = (u, Sv) = 0$.

Now the operator $\tilde{S} \equiv S|_{\tilde{H}^{\perp}}$is compact and symmetric. In addition $\sigma(\tilde{S}) = \{0\}$, since any non-zero eigenvalue of $\tilde{S}$ would be an eigenvalue of S as well. According to the lemma then, $(\tilde{S}u, u) = 0$ for all $u \in \tilde{H}^{\perp}$. But if $u, v \in \tilde{H}^{\perp}$,

$$2(\tilde{S}u, v) = (\tilde{S}(u + v), u + v) - (\tilde{S}u, u) - (\tilde{S}v, v) = 0.$$

Hence $\tilde{S} = 0$. Consequently $\tilde{H}^{\perp} \subset N(S) \subset \tilde{H}$, and so $\tilde{H}^{\perp} = \{0\}$. Thus $\tilde{H}$ is dense in H.

5. Choose an orthonormal basis for each subspace H_k ($k = 0, \dots$), noting that since H is separable, H_0 has a countable orthonormal basis. We obtain thereby an orthonormal basis of eigenvectors.

□

2.3.3 Eigenvalues of elliptic differential operators

In this section the spectral theory of compact and symmetric operators discussed above is applied to the case of elliptic differential operators, of which the eigenvalue problem discussed in Chapter 5 is a special case. The notation and setup of the problem follow those given in the article by Lamberti and Provenzano [10], which we have adapted to our setting.

Let Ω be an open set in $\mathbb{R}^n$ and $m \in \mathbb{N}$. As defined in Section 2.1, $H^m(\Omega)$ denotes the Sobolev space over the domain Ω, endowed with its standard scalar product (2.7) and $H_0^m(\Omega)$ denotes its subspace consisting of functions vanishing on the border $\partial\Omega$ in the sense given by the trace operators.

In the sequel, we shall always assume that $V(\Omega)$ is a fixed closed subspace of $H^m(\Omega)$ containing $H_0^m(\Omega)$ and such that the embedding $V(\Omega) \subset L^2(\Omega)$ is compact. What this last assumption means is that the restriction of the identity of $L^2(\Omega)$ over $V(\Omega)$ is a compact operator which we will denote by $i : V(\Omega) \to L^2(\Omega)$.

Moreover, we shall assume that $A_{\alpha\beta} \in L^\infty(\Omega)$ are fixed coefficients such that $A_{\alpha\beta} = A_{\beta\alpha}$ for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq m$.

We consider the following eigenvalue problem

$$\int_\Omega \sum_{0 \leq |\alpha|, |\beta| \leq m} A_{\alpha\beta} D^\alpha u D^\beta \phi \, dx = \lambda \int_\Omega u\phi \, dx \,, \quad \forall \phi \in V(\Omega) \,, \tag{2.27}$$

in the unknowns $u \in V(\Omega)$ (the eigenfunction) and $\lambda \in \mathbb{R}$ (the eigenvalue). Note that problem (2.27) is the weak formulation of the following

$$\mathcal{L}u = \lambda u \,, \tag{2.28}$$

where the general elliptic partial differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (A_{\alpha\beta} D^\beta u)$$

and subject to suitable homogeneous boundary conditions. The choice of the space $V(\Omega)$ is related to the boundary conditions in the classical formulation of the problem. For example, if $V(\Omega) = H_0^m(\Omega)$ we obtain Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \,, \quad \text{on } \partial\Omega \,. \tag{2.29}$$

If $V(\Omega) = H^m(\Omega)$ we obtain Neumann boundary conditions. If $V(\Omega) = H^m(\Omega) \cap W_0^{k,2}(\Omega)$, for some $k < m$, we obtain intermediate boundary conditions.

It is convenient to denote the left-hand side of equation (2.27) by $\mathcal{Q}[u, \phi]$.

We assume that the space $V(\Omega)$ and the coefficients $A_{\alpha\beta}$ are such that the Gårding's inequality holds, i.e., we assume that there exist $a, b > 0$ such that

$$a \|u\|^2_{H^m(\Omega)} \leq \mathcal{Q}[u,u] + b \|u\|^2_{L^2(\Omega)} \,, \tag{2.30}$$

for all $u \in V(\Omega)$. For classical conditions ensuring the validity of (2.30) in the case of Dirichlet boundary conditions we refer to Agmon [2, Thm. 7.6]. Moreover, we assume that there exists $c > 0$ such that

$$\mathcal{Q}[u,u] \leq c \|u\|^2_{H^m(\Omega)} \,, \tag{2.31}$$

for all $u \in V(\Omega)$. Note that since the coefficients $A_{\alpha\beta}$ are bounded, inequality (2.31) is always satisfied if Ω is a bounded open set with Lipschitz boundary.

Under assumptions (2.30), (2.31), we show that problem (2.27) has a divergent sequence of eigenvalues bounded below by $-b$, proving the following

Theorem 2.3.8. *Assume inequalities (2.30) and (2.31) are satisfied for some a, b, $c > 0$. Then the eigenvalues of equation (2.27) have finite multiplicity and can be represented by means of a divergent sequence λ_n, $n \in \mathbb{N}$ satisfying*

$$\lambda_n \geq a - b \,,$$

for all $n \in \mathbb{N}$. Furthermore the corresponding eigenfunctions form a complete orthogonal system in $L^2(\Omega)$.

Proof. In order to prove the theorem, we shall apply the spectral theory studied in the preceding sections to a suitably defined operator T. To this aim, we consider the bounded linear operator L from $V(\Omega)$ to its dual $V(\Omega)'$ which takes any $u \in V(\Omega)$ to the functional $L[u]$ defined by $L[u][\phi] = \mathcal{Q}[u,\phi]$, for all $\phi \in V(\Omega)$. Moreover, we consider the bounded linear operator I from $L^2(\Omega)$ to $V(\Omega)'$ which takes any $u \in L^2(\Omega)$ to the functional $I[u]$ defined by $I[u][\phi] = \int_\Omega u\,\phi$, for all $\phi \in V(\Omega)$.

By inequalities (2.30), (2.31) and by the boundedness of the coefficients $A_{\alpha\beta}$, it follows that the quadratic form defined by the right-hand side of (2.30) induces on $V(\Omega)$ a norm equivalent to the standard norm (2.7). Hence by the Riesz Theorem, it follows that the operator $L + bI$ is a linear homomorphism from $V(\Omega)$ onto $V(\Omega)'$. Thus, equation (2.27) is equivalent to the equation

$$(L + bI)^{(-1)} \circ I[u] = \mu u \tag{2.32}$$

where

$$\mu = (\lambda + b)^{-1} \,. \tag{2.33}$$

Indeed one can reach equation (2.32) from (2.27) through the following equivalences, where for clarity's sake we have written out the embedding operator $i : V(\Omega) \hookrightarrow L^2(\Omega)$

whenever an operator defined on $L^2(\Omega)$ would be applied to a function residing in $V(\Omega)$

$$\begin{aligned}
\mathcal{Q}[u,\phi] &= \lambda(iu, i\phi)_{L^2(\Omega)}, \qquad \forall \phi \in V(\Omega)\\
L[u][\phi] &= \lambda I[iu][\phi], \qquad \forall \phi \in V(\Omega)\\
Lu &= \lambda I iu\\
Lu + bIiu &= \lambda Iiu + bIiu\\
(L + bIi)u &= (\lambda + b)Iiu\\
(L + bIi)^{(-1)}Iiu &= (\lambda + b)^{-1}u\,.
\end{aligned}$$

Thus, it is natural to consider the operator T from $L^2(\Omega)$ to itself defined by

$$T := i \circ (L + bI)^{(-1)} \circ I\,, \tag{2.34}$$

where i is the embedding of $V(\Omega)$ into $L^2(\Omega)$. In the sequel, we shall omit i and we shall simply write $T = (L + bI)^{(-1)} \circ I$. Note that

$$\begin{aligned}
(Tu_1, u_2)_{L^2} &= I[u_2]\big[(L + bI)^{(-1)} \circ I[u_1]\big]\\
&= (L + bI)\big[(L + bI)^{(-1)} \circ I[u_1]\big]\big[(L + bI)^{(-1)} \circ I[u_2]\big]
\end{aligned}$$

for all $u_1, u_2 \in L^2(\Omega)$. Thus, since the operator $L + bI$ is symmetric it follows that T is a self-adjoint operator in $L^2(\Omega)$. Moreover, if the embedding $V(\Omega) \subset L^2(\Omega)$ is compact then the operator T is compact. By inequality (2.30), T is injective. Applying Theorem 2.3.5, it follows that the spectrum of T is discrete ad consists of a sequence μ_n, $n \in \mathbb{N}$ of positive eigenvalues of finite multiplicity converging to zero. Since T is self-adjoint we can also use Theorem 2.3.7 to obtain that the eigenvectors of T corm an orthonormal basis of $L^2(\Omega)$.

Inverting (2.33), all the eigenvalues of (2.27) are enumerated by sequence $\lambda_n = \mu^{-1} - b$, $n \in \mathbb{N}$. The fact that $\mu_n \to 0$ implies that the series is divergent, it only remains to be shown that $\lambda_n \geq a - b$ for all $n \in \mathbb{N}$.

Consider the n-th eigenfunction u_n is satisfying $Tu_n = \mu_n u_n$ then one also has that $u_n \in V(\Omega)$ and $\mathcal{Q}[u,\phi] = \lambda(u,\phi)_{L^2}$, $\forall \phi \in V(\Omega)$. We assume that $\|u_n\|_{L^2} = 1$ and apply Gårding's inequality (2.30) to u_n, obtaining

$$a \leq a\,\|u_n\|^2_{H^m} \leq \mathcal{Q}[u_n,u_n] + b\,\|u\|^2_{L^2} = \lambda(u_n,u_n)_{L^2} + b\,\|u\|^2_{L^2} = \lambda + b$$

so that $\lambda \geq a - b$, completing the proof. □

An application of Theorem 2.3.8 relevant to this work is given by the following

Example. We consider the case of the bi-harmonic operator. We fix $m = 2$ and choose the coefficients

$$A_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta}2/\alpha! & \text{if } |\alpha| = |\beta| = 2\\ 0 & \text{otherwise}\,,\end{cases}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ otherwise: This coefficients lead to $\mathcal{L} = \Delta^2$, as shown by the following equalities

$$\begin{aligned}
\mathcal{L}u &= \sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha (A_{\alpha\beta} D^\beta u) \\
&= \sum_{|\alpha|,|\beta|=m} D^\alpha \left(\frac{2}{\alpha!} \delta_{\alpha\beta} D^\beta u \right) \\
&= \sum_{|\alpha|=m} \frac{2}{\alpha!} (D^\alpha)^2 u \\
&= \sum_{j=1}^{N} \frac{2}{2} \left(\frac{\partial^2}{\partial x_j^2} \right)^2 u + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \frac{2}{1} \left(\frac{\partial^2}{\partial x_j x_k} \right)^2 u \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^2}{\partial x_j^2} \left(\frac{\partial^2}{\partial x_k^2} u \right) = \Delta(\Delta u) = \Delta^2 u \,.
\end{aligned}$$

Let $k \in \mathbb{N}$, $0 \le k \le 2$ and $V(\Omega) = H^2(\Omega) \cap H^2_0(\Omega)$. Note that (2.30) and (2.31) are satisfied for any $b > 0$ where a, $c > 0$ are suitable constants possibly depending on b. If $k = 0{,}1$ and the open set Ω is bounded and has a Lipschitz continuous boundary then the embedding $V(\Omega) \subset L^2(\Omega)$ is compact by Theorem 2.1.3. Under these assumptions all corresponding eigenvalues λ_n are well-defined and non-negative.

Note that if $k = 2$ then $V(\Omega) = H^2_0(\Omega)$ and by integrating by parts one can easily realize that the the bilinear form $\mathcal{Q}[u, \phi]$ can be written in the more familiar form

$$\mathcal{Q}[u, \phi] = \int_\Omega \Delta u \, \Delta \phi \, dx$$

for all $u, \phi \in H^2_0(\Omega)$, so that one as $\mathcal{Q}[u, \phi] = (u, \phi)_{H^2_0}$ as defined in (2.9). In this case we obtain the classic bi-harmonic operator $\mathcal{L} = \Delta^2$ subject to the Dirichlet boundary conditions (2.29). If $N = 2$, so that Ω is bidimensional, the Dirichlet problem can be used to model a clamped plate. In this case the eigenfunctions of (2.27) are called modes of oscillation of the plate, as they correspond to oscillating solutions of the corresponding evolutionary problem.

In the general case $k \le 2$, the classic formulation of the eigenvalue problem is

$$\begin{cases}
\Delta^2 u = \lambda u \,, & \text{in } \Omega \,, \\
\frac{\partial^j u}{\partial \nu^j} = 0, \forall j = 0, \dots, k-1 \,, & \text{on } \partial\Omega \,, \\
B_j u = 0, \forall j = 0, \dots, k-1 \,, & \text{in } \Omega \,,
\end{cases}$$

where B_j are uniquely defined operators satisfying the complementing conditions 2.2.1.

Chapter 3

The physical model

Several models have been used to study the oscillations of suspended bridges and walkways.

One crucial behaviour which a suitable model should be equipped to reflect in order to study the instability of the bridge is the emergence of torsional modes of oscillation along with the expected vertical ones. A significant way to convey this behaviour is by representing the deck of the bridge as a partially hinged rectangular plate. Specifically, we consider a long narrow rectangular thin plate hinged at two opposite sides an free on the remaining two sides. This reflects the situation of the bridge as the short sides are the ones fixed to the ground, while the long sides are free to move.

The free edges have length L and are represented horizontally, the short sides have length 2ℓ and are hinged; a realistic assumption is that $\ell \cong \frac{L}{100}$. In the following by scaling arguments we will assume that $L = \pi$, thus the plate can be identified with the planar domain

$$\Omega = (0, \pi) \times (-\ell, \ell) \,.$$

3.1 Linear model

3.1.1 Bending and total energy

We are interested in studying the vertical displacement of the plate, modelled as a function $u : \Omega \to \mathbb{R}$, $u = u(x, y)$.

According to classical Kirchhoff theory, the bending energy of the plate can be computed as

$$\mathbb{E}_B(u) = \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2 \right) \mathrm{d}x \, \mathrm{d}y, \tag{3.1}$$

where κ_1 and κ_2 are the principal curvatures of the graph of function u, d is the thickness of the plate and σ and E are the Poisson ratio and the Young modulus respectively and can be expressed in terms of the Lamé constants λ and μ as

$$\sigma = \frac{\lambda}{2(\lambda + \mu)}, \qquad E = 2\mu(1 + \sigma) \,.$$

Due to physical reasons, it always holds that $\mu > 0$ and $-1 < \sigma < \frac{1}{2}$. Oftentimes we shall further assume that $\lambda > 0$, so that

$$0 < \sigma < \frac{1}{2}\,, \tag{3.2}$$

this assumption is reasonable since we aim to model the deck of a bridge, which is a mixture of concrete and steel, whose Poisson ratio lies between 0.1 and 0.3. In any case this assumption will be explicitly stated.

Since we consider a smooth deformation u, the graph of this function is a smooth manifold parametrized by the map $(x, y) \in \Omega \mapsto (x, y, u(x, y)) \in \mathbb{R}^3$. We can then compute the mean curvature H and the gaussian curvature K of the manifold —which are related to the principal curvatures κ_1 and κ_2— as an expression of the function u itself and of its derivatives. Namely

$$\begin{aligned} H &= \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \\ K &= \kappa_1\kappa_2 = \frac{u_{xx}u_{yy} - u_{xy}^2}{\sqrt{1 + |\nabla u|^2}} \end{aligned}$$

where whe have taken $|\nabla u| = \sqrt{u_x^2 + u_y^2}$ to mean the euclidean $\mathbb{R}^2$ norm of the gradient of u.

Assuming that the magnitude of the plate's deformation is small, so that $u \sim 0$ and $|\nabla u| \ll 1$, we can apply the approximation $\sqrt{1 + |\nabla u|^2} \approx 1$ so that we can relate the principal curvatures to the second derivatives of u as follows

$$\begin{aligned} (\kappa_1 + \kappa_2)^2 = (2H)^2 &\approx [\operatorname{div}(\nabla u)]^2 = (\Delta u)^2 \\ \kappa_1\kappa_2 = K &\approx \det(D^2 u) = u_{xx}u_{yy} - u_{xy}^2\,. \end{aligned}$$

This entails that

$$\begin{aligned} \frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2 &= \frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \kappa_1\kappa_2 - \kappa_1\kappa_2 + \sigma\kappa_1\kappa_2 \\ &= \frac{1}{2}(\kappa_1 + \kappa_2)^2 + (\sigma - 1)\kappa_1\kappa_2 \\ &\approx \frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2 u). \end{aligned}$$

The total energy $\mathbb{E}_T(u)$ is obtained adding a therm which describes the load f, both dead and alive, resulting in

$$\begin{aligned} \mathbb{E}_T(u) &= \mathbb{E}_B(u) - \int_\Omega f u \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{Ed^3}{12(1 - \sigma^2)}\int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2 u)\right)\mathrm{d}x\,\mathrm{d}y - \int_\Omega f u \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.3}$$

In order to simplify the notation we rescale the total energy by a factor of $\frac{Ed^3}{12(1-\sigma^2)}$ and replace f with $\frac{Ed^3}{12(1-\sigma^2)}f$, so that one has

$$\mathbb{E}_T(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2u) - fu\right) \mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

Remark 3.1.1. The quadratic part of the functional (3.4) is positive. To see this it is sufficient to rewrite

$$\begin{aligned}
\frac{1}{2}(\Delta u)^2 + (\sigma-1)\det(D^2u) &= \frac{1}{2}(u_{xx}+u_{yy})^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) \\
&= \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + u_{xx}u_{yy} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) \\
&= \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \sigma u_{xx}u_{yy} + (1-\sigma)u_{xy}^2 \\
&= \frac{1}{2}\begin{pmatrix} u_{xx} \\ u_{yy} \\ u_{xy} \end{pmatrix}^t \underbrace{\begin{pmatrix} 1 & \sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & 2(1-\sigma) \end{pmatrix}}_{=:M} \begin{pmatrix} u_{xx} \\ u_{yy} \\ u_{xy} \end{pmatrix},
\end{aligned}$$

and to note that matrix M which we have just introduced is definite positive for any value of the Poisson Ratio $-1 < \sigma < \frac{1}{2}$ since its eigenvalues are $2(1-\sigma)$ and $1 \pm \sigma$.

3.1.2 Statement of the problem

By Hamilton's principle, the function u which describes the realization of the displacement of the plate is the unique minimizer of the total energy $\mathbb{E}_T$; the proper functional space over which such minimization should take place will be derived in Section 4. Under appropriate assumption on the regularity of f, this function satisfies the following Euler–Lagrange equation

$$\Delta^2 u = f \tag{3.5}$$

with the addition of boundary conditions which we now present.

The situation which we aim to describe is the deck of a suspended bridge, so we assume that the plate is hinged on its short edges to model the connection with the ground, which results in the conditions

$$u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 \qquad \forall y \in (-\ell,\ell)\,. \tag{3.6}$$

These conditions are named after Navier since their first appearance in [13].

On the other hand the horizontal edges of the plate are left free, modelling the suspended portion of the bridge. This leads to a different set of boundary conditions sometimes called in literature Neumann type boundary conditions

$$\begin{aligned}
u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) &= 0 \qquad \forall x \in (0,\pi)\,, \\
u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) &= 0 \qquad \forall x \in (0,\pi)\,.
\end{aligned} \tag{3.7}$$

The way in which this conditions arise is detailed in Section 4.2.3.

Putting together the Euler-Lagrange equation and its associated boundary conditions we obtain the complete classical formulation of the problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{3.8}$$

3.2 Nonlinear model

In this section we improve on the model obtained in Section 3.1 by introducing a nonlinear term to the energy functional of the plate, describing the localized action of the supporting hangers of the bridge and express the resulting static and dynamical energies, as well as the corresponding Euler-Lagrange equations.

We introduce the set $\omega := (0,\pi) \times [(-\ell,-\ell+\epsilon)] \cup (\ell-\epsilon,\ell)]$, with $\epsilon > 0$ small. This set is made of two thin strips around the horizontal edges of the plate Ω. Justified by the design of many real world suspension bridges we may assume that the suspensions hangers act on the walkway in a localized manner, so that their effect is concentrated on a region such as ω.

We denote by $\Upsilon(y)$ the characteristic function of $(-\ell,\ell+\epsilon)\cup(\ell-\epsilon,\ell)$, so that for any point $(x,y) \in \Omega$ it holds $(x,y) \in \omega \iff \Upsilon(y) = 1$ and we introduce a function $g : \mathbb{R} \to \mathbb{R}$ so that the restoring force exerted by the hangers can be described as

$$h(x,y,u) = \Upsilon(y) g(u + \gamma x(\pi - x))\,, \tag{3.9}$$

for a constant $\gamma > 0$. A valid set of assumptions for the function g is given by

$$g \in C^1(0,+\infty), \quad g(s) = 0 \text{ for any } a \leq 0, \quad g'(0^+) > 0, \quad g'(s) \geq 0 \text{ for any } s > 0\,.$$

The motivation behind the form (3.9) of the term h lies in the empirical observation that the hangers exert a stronger force around the center of the bridge $x = \pi/2$ comparatively to the sides which $x = 0$ and $x = \pi$ which are supported by the ground.

The force h can be integrated to obtain a corresponding potential energy of the form $\int_\Omega H(x,y,u)dxdy$, by defining $H(x,y,s) := \int_{\bar{s}}^{s} h(x,y,\tau)d\tau$ for all $s \in \mathbb{R}$. Adding this potential energy to the elastic energy of the plate, given in (3.4), which also includes the effect of static and dynamic external loads f, we obtain the total static energy of the plate:

$$\mathbb{E}_T(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{x,y}^2 - u_{xx}u_{yy}) + H(x,y,u) - fu \right) \mathrm{d}x\,\mathrm{d}y\,. \tag{3.10}$$

The vertical displacement u of the plate is then obtained by minimizing the convex functional (3.10) and satisfies the corresponding Euler-Lagrange equations reported here in their classical form:

$$\begin{cases} \Delta^2 u + h(x,y,u) = f & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{3.11}$$

Notice how (3.11) is obtained from (3.8) simply by adding the nonlinear term h to the equation.

A further refinement of the model is obtained when we consider an external force $f = f(x,y,t)$ also depending on time, so that it is no more significant to study an equilibrium position $u(x,y)$ and we must instead examine the evolution with respect to time of the position, describing a trajectory $u(x,y,t)$. Denoting by m the total mass of the bridge, the kinetic energy associated to such a deformation u is then given by the integral

$$\frac{m}{2|\Omega|}\int_\Omega u_t^2 dxdy\,.$$

We can set the coefficient $m|\Omega|^{-1}$ to 1 through an appropriate time scaling of the form $t \mapsto \sqrt{m|\Omega|^{-}1}t$. Adding this kinetic energy term to the nonlinear static energy (3.10) we obtain the total energy of a bridge according to the nonlinear dynamical model:

$$\mathcal{E}_u(t) := \int_\Omega \frac{1}{2}u_t^2 dxdy + \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma-1)(u_{x,y}^2 - u_{xx}u_{yy}) + H(x,y,u) - fu\right) \mathrm{d}x\,\mathrm{d}y\,. \tag{3.12}$$

Next we introduce the action functional obtained by subtracting the potential energy from the kinetic energy and integrating the result over a time interval $[0,T]$:

$$\begin{aligned} \mathcal{A}(u) :=& \int_0^T \left[\int_\Omega \frac{1}{2}u_t^2 dxdy\right] dt \\ &- \int_0^T \left[\int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma-1)(u_{x,y}^2 - u_{xx}u_{yy}) + H(x,y,u) - fu\right) \mathrm{d}x\,\mathrm{d}y\right] dt\,. \end{aligned}$$

To characterize the evolution of the deformation u we derive the equations of motion by imposing that the functional derivative of the action $\mathcal{A}$ vanishes:

$$u_{tt} + \Delta^2 u + h(x,y,u) = f \qquad \text{in } \Omega \times (0,T)\,.$$

Finally, a damping term δu_t, for a positive constant δ, is added to express internal friction and the equation for the displacement u in the nonlinear dynamical model is obtained

$$\begin{cases} u_{tt} + \delta u_t \Delta^2 u + h(x,y,u) = f & \text{in } \Omega \times (0,T) \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \times (0,T) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \times (0,T) \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \times (0,T) \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y) & \text{for } (x,y) \in \Omega\,. \end{cases} \tag{3.13}$$

This equation, which also arises from different settings, is sometimes called the Swift-Hohenberg equation.

Chapter 4

The Linear Stationary Problem

This chapter is dedicated to the analysis of the linear model derived in Section 3.1. We begin by introducing the space $H^2_*(\Omega)$ in Section 4.1, which is the natural setting in which to carry out our analysis, and proving some of its properties which are relevant to the treatment. In Section 4.2 we analyze the relation between solutions of the classical linear model (3.8) and of the corresponding weak formulation, and prove the well posedness of the latter. The main result of the section is Theorem 4.2.2, and proceed to prove it through several lemmas.

Section 4.3 is dedicated to the explicit characterization of solutions of (3.8), through a technique presented in [12, Chapter 2] which extends the classical separation of variables technique. Each subsection considers a different set of assumptions on the load function f and proceeds to compute explicitly the corresponding solution u of (3.8), from the more restrictive hypothesis that f does not depend on the variable y of Theorem 4.3.1, which was first presented in [8], to the adaptation of the result to loads of the form $yf(x)$ and $e^y f(x)$ given, respectively, in Theorems 4.3.2 and 4.3.3, culminating in the generalization to arbitrary load functions $f \in L^2$ provided by Theorem 4.3.4, which improves Theorem 3.2 from the paper [8]. To the best of our knowledge these last results were not obtained in previous works.

Finally Section 4.4 explores the link between the two-dimensional clamped plate model and the description of a one-dimensional simply supported beam, detailing as the solution to the first problem converges to the second's as the width ℓ of the plate tends to zero. As for Section 4.3 the results proven in [8] —specifically the article's Theorem 3.3 reported here as Theorem 4.4.1— are generalized to arbitrary load functions: whereas previously they had been proved only for those forcing terms f which are constant with respect to the variable y, we are able to relax this assumption. In this way Theorem 4.4.3 is a direct extension of Theorem 4.4.1.

4.1 The space $H^2_*(\Omega)$

We begin this section by defining the space $H^2_*(\Omega)$, which contains the possible displacement functions u which can describe the position of the plate, accounting for the hinged condition of its short edges. We proceed by endowing this space with a significant norm $\|\cdot\|_{H^2_*}$ and proving this norm's equivalence with the more usual norm $\|\cdot\|_{H^2}$. This result is instrumental for the analysis of the weak formulation of the problem given in Section 4.2.

The space $H^2_*(\Omega)$ is defined as follows

$$H^2_*(\Omega) := \left\{ w \in H^2(\Omega); w = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell) \right\}.$$

This is a closed subspace of the Sobolev space $H^2(\Omega)$. Given the bidimensional nature of the plate, as detailed in Section 2.1, we have $H^2_*(\Omega) \subset C^0(\overline{\Omega})$ so that functions $u \in H^2_*(\Omega)$ are those functions in $H^2(\Omega)$ which vanish on the short edges of Ω in the classical sense, i.e., pointwise (up to selecting a continuous representative).

We will call $\mathcal{H}(\Omega) = \left(H^2_*(\Omega)\right)'$ the dual space of $H^2_*(\Omega)$ and denote by $\langle \cdot, \cdot \rangle$ the corresponding duality. Thanks again to the fact that the domain Ω resides in the plane we also have that

$$L^p(\Omega) \subset \mathcal{H}(\Omega) \qquad \forall 1 \le p \le \infty\,. \tag{4.1}$$

As a consequence, if $f \in L^1(\Omega)$ then the functional $\mathbb{E}_T$ is well-defined over $H^2_*(\Omega)$. If instead $f \in \mathcal{H}(\Omega)$ then one has to replace the integral $\int_\Omega fu$ with the duality $\langle f, u \rangle$, this will be left implicit in the following.

Lemma 4.1.1. *Assume (3.2). On the space $H^2_*(\Omega)$ the two norms*

$$u \mapsto \|u\|_{H^2} = \left[\int_\Omega \left(|D^2u|^2 + |u|^2\right)\, dx\, dy\right]^{1/2},$$
$$u \mapsto \|u\|_{H^2_*} := \left[\int_\Omega \left[(\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right]\, dx\, dy\right]^{1/2}$$

are equivalent. Therefore, $H^2_*(\Omega)$ is a Hilbert space when endowed with the scalar product

$$(u, v)_{H^2_*} := \int_\Omega \left[\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\right]\, dx\, dy \tag{4.2}$$

Proof. First we show that for all $u \in H^2_*(\Omega)$, thanks to the fact that u vanishes on the short edges, it holds that $\|u\|_{L^2} < C \left\| |D^2u| \right\|_{L^2}$. The bidimensional nature of the domain Ω implies that $u \in H^2_*(\Omega) \implies u \in H^2(\Omega) \subset C^0(\overline{\Omega})$, moreover $\forall y \in (-\ell, \ell)$,

$u(0,y)=u(\pi,y)=0$. Given a point $(x,y)\in\Omega$ we have

$$\begin{aligned}|u(x,y)| &= \left|\int_0^x u_x(t,y)dt\right| \\ &\leq \int_0^x |u_x(t,y)|dt \\ &\leq \sqrt{\pi}\left[\int_0^\pi (u_x(t,y))^2 dt\right]^{1/2} \\ &\leq \sqrt{\pi}\left[-\int_0^\pi u_{xx}(t,y)u(t,y)dt\right]^{1/2}.\end{aligned}$$

To formally justify these inequalities we define the space $C^\infty_*(\overline{\Omega})$ in analogy to $H^2_*(\Omega)$ as follows

$$C^\infty_*(\overline{\Omega}) = \left\{\phi\in C^\infty(\overline{\Omega}); \phi=0 \text{ on } \{0,\pi\}\times(-\ell,\ell)\right\}.$$

It is clear that $C^\infty_*(\overline{\Omega})$ is a dense subspace of $H^2_*(\Omega)$, so that the function $u\in H^2_*(\Omega)$ can be approximated up to an arbitrarily small threshold with a smooth function $\phi\in C^\infty_*(\overline{\Omega})$. Then the following steps are sound:

$$\begin{aligned}|\phi(x,y)| &= \left|\int_0^x \phi_x(t,y)dt\right| \\ &\leq \int_0^x |\phi_x(t,y)|dt \\ &\leq \sqrt{\pi}\left[\int_0^\pi (\phi_x(t,y))^2 dt\right]^{1/2} \\ &\leq \sqrt{\pi}\left[-\int_0^\pi \phi_{xx}(t,y)\phi(t,y)dt\right]^{1/2}.\end{aligned}$$

Integrating over $(0,\pi)$ we are able to prove that $\|\phi\|_{L^2}\leq C\,\||D^2\phi|\|_{L^2}$ for some constant $C>0$. Letting ϕ tend to u in $H^2_*(\Omega)$ we obtain that also $\|u\|_{L^2}\leq C\,\||D^2u|\|_{L^2}$.

Finally to tie $\||D^2u|\|_{L^2}$ with $\|u\|_{H^2_*}$ we use the result of Remark 3.1.1, in which we proved that

$$(1-\sigma)|D^2u|^2 \leq (\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) \leq (1+\sigma)|D^2u|^2\,, \tag{4.3}$$

in which $|\cdot|$ denotes the euclidean norm of $\mathbb{R}^3$, so that one has

$$\begin{aligned}(1-\sigma)\left\||D^2u|\right\|_{L^2} &\leq \|u\|^2_{H^2_*} \\ &= \int_\Omega \left[(\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right] \mathrm{d}x\,\mathrm{d}y \\ &\leq (1+\sigma)\left\||D^2u|\right\|_{L^2}.\end{aligned}$$

□

4.2 Well posedness

In this section we state formally the relation between the energy minimization problem (3.4) and its weak formulation.

Differentiating the energy $\mathbb{E}_T(u)$ and requiring that u be a stationary point, we obtain

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, \mathrm{d}x\, \mathrm{d}y = \langle f, v\rangle, \quad \forall v \in H^2_*(\Omega)\,, \tag{4.4}$$

which motivates the following definition.

Definition 4.2.1. We say that a funtion $u \in H^2_*(\Omega)$ is a weak solution of (3.8) if it satisfies (4.4).

Then we show

Theorem 4.2.2. [8, Theorem 3.1] *Assume* (3.2) *and let $f \in \mathcal{H}(\Omega)$. Then there exists a unique $u \in H^2_*(\Omega)$ such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, dx\, dy = \langle f, v\rangle, \quad \forall v \in H^2_*(\Omega)\,; \tag{4.5}$$

moreover, u is the minimum point of the convex functional $\mathbb{E}_T$. Finally, if $f \in L^2(\Omega)$ then $u \in H^4(\Omega)$, and if $u \in C^4(\overline{\Omega})$ then u is a classical solution of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ u_{yyy}(x \pm \ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{4.6}$$

The rest of the section is dedicated to the proof of Theorem 4.2.2, carried by several lemmas. In particular, Lemma 4.2.3 pertains the existence of weak solutions of (3.8), while Lemma 4.2.4 states the solutions' gain of regularity. Finally Lemma 4.2.5 assures that smooth weak solutions of (3.8) are also solutions in the classical sense.

4.2.1 Well posedness of the weak problem

Using Lax–Milgram theorem we are able to prove existence and uniqueness of solutions of the weak formulation (4.4) and its equivalence to the energy minimization problem associated to $\mathbb{E}_T$.

Lemma 4.2.3. *Assume* (3.2) *and let $f \in \mathcal{H}(\Omega)$ then there exists an unique $u \in H^2_*(\Omega)$ such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, dx\, dy = \langle f, v\rangle \qquad \forall v \in H^2_*(\Omega); \tag{4.7}$$

moreover, u is the minimum point of the convex functional $\mathbb{E}_T$ and there is a constant $C > 0$ such that $\|u\|_{H^2_*} \leq C\,\|f\|_{\mathcal{H}(\Omega)}$.

Proof. The space $H^2(\Omega)$ is a Hilbert space if endowed with the scalar product

$$(u,v)_{H^2} := \int_\Omega (D^2u \cdot D^2v + uv)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for all } u,\, v \in H^2(\Omega).$$

On the closed subspace $H^2_*(\Omega)$ we may also define a different scalar product $(\cdot,\cdot)_{H^2_*}$, as detailed in Section 4.1. With this notation we can rewrite equation (4.4) as

$$(u,v)_{H^2_*} = \langle f, v\rangle_{L^2}$$

then the thesis follows by applying Lax–Milgram Theorem to obtain the existence of an unique solution to problem (4.4) as well as the equivalence o the minimization problem. □

4.2.2 L^p-Regularity of weak solutions

Having proved the existence of a weak solution u of problem (4.4) we commit to study its regularity, applying the theory of complementing conditions exposed in Section 2.2. In order to accomplish this, we prove the following Lemma.

Lemma 4.2.4. *Assume* (3.2) *and* $1 < p < \infty$*; let* $f \in L^p(\Omega)$ *and let* $u \in H^2_*(\Omega)$ *be a (weak) solution of* (3.8)*. Then* $u \in W^{4,p}(\Omega)$ *and there exists a constant* $C(\ell,\sigma,p)$ *depending only on* ℓ*,* σ *and* p *such that*

$$\|u\|_{W^{4,p}} \le C(\ell,\sigma,p)\,\|f\|_{L^p}\ . \tag{4.8}$$

Proof. First we note that assumption $f \in L^p(\Omega)$ is valid because of (4.1). We now apply the theory exposed in Section 2.2 so that we need to verify that the the complementing conditions (2.2.1) are satisfied by the boundary condition in (4.6).

On the short edges we have conditions (3.6), which are a case of Navier conditions. In order to verify the complementing conditions we must express the boundary conditions in the form of (2.16) by defining appropriate boundary operators $B_j(x,y;D)$. In this case we can see that B_j does not depend on the position (x,y). Specifically, defining

$$\begin{aligned} B_1(x,y;D) &= 1 \\ B_2(x,y;D) &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta\,, \end{aligned}$$

we have

$$\begin{aligned} B_1(x,y;D)u &= u \\ B_2(x,y;D)u &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \Delta u\,, \end{aligned}$$

which is what we wanted.

On the horizontal edges the boundary conditions are given by (3.7). These equations are encoded by the boundary operators

$$B_1(x,y;D) = \frac{\partial^2}{\partial y^2} + \sigma\frac{\partial^2}{\partial x^2}$$
$$B_2(x,y;D) = \frac{\partial^3}{\partial y^3} + (2-\sigma)\frac{\partial^3}{\partial x^2 \partial y}\,.$$

We see that, once more, the boundary operators do not depend on the position x so that we can rewrite them in the form $B_1(\alpha,\beta) = \sigma\alpha^2 + \beta^2$, $B_2(\alpha,\beta) = (2-\sigma)\alpha^2\beta + \beta^3$ which are easier to manipulate.

Let $\nu = (\nu_1, \nu_2)$ denote the unit normal to $\partial\Omega$ and let $\tau = (\tau_1, \tau_2)$ be any non-trivial vector tangent to $\partial\Omega$. Since we are considering an horizontal edge we have $\nu_1 = \tau_2 = 0$ and $\nu_2 = \operatorname{sign} y$ while τ_1 is an arbitrary non-zero number. We want to compute $B_j(x,y;\tau+t\nu)$ for $j = 1,2$ and $t \in \mathbb{R}$ and show that they are independent polynomials modulo $(t-i|\tau|)^2$. Using the fact that $\tau + t\nu = (\tau_1, t\operatorname{sign} y)$ we have

$$B_1(x,y;\tau+t\nu) = \sigma\tau_1^2 + t^2$$
$$B_2(x,y;\tau+t\nu) = (\operatorname{sign} y)t\big[t^2 + (2-\sigma)\tau_1^2\big]\,.$$

Taking the reminder modulo $(t-i|\tau_1|)^2 = t^2 - 2ut|\tau_1| - \tau_1^2$ one has

$$\begin{aligned}
B_1(x,y;\tau+t\nu) &= \big(t-i|\tau_i|\big)^2 + 2i|\tau_1|t + (\sigma+1)\tau_1^2 \\
&= 2i|\tau_1|t + (\sigma+1)\tau_1^2 \mod \big(t-i|\tau_i|\big)^2\,, \\
B_2(x,y;\tau+t\nu) &= (\operatorname{sign} y)\big(t+2i|\tau_i|\big)\big(t-i|\tau_i|\big)^2 \\
&\quad + (\operatorname{sign} y)\big[-(\sigma+1)\tau_1^2 t + 2i|\tau_1|^3\big] \\
&= (\operatorname{sign} y)\big[-(\sigma+1)\tau_1^2 t + 2i|\tau_1|^3\big] \mod \big(t-i|\tau_i|\big)^2\,.
\end{aligned}$$

To show that the complementing conditions are satisfied we have to prove that the polynomials $2it + (\sigma+1)|\tau_1|$ and $(\sigma+1)t - 2i|\tau_1|$, where we have left out the non-zero constant τ_1 in the first case and $-|\tau_1|(\operatorname{sign} y)$ in the second one, are linearly independent. Indeed we have

$$\begin{aligned}
\det\begin{pmatrix} 2i & (\sigma+1)|\tau_1| \\ (\sigma+1) & -2i|\tau_1| \end{pmatrix} &= |\tau_1|\det\begin{pmatrix} 2i & (\sigma+1) \\ (\sigma+1) & -2i \end{pmatrix} \\
&= |\tau_1|\Big(4 - (\sigma+1)^2\Big) \\
&= 0 \iff \sigma+1 = 2 \iff \sigma = 1\,,
\end{aligned}$$

so that the linear independence is satisfied due to the fact that σ cannot take the value 1. Hence, the complementing conditions are satisfied.

Unfortunately, Theorem 2.2.3 cannot be directly applied since it requires for the border $\partial\Omega$ of the domain to be of class C^k; we can overcome this limitation using the technique of odd extension.

We extend the rectangular domain $\Omega = (0, \pi) \times (-\ell, \ell)$ to $\tilde{\Omega} = [-\pi, 2\pi] \times [-\ell, \ell]$ by reflection on the short edges. Specifically, let $\Omega_- = [-\pi, 0] \times [-\ell, \ell]$ and $\Omega_+ = [\pi, 2\pi] \times [-\ell, \ell]$ denote the left and right reflections of Ω respectively, so that we have

$$\tilde{\Omega} = \Omega_- \cup \overline{\Omega} \cup \Omega_+ .$$

Given a weak solution $u : \Omega \to \mathbb{R}$ its odd extension to $\tilde{\Omega}$ is the function $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ whose piecewise definition is given by

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in \Omega \\ -u(-x, y) & \text{if } (x, y) \in \Omega_- \\ u(x - 2\pi, y) & \text{if } (x, y) \in \Omega_+ . \end{cases}$$

Thanks to the fact that u vanishes on the short sides of the rectangle, it can be shown that the resulting function $\tilde{u}$ is twice differentiable in weak sense so that we have $\tilde{u} \in H^2_*(\tilde{\Omega})$. Moreover, u is a solution to the extension of the weak formulation (4.4) taken over $\tilde{\Omega}$, that is

$$\int_{\tilde{\Omega}} [\Delta\tilde{u}\Delta\tilde{v} + (1-\sigma)(2\tilde{u}_{xy}\tilde{v}_{xy} - \tilde{u}_{xx}\tilde{v}_{yy} - \tilde{u}_{yy}\tilde{v}_{xx})] \, \mathrm{d}x \, \mathrm{d}y = \langle \tilde{f}, \tilde{v} \rangle \qquad \forall \tilde{v} \in H^2_*(\tilde{\Omega}) ,$$

where $\tilde{f}$ denotes the odd extension of the load f defined in analogy with $\tilde{u}$.

It is then possible to derive the desired bounds on u by covering the horizontal edges of $\tilde{\Omega}$ with half-spheres and applying Theorem 2.2.2 to each of them, thus avoiding the non-smooth vertices of the original rectangle Ω. □

4.2.3 Classical formulation

Lemma 4.2.5. *Let $u \in C^4(\overline{\Omega})$ be a solution to the weak equation (4.4). Then u also solves the classical equation (3.8).*

Proof. Let $v \in C^\infty_*(\overline{\Omega})$, then we have

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}]dxdy = \langle f, v \rangle . \tag{4.9}$$

Applying Green Gauss formula we can rewrite

$$\int_\Omega \Delta u \Delta v dxdy = \int_\Omega \Delta^2 u v dxdy + \int_{\partial\Omega} [\Delta u v_\nu - v(\Delta u)_\nu] ds. \tag{4.10}$$

The integral over the rectangle's border $\partial\Omega$ can be split over the rectangle's sides as follows

$$\begin{aligned}
&\int_{\partial\Omega}\big[\Delta u v_\nu - v(\Delta u)_\nu\big]ds = \\
&\qquad + \int_{-\ell}^{\ell}\big[(u_{xx}+\underbrace{u_{yy}}_{=0})(-v_x) + \underbrace{v}_{=0}(u_{xx}+u_{yy})_x\big]_{x=0}dy \\
&\qquad + \int_{-\ell}^{\ell}\big[(u_{xx}+\underbrace{u_{yy}}_{=0})v_x - \underbrace{v}_{=0}(u_{xx}+u_{yy})_x\big]_{x=\pi}dy \\
&\qquad + \int_{0}^{\pi}\big[(u_{xx}+u_{yy})(-v_y) + v(u_{xx}+u_{yy})_y\big]_{y=-\ell}dx \\
&\qquad + \int_{0}^{\pi}\big[(u_{xx}+u_{yy})v_y - v(u_{xx}+u_{yy})_y\big]_{y=\ell}dx \\
&\quad = + \int_{-\ell}^{\ell}\Big(\big[u_{xx}v_x\big]_{x=\pi} - \big[u_{xx}v_x\big]_{x=0}\Big)dy \\
&\qquad + \int_{0}^{\pi}\big[(u_{xx}+u_{yy})(-v_y) + v(u_{xx}+v_{yy})_y\big]_{y=-\ell}dy \\
&\qquad + \int_{0}^{\pi}\big[(u_{xx}+u_{yy})v_y - v(u_{xx}+v_{yy})_y\big]_{y=\ell}dy\,.
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\big[2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}\big]dxdy \\
&\quad = \int_{\Omega}\Big(\frac{\partial}{\partial x}\big[u_{xy}v_y - u_{yy}v_x\big] + \frac{\partial}{\partial y}\big[u_{xy}v_x - u_{xx}v_y\big]\Big)dxdy \\
&\quad = + \int_{-\ell}^{\ell} -\big[u_{xy}\underbrace{v_y}_{=0} - \underbrace{u_{yy}}_{=0}v_x\big]\Big|_{x=0}dy \\
&\qquad + \int_{-\ell}^{\ell}\big[u_{xy}\underbrace{v_y}_{=0} - \underbrace{u_{yy}}_{=0}v_x\big]\Big|_{x=\pi}dy \\
&\qquad + \int_{0}^{\pi} -\big[u_{xy}v_x - u_{xx}v_y\big]\Big|_{y=-\ell}dx \\
&\qquad + \int_{0}^{\pi}\big[u_{xy}v_x - u_{xx}v_y\big]\Big|_{y=\ell}dx\,.
\end{aligned}$$

Integrating by parts we get

$$\int_0^{\pi}\big[u_{xy}v_x\big]_{y=\ell}dx = \underbrace{\big[u_{xy}v\big]_{y=\ell}\Big|_{x=0}^{x=\pi}}_{=0} - \int_0^{\pi}\,[u_{xxy}v]_{y=\ell}dx,$$

and the same is true when replacing ℓ with $-\ell$.

Finally, adding together all the terms we have derived we obtain

$$
\begin{aligned}
&\int_\Omega \left[\Delta u \Delta v + (1-\sigma) 2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx} - f\right] dxdy \\
&= \int_\Omega (\Delta^2 u - f) v\, dxdy \\
&\quad + \int_{-\ell}^{\ell} \Big(\big[u_{xx}v_x\big]_{x=\pi} - \big[u_{xx}v_x\big]_{x=0} \Big) dy \\
&\quad + \int_0^\pi \big[u_{xxy}v + u_{yyy}v - u_{xx}v_y - u_{yy}v_y\big]\Big|_{y=-\ell} dx \\
&\quad + \int_0^\pi \big[-u_{xxy}v - u_{yyy}v + u_{xx}v_y + u_{yy}v_y\big]\Big|_{y=\ell} dx \\
&\quad + \int_0^\pi (1-\sigma)\big[u_{xxy}v + u_{xx}v_y\big]\Big|_{y=-\ell} dx \\
&\quad + \int_0^\pi (1-\sigma)\big[-u_{xxy}v - u_{xx}v_y\big]\Big|_{y=\ell} dx \\
&= \int_\Omega (\Delta^2 u - f) v\, dxdy \\
&\quad + \int_{-\ell}^{\ell} \Big(\big[u_{xx}v_x\big]_{x=\pi} - \big[u_{xx}v_x\big]_{x=0} \Big) dy \\
&\quad + \int_0^\pi \Big([(2-\sigma)u_{xxy} + u_{yyy}]v + [-\sigma u_{xx} - u_{yy}]v_y \Big)_{y=-\ell} dx \\
&\quad + \int_0^\pi \Big([-(2-\sigma)u_{xxy} - u_{yyy}]v + [\sigma u_{xx} + u_{yy}]v_y \Big)_{y=\ell} dx \\
&= 0\,.
\end{aligned}
$$

□

4.3 Explicit Resolution

This section aims to complement the theoretical results obtained in the previous section on the properties of solutions u of problem (4.4) with more practical descriptions of such solutions. Specifically, we provide various explicit, i.e., computable, formulations for the solution u depending on the specific form taken by the forcing term f. The results are organized from more specific to more general as follows. In Section 4.3.1 we consider a load function f which does not depend on the y coordinate of the position. Sections 4.3.2 and 4.3.3 we consider load functions depending on y in a linear and exponential fashion respectively. Finally Section 4.3.4 extends the results to arbitrary load functions $f = f(x,y)$ whose dependence on both variables is unrestricted.

4.3.1 Load depending only on x

In this section we shall assume that the forcing term does not depend on y. This assumption is justified since in the setting we are modelling the shape of the walkway can be well described by a long narrow rectangle, i.e, we have $\ell \ll \pi$. So, we now assume that

$$f = f(x), \qquad f \in L^2(0,\pi)\,. \tag{4.11}$$

We follow the method described in [8] and originally introduced in Section 2.2 of the book [12]. We extend the source f as an odd 2π-periodic function over $\mathbb{R}$ and we expand it in Fourier series

$$f(x) = \sum_{m=1}^{+\infty} \beta_m \sin(mx)\,, \qquad \beta_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)dx\,, \tag{4.12}$$

so that $\{\beta_m\} \in \ell^2$ and the series converges in $L^2(0,\pi)$ to f. Then we define the constants

$$A = A(m,\ell) := \frac{\sigma}{1-\sigma}\frac{\beta_m}{m^4}\frac{(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell}\,, \tag{4.13}$$

$$B = B(m,\ell) := \sigma\frac{\beta_m}{m^4}\frac{\sinh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell}\,, \tag{4.14}$$

and we state

Theorem 4.3.1. [8, Theorem 3.2] *Assume* (3.2) *and that f satisfies* (4.11)-(4.12). *Then the unique solution of* (3.8) *is given by*

$$u(x,y) = \sum_{m=1}^{+\infty}\left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right]\sin(mx)$$

where the constants A and B are defined in (4.13) *and* (4.14).

Proof. Assume that $f = f(x)$ does not depend on y. We can expand f into its Fourier series with Fourier coefficients $(\beta_m)_{\beta=1}^{\infty}$

$$f(x) = \sum_{m=1}^{\infty} \beta_m \sin(mx)\,, \qquad \beta_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)dx.$$

The function $\phi(x) = \sum_{m=1}^{\infty}\frac{\beta_m}{m^4}\sin(mx)$ solves the equation

$$\phi''''(x) = f(x) \text{ in } (0,\pi)\,, \qquad \phi(0) = \phi''(0) = \phi(\pi) = \phi''(\pi) = 0,$$

moreover $\phi'''' = f \in L^2(0,\pi)$ which implies that $\phi'' \in H^2(0,\pi)$ and its Fourier series is given by $\phi''(x) = -\sum_{m=1}^{\infty}\frac{\beta_m}{m^2}\sin(mx)$, which converges strongly in $H^2(0,\pi)$.

We study $v(x,y) = u(x,y) - \phi(x)$. If u solves the classical equation (3.8) then we have that v solves the following problem:

$$\begin{cases} \Delta^2 v = \Delta^2 u - \phi'''' = f - f = 0 & \text{in } \Omega \\ v = u - \phi = 0 \quad \text{and} \quad v_{xx} = u_{xx} - \phi'' = 0 & \text{on } \{0,\pi\}\times(-\ell,\ell) \\ v_{yy} + \sigma v_{xx} = u_{yy} + \sigma u_{xx} - \sigma\phi'' = -\sigma\phi'' & \text{on } (0,\pi)\times\{-\ell,\ell\} \\ v_{yyy} + (2-\sigma)v_{xxy} = u_{yyy} + (2-\sigma)u_{xxy} - 0 = 0 & \text{on } (0,\pi)\times\{-\ell,\ell\}\,. \end{cases}$$

We look for solutions v of this problem having separable variables, namely $v(x,y) = \sum_{m=1}^{\infty} Y_m(y)\sin(mx)$ for appropriate functions $Y_m = Y_m(y)$.

We note that

$$\begin{aligned}\Delta^2\big(Y_m(y)\sin(mx)\big) &= \Delta\big(Y_m''(y)\sin(mx) + Y_m(y)(-m^2)\sin(mx)\big) \\ &= Y_m''''(y)\sin(mx) - m^2 Y_m''(y)\sin(mx) \\ &\quad - m^2 Y_m''(y)\sin(mx) + m^4 Y_m(y)\sin(mx) \\ &= \left[Y_m''''(y) - 2m^2 Y_m''(y) + m^4 Y_m(y)\right]\sin(mx)\,,\end{aligned}$$

and so we have

$$\Delta^2 v(x,y) = \sum_{m=1}^{\infty}\left[Y_m''''(y) - 2m^2 Y_m''(y) + m^4 Y_m(y)\right]\sin(mx).$$

Since we require $\Delta^2 v$ to be identically zero we infer that Y_m must satisfy the constraint

$$Y_m''''(y) - 2m^2 Y_m''(y) + m^4 Y_m(y) = 0 \qquad \text{for } y \in (-\ell,\ell).$$

This is a fourth order linear ordinary differential equation with constant coefficients whose characteristic polynomial is

$$\lambda^4 - 2m\lambda^2 + m^4 = (\lambda^2 - m^2)^2 = (\lambda + m)^2(\lambda - m)^2$$

so any solution can be expressed as a linear combination of the four independent solutions $\cosh(my), \sinh(my), y\cosh(my), y\sinh(mx)$.

From the symmetry of the domain Ω and the uniqueness of the solution v (which we proved in Lemma 4.2.3) we know that if a solution exists then it must be even, therefore we can exclude $\sinh(my)$ and $y\cosh(my)$ so that Y_m is of the form

$$Y_m(y) = A\cosh(my) + Bmy\sinh(my),$$

Where $A$and B are coefficients to be determined by imposing the boundary conditions and the constant m has been factored in to the second term in order to simplify future calculations.

The derivatives of Y_m can be computed as follows

$$\begin{aligned}Y_m'(y) &= mA\sinh(my) + Bm^2 y\sinh(my) + Bmy\cosh(my) \\ &= m\big[(A+B)\sinh(my) + Bmy\cosh(my)\big] \\ Y_m''(y) &= m\big[m(A+B)\cosh(my) + Bm\cosh(my) + Bm^2 y\sinh(my)\big] \\ &= m^2\big[(A+2B)\sinh(my) + Bmy\cosh(my)\big] \\ Y_m'''(y) &= m^2\big[m(A+2B)\sinh(my) + Bm\sinh(my) + Bm^2 y\cosh(my)\big] \\ &= m^3\big[(A+3B)\sinh(my) + Bmy\cosh(my)\big]\,.\end{aligned}$$

We can now determine $A = A(m,\ell)$ and $B = B(m,\ell)$ imposing that v satisfies the boundary conditions

$$
\begin{aligned}
v_{yy} + \sigma v_{xx} = & \sum_{m=1}^{\infty} \left[Y_m''(y) - \sigma m^2 Y_m(y)\right] \sin(mx) \\
= & -\sigma\phi''(x) = \sum_{m=1}^{\infty} \sigma \frac{\beta_m}{m^2} \sin(mx)\,, \\
v_{yyy} + (2-\sigma) v_{xxy} = & \sum_{m=1}^{\infty} \left[Y_m'''(y) - (2-\sigma) m^2 Y_m'(y)\right] \sin(mx) = 0 \\
& \forall x \in (0,\pi) \text{ and } y \in \{-\ell, \ell\}\,.
\end{aligned}
$$

Since the trigonometric series must vanish for every value of x we deduce that their coefficients must all be zero, namely

$$
\begin{aligned}
Y_m''(y) - \sigma m^2 Y_m(y) &= \sigma \frac{\beta_m}{m^2} \\
Y_m'''(y) - (2-\sigma) m^2 Y_m'(y) &= 0 \\
& \text{when } y \in \{-\ell, \ell\}.
\end{aligned}
$$

By the evenness of function Y_m it is enough to impose the equality for $y = \ell$, indeed

$$
\begin{aligned}
Y_m''(-\ell) - \sigma m^2 Y_m(-\ell) &= Y_m''(\ell) - \sigma m^2 Y_m(\ell) \\
Y_m'''(-\ell) - (2-\sigma) m^2 Y_m'(-\ell) &= -Y_m'''(\ell) + (2-\sigma) m^2 Y_m'(\ell)\,.
\end{aligned}
$$

By substituting th explicit form of Y_m and its derivatives we find

$$
\begin{aligned}
Y_m''(\ell) - \sigma m^2 Y_m(\ell) &= m^2\big[(A+2B)\cosh(m\ell) + Bm\ell(\sinh(m\ell))\big] \\
&\quad - \sigma m^2\big[A\cosh(m\ell) + Bm\ell\sinh(m\ell)\big] \\
&= m^2 A(1-\sigma)\cosh(m\ell) + m^2 B\big[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)\big]
\end{aligned}
$$

$$
\begin{aligned}
Y_m'''(\ell) - (2-\sigma) m^2 Y_m'(\ell) = & \; m^3\big[(A+3B)\sinh(m\ell) + Bm\ell\cosh(m\ell)\big] \\
& - (2-\sigma) m^3\big[(A+B)\sinh(m\ell) + B,\ell\cosh(m\ell)\big] \\
= & - m^3 A(1+\sigma)\sinh(m\ell) \\
& - m^3 B\big[(-1-\sigma)\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)\big].
\end{aligned}
$$

We obtain the following linear system in the unknowns A and B

$$
\begin{cases}
(1-\sigma)\cosh(m\ell)A + \big(2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)\big)B = \sigma\frac{\beta_m}{m^4} \\
(1-\sigma)\sinh(m\ell)A + \big((1-\sigma)m\ell\cosh(m\ell) - (1+\sigma)\sinh(m\ell)\big)B = 0,
\end{cases}
$$

whose solutions are given by

$$
\begin{aligned}
A = A(m,\ell) &= \frac{\sigma}{1-\sigma}\frac{\beta_m}{m^4}\frac{(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell} \\
B = B(m,\ell) &= \sigma\frac{\beta_m}{m^4}\frac{\sinh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell}\,.
\end{aligned}
$$

We have shown that, as stated by the theorem, the function

$$u = v + \phi = \sum_{m=1}^{\infty} \left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my) \right] \sin(mx)$$

solves the equation. □

Figures 4.1 and 4.2 represent respectively the load function f and corresponding solution u in the specific case where $f(x, y) = \sin(3x)$, so that its Fourier coefficients are $\beta_m = \delta_{m,3}$.

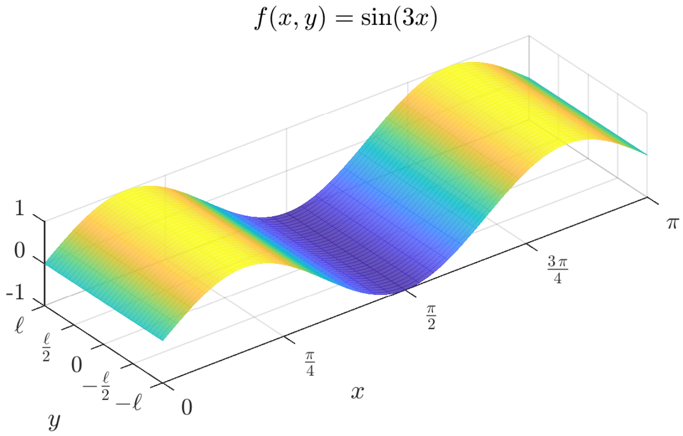

Figure 4.1. The load function $f(x, y) = \sin(3x)$

The same technique outlined in the previous statement can be extended, as sketched in [12, Section 2.2.2], in order to obtain explicit solutions to (3.8) in cases where the forcing term has the form $yf(x)$ and $e^{\alpha y} f(x)$ for some function $f \in L^2$ depending only on x and for any constant $\alpha \in \mathbb{R}$. Since most of the proof is the same as that of Theorem 4.3.1 we write out only the steps for which adaptations are needed.

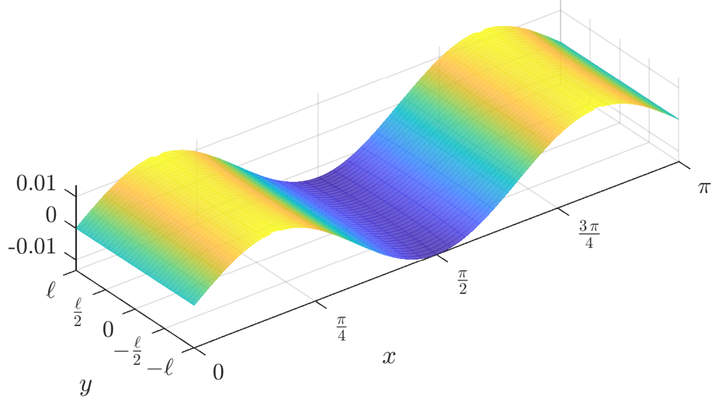

Figure 4.2. The solution $u(x,y)$ when $f(x,y) = \sin(3x)$

4.3.2 Load depending linearly on y

Consider a load f which can be expanded as follows

$$f(x,y) = y \sum_{m=1}^{+\infty} \beta_m \sin(mx)\,. \tag{4.15}$$

In analogy to the previous section we define the constants

$$C = -\frac{1}{1-\sigma}\frac{\beta_m}{m^5}\frac{(4-2\sigma+\sigma(1-\sigma)\ell^2m^2)\sinh(\ell m)+2(1-2\sigma)\ell m\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}\,, \tag{4.16}$$

$$D = \frac{\beta_m}{m^5}\frac{(2-\sigma)\sinh(\ell m)+\sigma\ell m\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}\,, \tag{4.17}$$

and prove the following.

Theorem 4.3.2. *Assume* (3.2) *and that f satisfies* (4.15)*. Then the unique solution of* (3.8) *is given by*

$$u(x,y) = \sum_{m=1}^{\infty}\left[y\frac{\beta_m}{m^4} + C\sinh(my) + Dmy\cosh(my)\right]\sin(mx)\,, \tag{4.18}$$

where the constants C and D are defined in (4.16)–(4.17).

Proof. We define $\phi(x) = \sum_{m=1}^{+\infty} \frac{\beta_m}{m^4}\sin(mx)$ and observe that $y\phi(x)$ solves equation (3.5) as well as the Navier boundary conditions on the short edges (3.6). Given the solution u

of problem (3.8), we obtain a solution $v(x,y) = u(x,y) - y\,\phi(x)$ of the system

$$\begin{cases} \Delta^2 v = 0 & \text{in } \Omega \\ v = 0 \quad \text{and} \quad v_{xx} = 0 & \text{on } \{0,\pi\} \times (-\ell,\ell) \\ v_{yy} + \sigma v_{xx} = -\sigma y \phi'' & \text{on } (0,\pi) \times \{-\ell,\ell\} \\ v_{yyy} + (2-\sigma) v_{xxy} = -(2-\sigma)\phi'' & \text{on } (0,\pi) \times \{-\ell,\ell\}\,. \end{cases}$$

Looking for solution of the form $v(x,y) = \sum_{m=1}^{\infty} Y_m(y)\sin(mx)$, we see that the coefficients Y_m must be linear combinations of the four solutions $\cosh(my)$, $\sinh(my)$, $y\cosh(my)$, $y\sinh(mx)$. Since the forcing term f is odd and by the symmetry of the domain Ω we can restrict our search to odd coefficients Y_m, so that one has

$$Y_m(y) = C\sinh(my) + Dmy\cosh(my),$$

for some choice of coefficients C and D depending on m and ℓ. Enforcing the boundary conditions we obtain

$$\begin{aligned} v_{yy} + \sigma v_{xx} &= \sum_{m=1}^{\infty} \left[Y_m''(y) - \sigma m^2 Y_m(y)\right]\sin(mx) \\ &= -\,\sigma y \phi''(x) = \sum_{m=1}^{\infty} \sigma \frac{\beta_m}{m^2} y \sin(mx)\,, \\ v_{yyy} + (2-\sigma)v_{xxy} &= \sum_{m=1}^{\infty} \left[Y_m'''(y) - (2-\sigma)m^2 Y_m'(y)\right]\sin(mx) \\ &= -\,(2-\sigma)\phi''(x) = \sum_{m=1}^{\infty} (2-\sigma)\frac{\beta_m}{m^2}\sin(mx)\,, \\ &\forall x \in (0,\pi) \text{ and } y \in \{-\ell,\ell\}\,. \end{aligned}$$

Equating the coefficients of $\sin(mx)$ for each value of $m \in \mathbb{N}$ leads to the following linear systems in the unknowns C and D.

$$\begin{cases} (1-\sigma)\sinh(m\ell)C + \big(2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)\big)D = \sigma\ell\frac{\beta_m}{m^4} \\ (1-\sigma)\cosh(m\ell)C + \big((1-\sigma)m\ell\sinh(m\ell) - (1+\sigma)\cosh(m\ell)\big)D = -(2-\sigma)\frac{\beta_m}{m^5}\,. \end{cases}$$

By solving the system we obtain the closed form of the coefficients $C = C(m\ell)$ and $D = D(m,\ell)$, which is given by (4.16)–(4.17).

This proves that when f depends linearly on the y coordinate, so that it can be put in form (4.15), the solution u of (3.8) can be expressed explicitly as the series (4.18), where the values for constants C and D are given in (4.16)–(4.17). □

To illustrate graphically this result, we consider Fourier coefficients given by $\beta_m = \delta_{m,3}$. The resulting load function is $f(x,y) = y\sin(3x)$ which we represent in Figure 4.3. We use the just stated formulas to calculate explicitly the corresponding solution $u(x,y)$ of (3.8) and show the result in Figure 4.4.

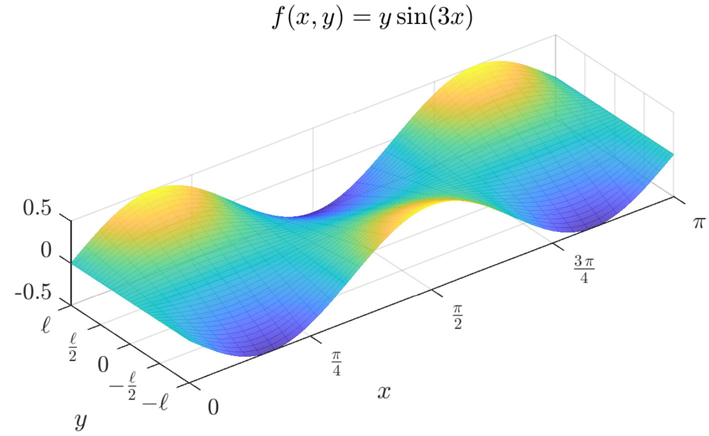

Figure 4.3. The load function $f(x, y) = y\sin(3x)$

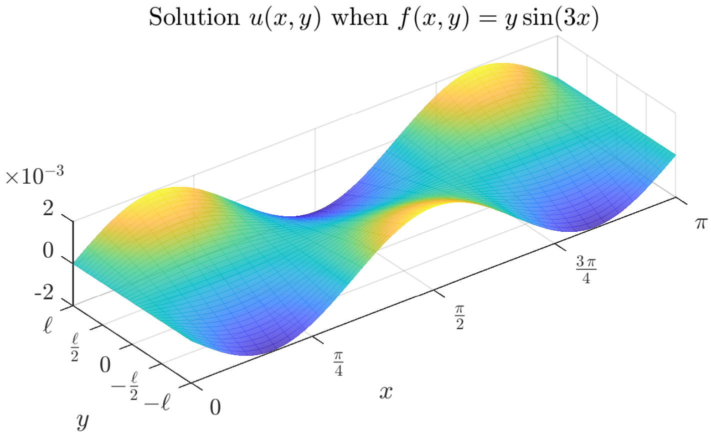

Figure 4.4. The solution $u(x, y)$ when $f(x, y) = y\sin(3x)$

4.3.3 Load depending exponentially on y

Now we consider a load f whose dependence on y is exponential as in $e^{\alpha y}$, for some constant $\alpha \in \mathbb{R}$, so that taking its Fourier expansion along the horizontal direction one obtains

$$f(x,y) = e^{\alpha y} \sum_{m=1}^{+\infty} \beta_m \sin(mx)\,. \tag{4.19}$$

As we did in the previous sections we introduce the constants

$$\begin{aligned}
A = {}& \frac{1}{1-\sigma}\frac{\beta_m}{m^3(m^2-\alpha^2)^2}\times \qquad (4.20)\\
&\times\left[m(\sigma m^2-\alpha^2)\cosh(\alpha\ell)\frac{(1+\sigma)\sinh(\ell m)-(1-\sigma)\ell m\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}+\right.\\
&\left.-\alpha\big((2-\sigma)m^2-\alpha^2\big)\sinh(\alpha\ell)\frac{(1-\sigma)\ell m\sinh(\ell m)+2\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}\right],\\
B = {}& \frac{\beta_m}{m^3(m^2-\alpha^2)^2}\times \qquad (4.21)\\
&\times\left[m(\sigma m^2-\alpha^2)\cosh(\alpha\ell)\frac{\sinh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}+\right.\\
&\left.-\alpha\big((2-\sigma)m^2-\alpha^2\big)\sinh(\alpha\ell)\frac{\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}\right],\\
C = {}& \frac{1}{1-\sigma}\frac{\beta_m}{m^3(m^2-\alpha^2)^2}\times \qquad (4.22)\\
&\times\left[m(\sigma m^2-\alpha^2)\sinh(\alpha\ell)\frac{(1+\sigma)\cosh(\ell m)-(1-\sigma)\ell m\sinh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}+\right.\\
&\left.-\alpha\big((2-\sigma)m^2-\alpha^2\big)\cosh(\alpha\ell)\frac{(1-\sigma)\ell m\cosh(\ell m)+2\sinh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m}\right],\\
D = {}& \frac{\beta_m}{m^3(m^2-\alpha^2)^2}\times \qquad (4.23)\\
&\times\left[m(\sigma m^2-\alpha^2)\sinh(\alpha\ell)\frac{\cosh(\ell m)}{(3+\sigma)\cosh(\ell m)\sinh(\ell m)-(1-\sigma)\ell m}+\right.\\
&\left.-\alpha\big((2-\sigma)m^2-\alpha^2\big)\cosh(\alpha\ell)\frac{\sinh(\ell m)}{(3+\sigma)\cosh(\ell m)\sinh(\ell m)-(1-\sigma)\ell m}\right].
\end{aligned}$$

Which allow us to state the following.

Theorem 4.3.3. *Assume* (3.2) *and that* f *satisfies* (4.19)*. Then the unique solution of*

(3.8) *is given by*

$$u(x,y)=\sum_{m=1}^{\infty}\left[\frac{\beta_m e^{\alpha y}}{(m^2-\alpha^2)^2}+\right.$$
$$\left.+A\cosh(my)+Bmy\sinh(my)+C\sinh(my)+Dmy\cosh(my)\right]\sin(mx) \qquad (4.24)$$

where the constants A, B, C and D are defined in (4.20)–(4.23).

Proof. In this case a particular solution of (3.5) as well as the Navier boundary conditions (3.6) is given by the function $e^{\alpha y}\phi(x)$, where the definition of ϕ is a slight modification of the homonymous function found in the preceding cases, namely

$$\phi(x)=\sum_{m=1}^{+\infty}\frac{\beta_m}{(m^2-\alpha^2)^2}\sin(mx)\,,$$

so that if a function u solves (3.8), then the difference $v:=u-e^{\alpha y}\phi(x)$ must satisfy the following system

$$\begin{cases}\Delta^2 v=0 & \text{in } \Omega\\ v=0 \quad \text{and} \quad v_{xx}=0 & \text{on } \{0,\pi\}\times(-\ell,\ell)\\ v_{yy}+\sigma v_{xx}=-\alpha^2 e^{\alpha y}\phi(x)-\sigma e^{\alpha y}\phi''(x) & \text{on } (0,\pi)\times\{-\ell,\ell\}\\ v_{yyy}+(2-\sigma)v_{xxy}=-\alpha^3 e^{\alpha y}\phi(x)-(2-\sigma)\alpha e^{\alpha y}\phi''(x) & \text{on } (0,\pi)\times\{-\ell,\ell\}\,.\end{cases}$$

As before we suppose that the solution can be expressed as a sum of the form $v(x,y)=\sum_{m=1}^{\infty}Y_m(y)\sin(mx)$, where the coefficients Y_m are linear combinations of the four base functions $\cosh(my)$, $\sinh(my)$, $y\cosh(my)$, $y\sinh(mx)$, namely

$$Y_m(y)=A\cosh(my)+Bmy\sinh(my)+C\sinh(my)+Dmy\cosh(my)\,, \qquad (4.25)$$

where the factors m are introduced for convenience and once gain we omit the dependence of the coefficients over m, α and ℓ.

Since in this case f does not have any symmetry, all four coefficients are generally non-zero and they have to be determined by imposing the Navier boundary conditions on the two sides $y=\pm\ell$. This leads to a linear system in the unknowns A, B, C, D whose solution is indeed given by (4.20)–(4.23), as one can directly check.

Plugging this back into the solution u we find that (4.24) is the explicit form of the solution of problem (3.8). □

We illustrate this result with a specific example. Consider the case $\alpha=2$ and $\beta_m=\delta_{m,3}$, so that $f(x,y)=e^{2y}\sin(3x)$ as shown in Figure 4.5. Then the solution $u(x,y)$ can be computed using (4.24), and the result is depicted in Figure 4.6.

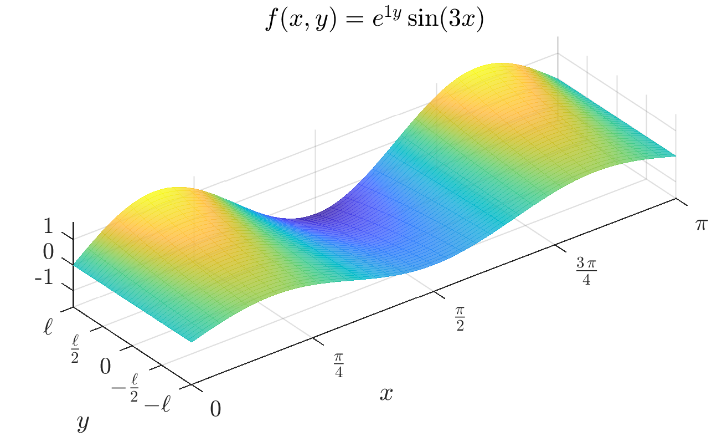

Figure 4.5. The load function $f(x, y) = e^{2y}\sin(3x)$

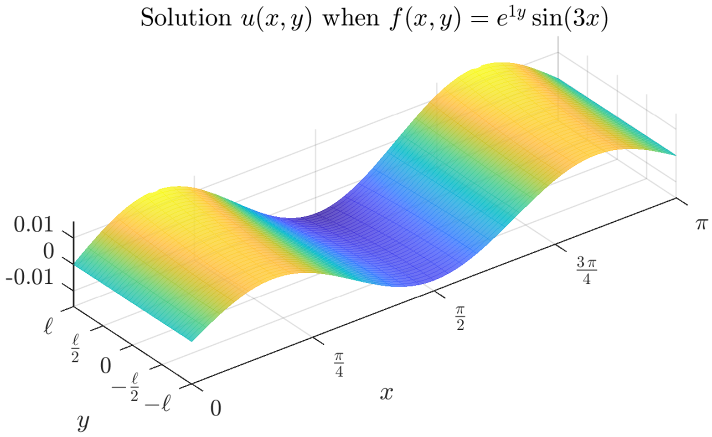

Figure 4.6. The solution $u(x, y)$ when $f(x, y) = e^{2y}\sin(3x)$

4.3.4 General explicit resolution

A remarkable consequence of the result obtained in Section 4.3.3 is that it allows one to write out an explicit representation of the solution u of problem (3.8) for an arbitrary load function $f = f(x, y) \in L^2(\Omega)$. To see this, we first introduce some notations. For $m \in \mathbb{N}$ and for any value of α, we consider a forcing term $f^{\text{exp}}_{m,\alpha}(x, y) = e^{\alpha y}\sin(mx)$. We see that this is a special case of (4.19) when $\beta_k = \delta_{k,m}$ so that the corresponding solution $u^{\exp}_{m,\alpha}$ of (3.8) is given by

$$u^{\exp}_{m,\alpha}(x,y) = \Bigg[\frac{e^{\alpha y}}{(m^2-\alpha^2)^2} + \\ + A^{\exp}\cosh(my) + B^{\exp} my\sinh(my) + C^{\exp}\sinh(my) + D^{\exp} my\cosh(my)\Bigg]\sin(mx)\,,$$

where the constants $A^{\exp}$, $B^{\exp}$, $C^{\exp}$, $D^{\exp}$ are those given by (4.20)–(4.23), taking $\beta_m = 1$. We observe that the solution is still valid if we allow α to assume complex values.

For any value of $n \in \mathbb{N}$ we define the angular frequency of the n-th harmonic of the segment $[-\ell, \ell]$ as $\omega_n = \pi n/\ell$. By the linearity of (3.8), considering a forcing term

$$f^{\sin}_{m,n} = \sin(mx)\sin(\omega_n y) = \frac{f^{\exp}_{m,i\omega_n} - f^{\exp}_{m,-i\omega_n}}{2i}\,,$$

the corresponding solution $u^{\sin}_{m,n}$ is given by

$$u^{\sin}_{m,n}(x,y) = \frac{u^{\exp}_{m,i\omega_n} - u^{\exp}_{m,-i\omega_n}}{2i} = \Bigg[\frac{\sin(\omega_n y)}{(m^2+\omega_n^2)^2} + \\ + A^{\sin}\cosh(my) + B^{\sin} my\sinh(my) + C^{\sin}\sinh(my) + D^{\sin} my\cosh(my)\Bigg]\sin(mx)\,,$$

where the coefficients $A^{\sin}$, $B^{\sin}$, $C^{\sin}$, $D^{\sin}$ are obtained as (explicitly writing the dependency of $A^{\sin}$, $B^{\sin}$, $C^{\sin}$ on n as well as the one of $A^{\exp}$, $B^{\exp}$, $C^{\exp}$, $D^{\exp}$ on α)

$$\begin{aligned}
A^{\sin}(n) &= \frac{A^{\exp}(i\omega_n) - A^{\exp}(-i\omega_n)}{2i} = \frac{A^{\exp}(i\omega_n) - A^{\exp}(i\omega_n)^*}{2i} = \Im[A^{\exp}(i\omega_n)]\,,\\
B^{\sin}(n) &= \frac{B^{\exp}(i\omega_n) - B^{\exp}(-i\omega_n)}{2i} = \frac{B^{\exp}(i\omega_n) - B^{\exp}(i\omega_n)^*}{2i} = \Im[B^{\exp}(i\omega_n)]\,,\\
C^{\sin}(n) &= \frac{C^{\exp}(i\omega_n) - C^{\exp}(-i\omega_n)}{2i} = \frac{C^{\exp}(i\omega_n) - C^{\exp}(i\omega_n)^*}{2i} = \Im[C^{\exp}(i\omega_n)]\,,\\
D^{\sin}(n) &= \frac{D^{\exp}(i\omega_n) - D^{\exp}(-i\omega_n)}{2i} = \frac{D^{\exp}(i\omega_n) - D^{\exp}(i\omega_n)^*}{2i} = \Im[D^{\exp}(i\omega_n)]\,,
\end{aligned}$$

where we use the fact that $A^{\exp}(\alpha^*) = A^{\exp}(\alpha)^*$ and the same holds for $B^{\exp}$, $C^{\exp}$ and $D^{\exp}$, due to the fact that $A^{\exp}$, $B^{\exp}$, $C^{\exp}$ and $D^{\exp}$ are rational functions of α, $\cosh(\alpha\ell)$ and $\sinh(\alpha\ell)$ having real coefficients and that $\cosh(\alpha^*\ell) = \cosh(\alpha\ell)^*$ as well as $\sinh(\alpha^*\ell) = \sinh(\alpha\ell)^*$. Carrying out the computations and taking into account the equations $\sinh(i\omega_n\ell) = i\sin(\omega_n\ell) = i\sin(\pi n) = 0$ and $\cosh(i\omega_n\ell) = \cos(\omega_n\ell) = \cos(\pi n) =$

$(-1)^n$, one obtains

$$\begin{aligned}
A^{\sin} &= 0 \\
B^{\sin} &= 0 \\
C^{\sin} &= -\frac{(-1)^n \omega_n \big(2\sinh(\ell m) + (1-\sigma)\cosh(\ell m)\ell m\big)\big((2-\sigma)m^2 + \omega_n^2\big)}{m^3(1-\sigma)(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m) + (1-\sigma)\ell m\big)} \\
D^{\sin} &= \frac{(-1)^n \omega_n \sinh(\ell m)\big((2-\sigma)m^2 + \omega_n^2\big)}{m^3(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m) + (1-\sigma)\ell m\big)} .
\end{aligned}$$

Notice how $A^{\sin} = B^{\sin} = 0$ is due to the fact that all functions $f^{\sin}_{m,n}$ are odd with respect to the variable y, so that that only the odd functions $\sinh(my)$ and $my\cosh(my)$ may be present in the corresponding solution $u^{\sin}_{m,n}$.

To better illustrate this result, Figure 4.7 represents the load function $f^{\sin}_{m,n}$ for $m = 3$ and $n = 1$ while Figure 4.8 depicts the corresponding solution $u^{\sin}_{m,n}$. Notice how the odd nature of the function $f^{\sin}_{m,n}$ with respect to the variable y results in an uneven distribution of the load, so that the solution $u^{\sin}_{m,n}$ presents a large torsional component along the longitudinal direction.

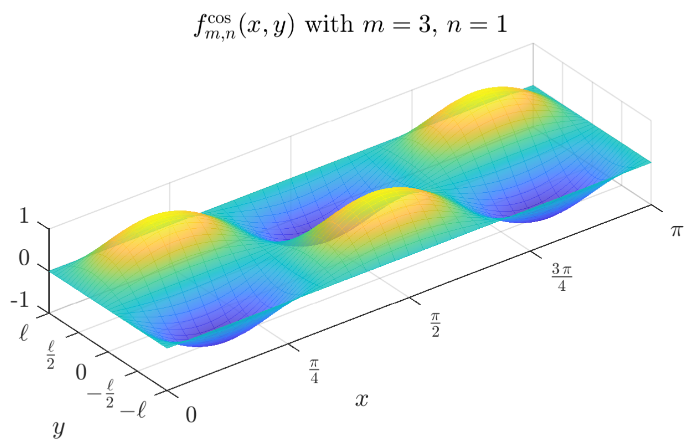

Figure 4.7. The load function $f^{\sin}_{m,n}$ for $m = 3$ and $n = 1$

The same approach can be adopted to solve problem (3.8) in the case of a load function of the form

$$f^{\cos}_{m,n} = \sin(mx)\cos(\omega_n y) = \frac{f^{\exp}_{m,i\omega_n} + f^{\exp}_{m,-i\omega_n}}{2} .$$

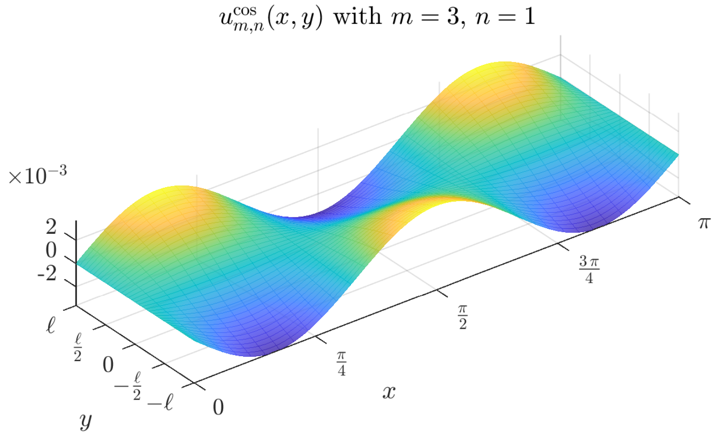

Figure 4.8. The solution $u^{\sin}_{m,n}$ for $m = 3$ and $n = 1$

In this case the corresponding solution $u^{\cos}_{m,n}$ can be computed as

$$u^{\cos}_{m,n}(x,y) = \frac{u^{\exp}_{m,i\omega_n} + u^{\exp}_{m,-i\omega_n}}{2} = \left[\frac{\cos(\omega_n y)}{(m^2+\omega_n^2)^2} + A^{\cos}\cosh(my) + \right.$$
$$\left. + B^{\cos} my\sinh(my) + C^{\cos}\sinh(my) + D^{\cos} my\cosh(my)\right]\sin(mx)\,,$$

where the coefficients $A^{\cos}$, $B^{\cos}$, $C^{\cos}$, $D^{\cos}$ are derived analogously to the previous case as (explicitly writing the dependency of $A^{\cos}$, $B^{\cos}$, $C^{\cos}$ on n as well as the one of $A^{\exp}$, $B^{\exp}$, $C^{\exp}$, $D^{\exp}$ on α)

$$\begin{aligned}
A^{\cos}(n) &= \frac{A^{\exp}(i\omega_n) + A^{\exp}(-i\omega_n)}{2} = \frac{A^{\exp}(i\omega_n) + A^{\exp}(i\omega_n)^*}{2} = \Re[A^{\exp}(i\omega_n)]\,,\\
B^{\cos}(n) &= \frac{B^{\exp}(i\omega_n) + B^{\exp}(-i\omega_n)}{2} = \frac{B^{\exp}(i\omega_n) + B^{\exp}(i\omega_n)^*}{2} = \Re[B^{\exp}(i\omega_n)]\,,\\
C^{\cos}(n) &= \frac{C^{\exp}(i\omega_n) + C^{\exp}(-i\omega_n)}{2} = \frac{C^{\exp}(i\omega_n) + C^{\exp}(i\omega_n)^*}{2} = \Re[C^{\exp}(i\omega_n)]\,,\\
D^{\cos}(n) &= \frac{D^{\exp}(i\omega_n) + D^{\exp}(-i\omega_n)}{2} = \frac{D^{\exp}(i\omega_n) + D^{\exp}(i\omega_n)^*}{2} = \Re[D^{\exp}(i\omega_n)]\,,
\end{aligned}$$

thanks again to the fact that $A^{\exp}(\alpha^*) = A^{\exp}(\alpha)^*$, and $B^{\exp}$, $C^{\exp}$, $D^{\exp}$ all satisfy similar equations. The resulting values can be computed as

$$\begin{aligned}
A^{\cos} &= \frac{(-1)^n(\sigma m^2+\omega_n^2)\big((1+\sigma)\sinh(\ell m)-(1-\sigma)\ell m\cosh(\ell m)\big)}{m^2(1-\sigma)(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m\big)}\\
B^{\cos} &= \frac{(-1)^n(\sigma m^2+\omega_n^2)\sinh(\ell m)}{m^2(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m\big)}\\
C^{\cos} &= 0\\
D^{\cos} &= 0\,.
\end{aligned}$$

Since all functions $f^{\cos}_{m,n}$ are even in the variable y, by symmetry the corresponding solutions $u^{\cos}_{m,n}$ must be even as well and so the coefficients $C^{\cos}$ and $D^{\cos}$, which correspond to the odd functions $\sinh(my)$ and $y\cosh(my)$ respectively, vanish as expected.

A graphical representation of this result is given by Figure 4.9, which represents the load function $f^{\cos}_{m,n}$ for $m = 3$ and $n = 1$, and Figure 4.10 depicting the corresponding solution $u^{\cos}_{m,n}$. Notice how the shape of the solution $u^{\cos}_{m,n}$ closely resembles that of the load $f^{\cos}_{m,n}$, sharing the same local maxima and minima.

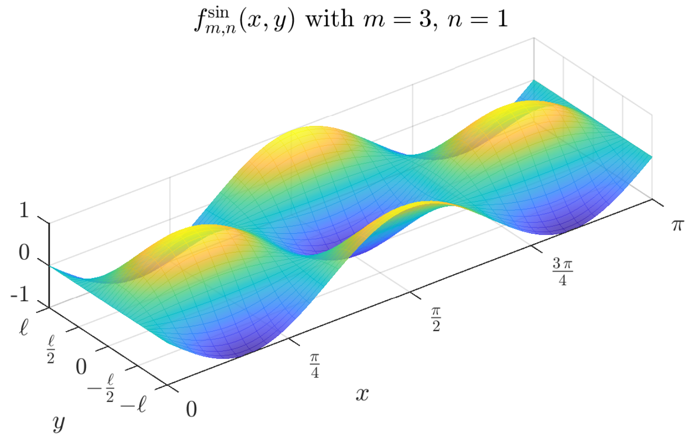

Figure 4.9. The load function $f^{\cos}_{m,n}$ for $m = 3$ and $n = 1$

It is well known that, since the domain Ω is represented by the rectangle $(0,\pi)\times(-\ell,\ell)$, the system

$$\begin{aligned}
&\{f^{\sin}_{m,n} : m,n \geq 1\} \cup \{f^{\cos}_{m,n} : m \geq 1, n \geq 0\} = \\
&\qquad \{\sin(mx)\sin(\omega_n y) : m,n \geq 1\} \cup \{\sin(mx)\cos(\omega_n y) : m \geq 1, n \geq 0\}\,, \quad (4.26)
\end{aligned}$$

where once again $\omega_n = \pi n/\ell$, forms an orthogonal basis of $L^2(\Omega)$. This entails that an

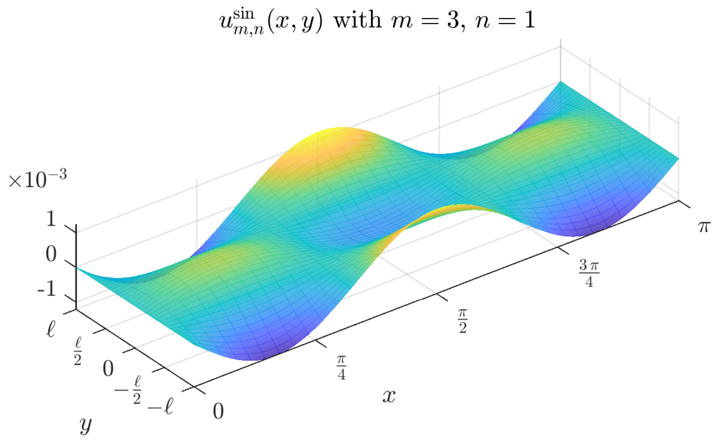

Figure 4.10. The solution $u^{\cos}_{m,n}$ for $m=3$ and $n=1$

arbitrary load function $f = f(x,y)$ admits a representation as a double Fourier series

$$\begin{aligned} f(x,y) &= \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\left[\beta^{\sin}_{m,n}\sin(\omega_n y) + \beta^{\cos}_{m,n}\cos(\omega_n y)\right]\sin(mx) \qquad (4.27)\\ &= \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\beta^{\sin}_{m,n}f^{\sin}_{m,n}(x,y) + \beta^{\cos}_{m,n}f^{\cos}_{m,n}(x,y) \end{aligned}$$

for an appropriate choice of coefficients $\beta^{\sin}_{m,n}$, $\beta^{\cos}_{m,n} \in \ell^2(\mathbb{N}^2)$. More specifically, one can compute the values of $\beta^{\sin}_{m,n}$ and $\beta^{\cos}_{m,n}$ for all m, $n \in \mathbb{N}$ as

$$\begin{aligned} \beta^{\sin}_{m,n} &= \frac{2}{\pi\ell}\int_\Omega f\, f^{\sin}_{m,n}\, dx\,,\\ \beta^{\cos}_{m,0} &= \frac{1}{\pi\ell}\int_\Omega f\, f^{\cos}_{m,0}\, dx\,,\\ \beta^{\cos}_{m,n} &= \frac{2}{\pi\ell}\int_\Omega f\, f^{\cos}_{m,n}\, dx\,, \qquad \text{for } n \geq 1\,. \end{aligned}$$

For convenience, we include in the sum the trivial functions $f^{\sin}_{m,0} \equiv 0$ for all $m \geq 1$, and the corresponding coefficients $\beta^{\sin}_{m,0}$ which are all equal to zero by definition.

Since for each load function $f^{\sin}_{m,n}$ and $f^{\cos}_{m,n}$ present in the decomposition (4.27) the corresponding solution $u^{\sin}_{m,n}$, respectively $u^{\cos}_{m,n}$, is known explicitly and applying once again the linearity of (3.8) as well as its well posedness, we reach the following explicit representation for the solution u corresponding to f

$$u(x,y) = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\beta^{\sin}_{m,n}u^{\sin}_{m,n}(x,y) + \beta^{\cos}_{m,n}u^{\cos}_{m,n}(x,y).$$

In order to give a more readable formulation we combine the coefficients $\beta^{\cos}_{m,n}$ with the constants $A^{\cos}$ and $B^{\cos}$ as well as $\beta^{\sin}_{m,n}$ with $C^{\sin}$ and $D^{\sin}$ defining the constants A, B, C and D as follows

$$\begin{aligned}
A &= \frac{\beta^{\cos}_{m,n}(-1)^n(\sigma m^2+\omega_n^2)\big((1+\sigma)\sinh(\ell m)-(1-\sigma)\ell m\cosh(\ell m)\big)}{m^2(1-\sigma)(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m\big)}\\
B &= \frac{\beta^{\cos}_{m,n}(-1)^n(\sigma m^2+\omega_n^2)\sinh(\ell m)}{m^2(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)-(1-\sigma)\ell m\big)}\\
C &= -\frac{\beta^{\sin}_{m,n}(-1)^n\omega_n\big(2\sinh(\ell m)+(1-\sigma)\cosh(\ell m)\ell m\big)\big((2-\sigma)m^2+\omega_n^2\big)}{m^3(1-\sigma)(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)+(1-\sigma)\ell m\big)}\\
D &= \frac{\beta^{\sin}_{m,n}(-1)^n\omega_n\sinh(\ell m)\big((2-\sigma)m^2+\omega_n^2\big)}{m^3(m^2+\omega_n^2)^2\big((3+\sigma)\sinh(\ell m)\cosh(\ell m)+(1-\sigma)\ell m\big)}\,.
\end{aligned} \tag{4.28}$$

Finally to summarize this result in analogy with Theorem 4.3.1 we state the following

Theorem 4.3.4. *Assume* (3.2) *and let* $f\in L^2(\Omega)$ *be an arbitrary load function and let* (4.27) *be its double Fourier series. Then the unique solution of* (3.8) *is given by*

$$u(x,y)=\sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\Bigg[\frac{\beta^{\sin}_{m,n}\sin(\omega_n y)+\beta^{\cos}_{m,n}\cos(\omega_n y)}{(\omega_n^2+m^2)^2}+ \\ +A\cosh(my)+Bmy\sinh(my)+C\sinh(my)+Dmy\cosh(my)\Bigg]\sin(mx)\,, \tag{4.29}$$

where $\omega_n=\pi n/\ell$ *and the coefficients* A, B, C, D *depend on both* m *and* n *and are defined in* (4.28).

Once we have obtained all the results above, the only thing left to check in order to see that the series (4.29) is a valid solution of (3.8), thus proving Theorem 4.3.4, is to verify that the series converges in the space $H^2_*(\Omega)$. Since the techniques needed to carry out this step are detailed extensively in Section 4.4, to aid the exposition and reduce repetition the proof of Theorem 4.3.4 is reported in Section 4.5 at the end of this chapter.

It is easy to see that we can recover Theorem 4.3.1 from the more general Theorem 4.3.4. Indeed if the load f depends only on the coordinate x of the position, then its double Fourier series (4.27) will have all vanishing coefficients $\beta^{\sin}_{m,n}$ and $\beta^{\sin}_{m,n}$ for $m,n\in\mathbb{N}$, except possibly for $\beta^{\cos}_{m,0}$ with $m\geq 1$. Then if we rename $\beta_m=\beta^{\cos}_{m,0}$ for all $m\geq 1$ we find that the (4.27) actually coincides with the single Fourier expansion (4.12). Finally if we set $n=0$ and account for the fact that $\omega_0=0$ we see that the constants A and B from (4.28) reduce to the ones provided in equations (4.13)–(4.14) while $C=D=0$, so that in this special case the form of the solution (4.29) is exactly the one obtained in Theorem 4.3.1.

4.4 Convergence to 1-dimensional beam

As we mentioned in Chapter 1, many models proposed for modeling suspension bridges are one dimensional in nature, it is therefore interesting to study how our bidimensional model is related to a one dimensional one, namely the simple supported beam model. As we did in the previous section we treat separately the case in which the load function does not depend on the y coordinate of the position, which was first treated in [8]. Thanks to the results obtained in subsection 4.3.4 we are able to relax this assumption, extending the result to arbitrary load functions.

4.4.1 Load depending only on x

When $\ell \to 0$, the plate Ω tends to become a one dimensional beam of length π. We wish to analyse the behaviour of the solution and of the energy in this limit situation. To this end, we re-introduce the constants appearing in (3.3) that were normalized in (3.4). Let $f \in L^2(\Omega)$ be as in (4.11) and let u^ℓ be a solution of the problem

$$\begin{cases} \frac{Ed^3}{12(1-\sigma^2)}\Delta^2 u^\ell = f & \text{in } \Omega \\ u^\ell(0,y) = u^\ell_{xx}(0,y) = u^\ell(\pi,y) = u^\ell_{xx}(\pi,y) & \text{for } y \in (-\ell,\ell) \\ u^\ell_{yy}(x,\pm\ell) + \sigma u^\ell_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ u^\ell_{yyy}(x,\pm\ell) + (2-\sigma)u^\ell_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{4.30}$$

whose total energy is given by (3.3). Obviously, $u^\ell = \frac{12(1-\sigma^2)}{Ed^3}u$, where u is the unique solution of (3.8) found in Theorem 3.2. If we view the plate as a parallelepiped-shaped beam $(0,\pi)\times(-\ell,\ell)\times(-d/2,d/2)$ we are led to the problem

$$EI\psi'''' = 2\ell f \quad \text{in } (0,\pi)\,, \qquad \psi(0) = \psi''(0) = \psi(\pi) = \psi''(\pi) = 0\,. \tag{4.31}$$

Here the forcing term $2\ell f$ represents a force per unit of length and $I = \frac{d^3\ell}{6} = \int_{(-\ell.\ell)\times(-d/2,d/2)} z^2 dydz$ is the moment of inertia of the section of the beam with respect to its middle line parallel to the y-axis. Then (4.31) reduces to $\frac{Ed^3}{12}\psi'''' = f$, the function ψ is independent of ℓ but the corresponding total energy of the beam depends on ℓ:

$$\mathbb{E}_T(\psi) = \frac{Ed^3\ell}{12}\int_0^\pi \psi''(x)^2 dx - 2\ell\int_0^\pi f(x)\psi(x)dx = -\left(\frac{6\pi}{Ed^3}\sum_{m=1}^{+\infty}\frac{\beta_m^2}{m^4}\right)\ell\,. \tag{4.32}$$

The relationship between problems (4.30) and (4.31) is stated in the following

Theorem 4.4.1. [8, Theorem 3.3] *Assume* (3.2) *and let $f \in L^2(\Omega)$ be a vertical load per unit of surface depending only on x, see* (4.11)-(4.12). *Let u^ℓ and ψ be respectively the unique solutions to* (4.30) *and* (4.31). *Then*

$$\lim_{\ell\to 0}\sup_{(x,y)\in\Omega}|u^\ell(x,y) - \psi(x)| = 0 \qquad \textit{and} \qquad \mathbb{E}_T(u^\ell) = \mathbb{E}_T(\psi) + o(\ell) \quad \textit{as } \ell \to 0\,, \tag{4.33}$$

where $\mathbb{E}_T(u^\ell)$ is given by (3.3) *and $\mathbb{E}_T(\psi)$ is given by* (4.32).

Let $u \in H^2_*(\Omega)$ be the solution of (3.8), see Theorem 4.3.1. By (4.13)-(4.14) we have

$$|A\cosh(my)| \le C\frac{\beta_m}{m^3} \quad \text{and} \quad Bmy\sinh(my) \le C\frac{\beta_m}{m^3}$$

for any $y \in (-\ell, \ell)$, $\ell \in (0,1)$ and $m \geq 1$ for some constant $C > 0$ depending on σ but independent of y, ℓ and m. Moreover we also have

$$\lim_{\ell \to 0} A(m,\ell) = \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4}, \quad \lim_{\ell \to 0} B(m,\ell) = \frac{\sigma}{2(1+\sigma)}\frac{\beta_m}{m^4} \quad \text{for any } m \in \mathbb{N}\,. \tag{4.34}$$

This implies that for any $N \in \mathbb{N}$

$$\begin{aligned}
&\limsup_{\ell \to 0} \sup_{(x,y)\in\Omega} \left| u(x,y) - \frac{1}{1-\sigma^2}\varphi(x) \right| \\
&\leq \limsup_{\ell \to 0} \sup_{y \in (-\ell,\ell)} \sum_{m=1}^{+\infty} \left| A\cosh(my) - \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4} + Bmy\sinh(my) \right| \\
&\leq \lim_{\ell \to 0} \sum_{m=1}^{N} \Bigg[\left| A - \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4} \right| \cosh(m\ell) + \\
&+ \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4}(\cosh(m\ell)-1) + Bm\ell\sinh(m\ell) \Bigg] + C \sum_{m=N+1}^{+\infty} \frac{\beta_m}{m^3} \leq C \sum_{m=N+1}^{+\infty} \frac{\beta_m}{m^3}\,.
\end{aligned}$$

Letting $N \to +\infty$, we obtain

$$\lim_{\ell \to 0} \sup_{(x,y)\in\Omega} \left| u(x,y) - \frac{1}{1-\sigma^2}\varphi(x) \right| = 0\,. \tag{4.35}$$

Let us now recall a well-known result about Fourier series which will be repeatedly used in the sequel.

Lemma 4.4.2. *Let $\{a_m\}$, $\{b_m\} \in \ell^2$ and let*

$$a(x) = \sum_{m=1}^{+\infty} a_m \sin(mx), \qquad b(x) = \sum_{m=1}^{+\infty} b_m \sin(mx),$$

Then $a, b \in L^2(0,\pi)$ and

$$\int_0^\pi a(x)b(x)dx = \frac{\pi}{2}\sum_{m=1}^{+\infty} a_m b_m\,, \qquad \int_0^\pi a(x)^2 dx = \frac{\pi}{2}\sum_{m=1}^{+\infty} a_m^2\,.$$

By differentiating the solution u we find

$$\begin{aligned}
u_{xx}(x,y) &= -\sum_{m=1}^{+\infty} \left[\frac{\beta_m}{m^2} + Am^2\cosh(my) + Bm^3\sinh(my) \right] \sin(mx)\,, \\
u_{yy}(x,y) &= -\sum_{m=1}^{+\infty} m^2 \left[(A+2B)\cosh(my) + Bmy\sinh(my) \right] \sin(mx)\,, \\
u_{xy}(x,y) &= -\sum_{m=1}^{+\infty} m^2 \left[(A+B)\cosh(my) + Bmy\sinh(my) \right] \cos(mx)\,,
\end{aligned}$$

and therefore

$$\Delta u(x,y) = \sum_{m=1}^{+\infty}\left[-\frac{\beta_m}{m^2} + 2Bm^2\cosh(my)\right]\sin(my)\,.$$

Then, by Lemma 4.4.2, we obtain

$$\begin{aligned}\int_\Omega |\Delta u|^2 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}\left[-\frac{\beta_m}{m^2}+2Bm^2\cosh(my)\right]^2 dy \\ &= \pi\sum_{m=1}^{+\infty}\left[\frac{\beta_m^2}{m^4}\ell + 2B^2m^4\ell - 4\frac{B\beta_m}{m}\sinh(m\ell) + B^2m^3\sinh(2m\ell)\right].\end{aligned} \tag{4.36}$$

Moreover, Lemma 4.4.2 also yields

$$\begin{aligned}\int_\Omega u_{xx}u_{yy} =& -\frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}\left[\beta_m + Am^4\cosh(my) + Bm^5\sinh(my)\right]\times \\ & \times\left[(A+2B)\cosh(my) + Bmy\sinh(my)\right]dy \\ =& -\pi\sum_{m=1}^{+\infty}\left[\frac{\beta_m}{m}[(A+B)\sinh(m\ell) + Bm\ell\cosh(m\ell)] + \right. \\ & + \frac{B(2A+B)m^3}{4}m\ell\cosh(2m\ell) + \\ & + \left(\frac{2A^2+2AB-B^2}{8} + \frac{B^2}{4}m^2\ell^2\right)m^3\sinh(2m\ell) + \\ & \left. + \frac{A(A+2B)}{2}m^4\ell - \frac{B^2}{6}m^6\ell^3\right]\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}\int_\Omega u_{x,y}^2 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}m^4\int_{-\ell}^{\ell}[(A+B)\sinh(my) + Bmy\cosh(my)]^2 dy \\ &= \pi\sum_{m=1}^{+\infty}m^3\left[\frac{2A^2+2AB+B^2}{8}\sinh(2m\ell) + \right. \\ &\quad + \frac{B(2A+B)}{4}m\ell\cosh(2m\ell) + \frac{B^2}{4}m^2\ell^2\sinh(2m\ell) + \\ &\quad \left. + \frac{B^2}{6}m^3\ell^3 - \frac{(A+B)^2}{2}m\ell\right].\end{aligned} \tag{4.38}$$

Finally, by (4.12) and a further application of Lemma 4.4.2,

$$\begin{aligned}\int_\Omega fu &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\beta_m\int_{-\ell}^{\ell}\left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right]dy \\ &= \pi\sum_{m=1}^{+\infty}\beta_m\left[\frac{\beta_m}{m^4}\ell + \frac{A-B}{m}\sinh(m\ell) + B\ell\cosh(m\ell)\right].\end{aligned} \tag{4.39}$$

Since u solves (3.8), the corresponding energy is given by (3.4) and hence collecting (4.36)-(4.39) we obtain

$$\mathbb{E}_T(u) = \pi \sum_{m=1}^{+\infty} \left\{ -\frac{\beta_m^2 \ell}{2m^4} + \frac{\sigma+1}{2} B^2 m^4 \ell - \frac{\sigma \beta_m (A+B)}{m} \sinh(m\ell) + \right.$$
$$+ \frac{B^2 m^3}{2} \sinh(2m\ell) - \sigma \beta_m B \ell \cosh(m\ell) + \frac{1-\sigma}{2} \Big[A(A+B) m^3 \sinh(2m\ell)$$
$$\left. + B^2 m^5 \ell^2 \sinh(2m\ell) + B(2A+B) m^4 \ell \cosh(2m\ell) \Big] \right\} =: \sum_{m=1}^{+\infty} a(m,\ell)\,.$$

With a direct computation one can see that by (4.13) and (4.14) we get

$$\frac{a(m,\ell)}{\ell} \le C \frac{\beta_m^2}{m^3} \qquad \text{for any } \ell \in (0,1) \text{ and } m \ge 1$$

for some constant $C > 0$ depending on σ but independent of ℓ and m.

This implies that for any $N \in \mathbb{N}$ we have

$$\sum_{m=1}^{N} \frac{a(m,\ell)}{\ell} - \sum_{m=N+1}^{+\infty} C \frac{\beta_m^2}{m^3} \le \sum_{m=1}^{+\infty} \frac{a(m,\ell)}{\ell} \le \sum_{m=1}^{N} \frac{a(m,\ell)}{\ell} + \sum_{m=N+1}^{+\infty} C \frac{\beta_m^2}{m^3}\,.$$

Letting $\ell \to 0$ we obtain

$$\sum_{m=1}^{N} \lim_{\ell \to 0} \left(\frac{a(m,\ell)}{\ell} \right) - \sum_{m=N+1}^{+\infty} C \frac{\beta_m^2}{m^3} \le \liminf_{\ell \to 0} \sum_{m=1}^{+\infty} \frac{a(m,\ell)}{\ell}$$
$$\le \limsup_{\ell \to 0} \sum_{m=1}^{+\infty} \frac{a(m,\ell)}{\ell} \le \sum_{m=1}^{N} \lim_{\ell \to 0} \left(\frac{a(m,\ell)}{\ell} \right) + \sum_{m=N+1}^{+\infty} C \frac{\beta_m^2}{m^3}\,.$$

Letting $N \to +\infty$, by (4.34), we deduce that

$$\lim_{\ell \to 0} \frac{\mathbb{E}_T(u)}{\ell} = \lim_{\ell \to 0} \sum_{m=1}^{+\infty} \frac{a(m,\ell)}{\ell} = \sum_{m=1}^{+\infty} \lim_{\ell \to 0} \frac{a(m,\ell)}{\ell} = -\frac{\pi}{2(1-\sigma^2)} \sum_{m=1}^{+\infty} \frac{\beta_m^2}{m^4}$$

and, in turn,

$$\mathbb{E}_T(u) = -\left(\frac{\pi}{2(1-\sigma^2)} \sum_{m^1}^{+\infty} \frac{\beta_m^2}{m^4} \right) \ell + o(\ell) \quad \text{as } \ell \to 0\,. \tag{4.40}$$

Consider now u^ℓ and ψ as in (4.30) and (4.31); recall that $u^\ell = \frac{12(1-\sigma)}{Ed^3} u$ where u solves (3.8) and that $\psi = \frac{12}{Ed^3}\varphi$. Then, from (4.35) we deduce the first of (4.33). Since u^ℓ solves (4.30), the corresponding energy is given by (3.3) and hence, by (4.40) and the identity $u^\ell = \frac{12(1-\sigma)}{Ed^3} u$, we obtain

$$\mathbb{E}_T(u^\ell) = \frac{12(1-\sigma^2)}{Ed^3} \int_\Omega \left[\frac{1}{2} (\Delta u)^2 + (\sigma - 1) \det(D^2 u) - fu \right] dxdy$$
$$= \frac{12(1-\sigma^2)}{Ed^3} \mathbb{E}_T(u) = \frac{12(1-\sigma^2)}{Ed^3} \left[-\left(\frac{\pi}{2(1-\sigma^2)} \sum_{m=1}^{+\infty} \frac{\beta_m^2}{m^4} \right) \ell + o(\ell) \right]$$

and the second of (4.33) follows.

4.4.2 Generalization to arbitrary load functions

Thanks to the explicit resolution provided by Theorem 4.3.4 we can prove a generalization of Theorem 4.4.1 dropping assumption (4.11), which requires the load function f to only depend on the x coordinate.

The first obstacle one encounters in attempting such a generalization is the fact that we require the height ℓ of the rectangular domain Ω to vary, so that if the load function f depends on the variable y then the definition of the function must also vary as so does its domain. While in Theorem 4.4.1 this problem is solved by imposing for f not to depend on y, in this generalization we have chosen to vary the function f along with its domain Ω by preserving its overall shape, more specifically its Fourier coefficients.

Consider the double Fourier expansion of f reported in (4.27). We formalize the assumption on the shape of f by requiring that the coefficients $\beta_{m,n}^{\cos}$ and $\beta_{m,n}^{\sin}$, for $m \geq 1$ and $n \geq 0$, do not depend on ℓ. At the end of this section we will present a situation where this assumption is indeed satisfied so as to make this assumption consistent.

It proves useful to distinguish the cases $n = 0$ and $n \geq 1$ in (4.27), since in the first case the frequency $\omega_0 = 0$ does not depend on ℓ while for the latter one has that ω_n tends to $+\infty$ as ℓ approaches 0. To emphasize this distinction we rename the coefficients $\beta_{m,0}^{\cos}$ to β_m, then the Fourier expansion of f becomes

$$f(x,y) = \sum_{m=1}^{+\infty} \beta_m \sin(mx) + \sum_{m,n=1}^{+\infty} \left[\beta_{m,n}^{\sin} \sin(\omega_n y) + \beta_{m,n}^{\cos} \cos(\omega_n y)\right] \sin(mx) \,. \tag{4.41}$$

When ℓ tends to zero, the oscillations of the load f along the y direction become irrelevant and the position of the plate is only affected by the mean value around which such oscillations occur. Following this observation we define $f^0 : [0,\pi] \to \mathbb{R}$ as the function

$$f^0(x) = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x,y) dy \,,$$

so that for each $x \in (0,\pi)$ the one dimensional load $f^0(x)$ describes the average value of the load function $f(x,y)$ along the y direction.

With this notation we obtain that the one dimensional load f^0 can be expressed as a Fourier series as follows

$$f^0(x) = \sum_{m=1}^{+\infty} \beta_m \sin(mx) \,,$$

As for Theorem 4.4.1 we consider the solution $u^\ell = u^\ell(x,y)$ of problem (4.30), which represent the vertical displacement of the walkway according to a partially hinged plate model, inclusive of the constant $\frac{Ed^3}{12(1-\sigma^2)}$, and confront it with the solution $\psi = \psi(x)$ of (4.31), where the role of f is played by the one dimensional load f^0, which is the equation one obtains when studying a parallepiped-shaped beam clamped at the two ends.

By taking account of the Fourier expansion of f^0, the solution ψ to the beam equation (4.31) with $f = f^0$ can be computed as $\psi = \frac{12}{Ed^3}\phi$ where ϕ is defined as

$$\phi(x) = \sum_{m=1}^{+\infty} \frac{\beta_m}{m^4} \sin(mx) \,,$$

the energy associated to solution ψ is then given by (4.32).

We then prove the following theorem, which extends the result of Theorem 4.4.1 to arbitrary load functions f which may depend on y as well as on x.

Theorem 4.4.3. *Assume (3.2), let $f \in L^2(\Omega)$ be a vertical load per unit of surface admitting a Fourier expansion as in (4.41), with coefficients not depending on ℓ. Let u^ℓ and ψ be respectively as in (4.30) and (4.31). Then*

$$\lim_{\ell\to 0} \sup_{(x,y)\in\Omega} |u^\ell(x,y) - \psi(x)| = 0 \tag{4.42}$$

and

$$\mathbb{E}_T(u^\ell) = \mathbb{E}_T(\psi) + o(\ell) \quad \text{as } \ell \to 0\,, \tag{4.43}$$

where $\mathbb{E}_T(u^\ell)$ is given by (3.3) and $\mathbb{E}_T(\psi)$ is given by (4.32).

Proof. The proof follows the same lines of the one proposed in [8], that we reported in the proof of Theorem 4.4.1, but with some key differences which allow us to generalize that result. We start by proving the first part of the statement, namely the validity of (4.42).

The solution u^ℓ of (4.30) coincides with $\frac{12(1-\sigma^2)}{Ed^3}u$, where u is the solution of (3.8) which we provided explicitly in Theorem 4.3.4. The difference $u^\ell - \psi$ is then proportional to $u - \frac{1}{1-\sigma^2}\phi$, indeed we have

$$u^\ell - \psi = \frac{12(1-\sigma^2)}{Ed^3}u - \frac{12}{Ed^3}\phi = \frac{12(1-\sigma^2)}{Ed^3}\left(u - \frac{1}{1-\sigma^2}\phi\right)\,,$$

so that we may focus on the latter, which we write explicitly as

$$\begin{aligned} u - \frac{1}{1-\sigma^2}\phi = \\ & \sum_{m=1}^{+\infty}\left[\frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} + A\cosh(my) + Bmy\sinh(my)\right]\sin(mx) + \\ & + \sum_{m,n=1}^{+\infty}\left[\frac{\beta^{\sin}_{m,n}\sin(\omega_n y) + \beta^{\cos}_{m,n}\cos(\omega_n y)}{(\omega_n^2 + m^2)^2} + \right. \\ & \left. + A\cosh(my) + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)\right]\sin(mx)\,, \end{aligned}$$

where A, B, C and D are defined in (4.28) and, as suggested above, we have separated the terms in the sum for which $n = 0$ from the ones where $n \geq 1$.

In order to study the behaviour of this sum when ℓ tends to 0, we note that the following inequalities are satisfied for a sufficiently large $\mathcal{C} > 0$ depending on σ but not on ℓ, m and n and for all $\ell \in (0,1)$, $m \geq 1$, $n \geq 0$, $y \in (-\ell,\ell)$.

$$\begin{aligned} |A\cosh(my)| &\leq \mathcal{C}\frac{\left|\beta^{\cos}_{m,n}\right|}{m(m^2+\omega_n^2)}\,, & |Bmy\sinh(my)| &\leq \mathcal{C}\frac{\left|\beta^{\cos}_{m,n}\right|}{m(m^2+\omega_n^2)}\,, \\ |C\sinh(my)| &\leq \mathcal{C}\frac{\left|\beta^{\cos}_{m,n}\right|\omega_n}{m^2(m^2+\omega_n^2)}\,, & |Dmy\cosh(my)| &\leq \mathcal{C}\frac{\left|\beta^{\sin}_{m,n}\right|\omega_n}{m^2(m^2+\omega_n^2)}\,. \end{aligned} \tag{4.44}$$

Moreover one can check that the following limits hold for all $m \geq 1$, $n \geq 0$.

$$\lim_{\ell\to 0} A = \begin{cases} \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.45}$$

$$\lim_{\ell\to 0} B = \begin{cases} \frac{\sigma}{2(1+\sigma)}\frac{\beta_m}{m^4} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.46}$$

$$\lim_{\ell\to 0} C = \lim_{\ell\to 0} D = 0\,. \tag{4.47}$$

We give a proof only for those statements concerning coefficient A, as all the others can be accomplished in a very similar manner. Notice that in the formula which defines A in (4.28) one can isolate an extensive sub-expression which depends only on σ and the product ℓm. To be able to express the following computations in a more concise manner we will associate this expression with a function $g = g(\ell m) : \mathbb{R} \to \mathbb{R}$, which we define as

$$g(\ell m) = \frac{(1+\sigma)\sinh(\ell m) - (1-\sigma)\ell m\cosh(\ell m)}{(3+\sigma)\sinh(\ell m)\cosh(\ell m) - (1-\sigma)\ell m}\,,$$

so that the definition of A can be rewritten as

$$A = \frac{\beta^{\cos}_{m,n}(-1)^n(\sigma m^2 + \omega_n^2)}{m(1-\sigma)(m^2+\omega_n^2)^2}\frac{g(\ell m)}{m}\,. \tag{4.48}$$

In this form it is clear that in order to prove the first inequality of (4.44) it is sufficient to show that $\left|\frac{1}{m}g(\ell m)\cosh(\ell m)\right| \leq \mathcal{C}$ for some constant $\mathcal{C} > 0$ depending only on σ. The reason for this is that the first fraction in (4.48) can be readily bounded as in

$$\left|\frac{\beta^{\cos}_{m,n}(-1)^n(\sigma m^2 + \omega_n^2)}{m(1-\sigma)(m^2+\omega_n^2)^2}\right| \leq \frac{1}{1-\sigma}\frac{\left|\beta^{\cos}_{m,n}\right|}{m(m^2+\omega_n^2)}\,,$$

and $\frac{1}{1-\sigma} > 0$ since we are assuming (3.2). Considering the fact that $m \geq 1$ and $\ell \in (0,1)$ so that $\frac{1}{m} \leq 2\frac{1}{\sqrt{1+(\ell m)^2}}$, we are lead to study the absolute value of the function of one variable given by $\ell m \mapsto \frac{1}{\sqrt{1+(\ell m)^2}}g(\ell m)\cosh(\ell m)$. For any value of σ This map is smooth over $\mathbb{R}$ and presents an horizontal asymptote for $\ell m \to +\infty$, namely $-\frac{1-\sigma}{3+\sigma}$, as seen in Figure 4.11, thus it admits a maximum $\mathcal{C}$ over $(0, +\infty)$ which is the constant we were looking for.

The expression (4.48) for A also allows to prove the limit (4.45). Indeed $g(\ell m)$ can be extended by continuity over $\mathbb{R}$ and for $\ell = 0$ it assumes the value $g(0) = \frac{\sigma}{1+\sigma}$. If $n = 0$ then $\omega_n = 0$ and $\beta^{\cos}_{m,n} = \beta_m$ so that letting ℓ tend to 0 in (4.48) we obtain

$$\lim_{\ell\to 0} A = \frac{\sigma\beta_m}{m^4(1-\sigma)}\frac{\sigma}{1+\sigma} = \frac{\sigma^2}{1-\sigma^2}\frac{\beta_m}{m^4}\,.$$

On the other hand if we have that $n > 0$ then ω_n diverges to $+\infty$ as ℓ tends to 0, consequently in this case it holds that $\lim_{\ell\to 0} A = 0$.

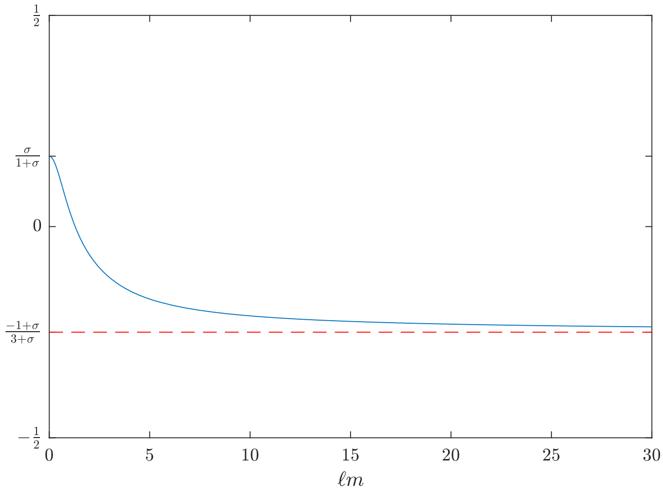

Figure 4.11. The map $\ell m \to \frac{1}{\sqrt{1+(\ell m)^2}} g(\ell m)\cosh(\ell m)$ and its horizontal asymptote $-\frac{1-\sigma}{3+\sigma}$, for $\sigma = 0.2$.

In proving (4.42) we are lead to the following computations

$$
\begin{aligned}
&\lim_{\ell\to 0}\sup_{(x,y)\in\Omega}\left|u - \frac{1}{1-\sigma^2}\phi\right| = \\
&\quad = \lim_{\ell\to 0}\sup_{(x,y)\in\Omega}\left|\sum_{m=1}^{+\infty}\left[A\cosh(my) - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} + Bmy\sinh(my)\right]\sin(mx) + \right. \\
&\qquad + \sum_{m,n=1}^{+\infty}\left[\frac{\beta^{\sin}_{m,n}\sin(\omega_n y) + \beta^{\cos}_{m,n}\cos(\omega_n y)}{(\omega_n^2+m^2)^2} + \right. \\
&\qquad \left.\left. + A\cosh(my) + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)\right]\sin(mx)\right| \\
&\quad \le \lim_{\ell\to 0}\sum_{m=1}^{+\infty}\left[\left|A\cosh(m\ell) - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2}\right| + |Bm\ell\sinh(m\ell)|\right] + \\
&\qquad + \lim_{\ell\to 0}\sum_{m,n=1}^{+\infty}\frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(\omega_n^2+m^2)^2} + \\
&\qquad + \lim_{\ell\to 0}\sum_{m,n=1}^{+\infty}\left[|A\cosh(m\ell)| + |Bm\ell\sinh(m\ell)| + |C\sinh(m\ell)| + |Dm\ell\cosh(m\ell)|\right],
\end{aligned}
$$

where we have used the inequalities $\sin(mx) \le 1$, $|\sinh(my)| \le \sinh(m\ell)$ and $|\cosh(my)| \le \cosh(m\ell)$.

We analyze each of the three resulting limits, showing in turn that they all converge to zero. To this aim we state the following Lemma, which is simply a special case of the well known dominated convergence theorem for discrete measure spaces.

Lemma 4.4.4. *Let M be a countable set and consider a family of sequences $a^\ell = \{a^\ell_m\}_{m\in M}$ depending on $\ell \in (0,1)$ and converging pointwise to a sequence a so that for every $m \in M$ one has $\lim_{\ell\to 0} a^\ell_m = a_m$. Suppose that there is a sequence $b = \{b_m\}_{m\in M}$, independent on ℓ, such that every a^ℓ is dominated by b, that is,*

$$\left|a^\ell_m\right| \le |b_m| \;, \qquad \forall m \in M \,, \, \forall \ell \in (0,1) \,,$$

and assume that b has finite sum, i.e., $\sum_{m\in M} |b_m| < +\infty$.

Then the sequences a and a^ℓ have finite sum for every $\ell \in (0,1)$ and it holds

$$\lim_{\ell\to 0} \sum_{m\in M} a^\ell_m = \sum_{m\in M} a_m \,,$$

that is, the limit of the sum of a^ℓ coincides with the sum of its pointwise limit.

We begin with the first therm which is

$$\lim_{\ell\to 0} \sum_{m=1}^{+\infty} \left[\left| A\cosh(m\ell) - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} \right| + |Bm\ell \sinh(m\ell)| \right] . \tag{4.49}$$

Thanks to inequalities (4.44) we can dominate the m-th term of the sum, for every $m \ge 1$ with

$$\left| A\cosh(m\ell) - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} \right| + |Bm\ell \sinh(m\ell)| \le \mathcal{C}\frac{|\beta_m|}{m^3}$$

for some sufficiently large constant $\mathcal{C} > 0$ not depending on ℓ or m. Since we assumed that $f \in L^2(\Omega)$ it follows that the sequence $\{\beta_m\}_{m\ge 1}$ lies in $\ell^2(\mathbb{N})$. By Schwarz's inequality, we obtain that the sum $\sum_{m=1}^{+\infty} \mathcal{C}\frac{|\beta_m|}{m^3}$ is finite, so that the hypothesis of Lemma 4.4.4 is satisfied and we may switch the order of the limit and the summation in (4.49) obtaining

$$\begin{aligned} &\sum_{m=1}^{+\infty} \lim_{\ell\to 0} \left[\left| A\cosh(m\ell) - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} \right| + |Bm\ell \sinh(m\ell)| \right] \\ &\le \sum_{m=1}^{+\infty} \lim_{\ell\to 0} \left[\left| A - \frac{\beta_m}{m^4}\frac{\sigma^2}{1-\sigma^2} \right| \cosh(m\ell) + \frac{|\beta_m|}{m^4}\frac{\sigma^2}{1-\sigma^2}\big(\cosh(m\ell) - 1\big) + \right. \\ &\left. + |Bm\ell \sinh(m\ell)| \right] = \sum_{m=1}^{+\infty} 0 = 0 \,. \end{aligned}$$

For the second term, which is

$$\lim_{\ell\to 0} \sum_{m,n=1}^{+\infty} \frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(\omega_n^2 + m^2)^2}$$

we give the following bound: notice that $\omega_n = \pi n/\ell \geq n$ since we consider $\ell < 1 < \pi$. This entails that for every $\ell \in (0,1)$ and for every m, $n \in \mathbb{N}$ it holds

$$\frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(\omega_n^2 + m^2)^2} \leq \frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(n^2 + m^2)^2}\,.$$

The bounding series can be interpreted as the scalar product of the two sequences $\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|$ and $(m^2+n^2)^{-2}$, which both belong to $\ell^2(\mathbb{N}\times\mathbb{N})$, thus the sum converges and Lemma 4.4.4 can be applied. Observing that for every $m, n \geq 1$ we have that $\lim_{\ell\to 0} \omega_n = +\infty$ it follows that

$$\lim_{\ell\to 0} \sum_{m,n=1}^{+\infty} \frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(\omega_n^2 + m^2)^2} = \sum_{m,n=1}^{+\infty} \lim_{\ell\to 0} \frac{\left|\beta^{\sin}_{m,n}\right| + \left|\beta^{\cos}_{m,n}\right|}{(\omega_n^2 + m^2)^2} = 0\,.$$

Finally, the third term which is

$$\lim_{\ell\to 0} \sum_{m,n=1}^{+\infty} \Big[|A\cosh(m\ell)| + |Bm\ell\sinh(m\ell)| + |C\sinh(m\ell)| + |Dm\ell\cosh(m\ell)| \Big]$$

is dealt with in a similar manner to the preceding ones, so that making use of inequalities (4.44) and Lemma 4.4.4 one can readily prove that this too converges to zero. This concludes the first part of the proof

We pass now to prove (4.43). Since u is a solution of (3.8) its associated energy is defined in equation (3.4). Remembering the definition of the norm $\|\cdot\|_{H^2_*}$ provided in (4.2) the energy $\mathbb{E}_T(u)$ can be rewritten as

$$\mathbb{E}_T(u) = \frac{1}{2}\,\|u\|^2_{H^2_*} - (u,f)_{L^2}\,.$$

This formulation is useful as it allows one to exploit known properties of norms and scalar products, especially in relation to sum of functions. In this optic, it proves beneficial to separate the solution u in two additive pieces u_1, u_2 and similarly to split the load f as the sum of two terms f_1 and f_2. Specifically we give the following definitions

$$u_1(x,y) = \sum_{m=1}^{+\infty} \left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right]\sin(mx) \tag{4.50}$$

$$u_2(x,y) = \sum_{m,n=1}^{+\infty} \left[\frac{\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)}{\left(m^2+\omega_n^2\right)^2} + A\cosh(my) + \right. \tag{4.51}$$
$$\left. + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)\right]\sin(mx)$$

$$f_1(x,y) = \sum_{m=1}^{+\infty} \beta_m \sin(mx) \tag{4.52}$$

$$f_2(x,y) = \sum_{m,n=1}^{+\infty} \left[\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)\right]\sin(mx)\,, \tag{4.53}$$

in this way we have the decompositions $u = u_1 + u_2$ and $f = f_1 + f_2$. Substituting these into (4.4.2) we obtain

$$\begin{aligned} \mathbb{E}_T(u) &= \frac{1}{2} \|u_1 + u_2\|^2_{H^2_*} - (u_1 + u_2, f)_{L^2} \\ &= \frac{1}{2} \|u_1\|^2_{H^2_*} - (u_1, f_1)_{L^2} - (u_1, f_2)_{L^2} + \qquad (4.54) \\ &\quad + \frac{1}{2} \|u_2\|^2_{H^2_*} + (u_1, u_2)_{H^2_*} - (u_2, f)_{L^2} \qquad (4.55) \end{aligned}$$

and we shall analyze each of the resulting terms individually.

First we prove that

$$\frac{1}{2} \|u_1\|^2_{H^2_*} - (u_1, f_1)_{L^2} = \frac{Ed^3}{12(1-\sigma^2)} \mathbb{E}_T(\psi) + o(\ell) \quad \text{as } \ell \to 0\,. \qquad (4.56)$$

To compute this we first evaluate the Hessian matrix of u_1, and using the notation $u_{1,xx} = \frac{\partial^2}{\partial x^2} u_1$ and similarly for the other derivatives we obtain the following

$$\begin{aligned} u_{1,xx}(x,y) &= -\sum_{m=1}^{+\infty} m^2 \left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right] \sin(mx) \\ u_{1,xy}(x,y) &= \sum_{m=1}^{+\infty} m^2 \left[(A+B)\sinh(my) + Bmy\cosh(my)\right] \cos(mx) \\ u_{1,yy}(x,y) &= \sum_{m=1}^{+\infty} m^2 \left[(A+2B)\cosh(my) + Bmy\sinh(my)\right] \sin(mx)\,. \end{aligned}$$

From this we obtain the laplacian Δu_1 which is

$$\Delta u_1(x,y) = \sum_{m=1}^{+\infty} \left[-\frac{\beta_m}{m^2} + 2Bm^2\cosh(my)\right] \sin(my)\,.$$

Since all of this results are expressed in the form of single Fourier series relative to the direction x, we can apply Lemma 4.4.2 to compute the following integrals, which appear in $\|u_1\|_{H^2_*}$

$$\begin{aligned}
\int_\Omega (\Delta u_1)^2 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}\left[-\frac{\beta_m}{m^2}+2Bm^2\cosh(my)\right]^2 \\
&= \pi\sum_{m=1}^{+\infty}\left[\frac{\beta_m^2}{m^4}\ell+2B^2m^4\ell-4\frac{B\beta_m}{m}\sinh(m\ell)+B^2m^3\sinh(2m\ell)\right] \\
\int_\Omega (u_{1,xy})^2 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}m^4\int_{-\ell}^{\ell}\left[(A+B)\sinh(my)+Bmy\cosh(my)\right]^2 dy \\
&= \pi\sum_{m=1}^{+\infty}m^3\Bigg[\frac{2A^2+2AB+B^2}{8}\sinh(2m\ell)+ \\
&\quad+\frac{B(2A+B)}{4}m\ell\cosh(2m\ell)+\frac{B^2}{4}m^2\ell^2\sinh(2m\ell)+ \\
&\quad+\frac{B^2}{6}m^3\ell^3-\frac{(A+B)^2}{2}m\ell\Bigg] \\
\int_\Omega u_{2,xx}\,u_{2,yy} &= -\frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}\Big[\beta_m+Am^4\cosh(my)+Bm^5\sinh(my)\Big]\times \\
&\quad\times\left[(A+2B)\cosh(my)+Bmy\sinh(my)\right]dy \\
&= -\pi\sum_{m=1}^{+\infty}\Bigg[\frac{\beta_m}{m}\left[(A+B)\sinh(m\ell)+Bm\ell\cosh(m\ell)\right]+ \\
&\quad+\frac{B(2A+B)m^3}{4}m\ell\cosh(2m\ell)+ \\
&\quad+\left(\frac{2A^2+2AB-B^2}{8}+\frac{B^2}{4}m^2\ell^2\right)m^3\sinh(2m\ell)+ \\
&\quad+\frac{A(A+2B)}{2}m^4\ell-\frac{B^2}{6}m^6\ell^3\Bigg].
\end{aligned}$$

Moreover, the definition of f_1 provided in (4.52) is also in the form of a trigonometric series so that applying again Lemma 4.4.2 we can compute the scalar product $(u_1, f_1)_{L^2}$ as

$$\begin{aligned}
(u_1,f_1)_{L^2} = \int_\Omega f_1u_1 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\beta_m\int_{-\ell}^{\ell}\left[\frac{\beta_m}{m^4}+A\cosh(my)+Bmy\sinh(my)\right]dy \\
&= \pi\sum_{m=1}^{+\infty}\beta_m\left[\frac{\beta_m}{m^4}\ell+\frac{A-B}{m}\sinh(m\ell)+B\ell\cosh(m\ell)\right].
\end{aligned}$$

Collecting these results together we obtain that

$$\frac{1}{2}\|u_1\|_{H^2_*} - (u_1, f_1)_{L^2} = \pi \sum_{m=1}^{+\infty} \left\{ -\frac{\beta_m^2 \ell}{2m^4} + \frac{\sigma+1}{2} B^2 m^4 \ell - \frac{\sigma \beta_m (A+B)}{m} \sinh(m\ell) + \right.$$
$$+ \frac{B^2 m^3}{2} \sinh(2m\ell) - \sigma \beta_m B \ell \cosh(m\ell) + \frac{1-\sigma}{2} \Big[A(A+B) m^3 \sinh(2m\ell)$$
$$\left. + B^2 m^5 \ell^2 \sinh(2m\ell) + B(2A+B) m^4 \ell \cosh(2m\ell) \Big] \right\} =: \sum_{m=1}^{+\infty} a(m,\ell)\,.$$

Using the same technique applied extensively in the first part of the proof we give a bound on the m-th term of the series thanks to the inequalities (4.44), resulting in

$$\frac{|a(m,\ell)|}{\ell} \le \mathcal{C} \frac{|\beta_m|}{m^2}\,, \qquad \text{for any } \ell \in (0,1) \text{ and } m \ge 1$$

where the constant $\mathcal{C}$ depends only on σ. Exploiting again the fact that β_m forms a sequence in $\ell^2(\mathbb{N})$ and so the bounding series is convergent we can then apply Lemma 4.4.4 to justify the switching of the limit from applying to the whole series to being term by term in the following equation chain

$$\lim_{\ell\to 0} \frac{1}{\ell} \left[\frac{1}{2} \|u_1\|_{H^2_*} - (u_1, f_1)_{L^2} \right] = \lim_{\ell\to 0} \sum_{m=1}^{+\infty} \frac{a(m,\ell)}{\ell} = \sum_{m=1}^{+\infty} \lim_{\ell\to 0} \frac{a(m,\ell)}{\ell}$$
$$= -\frac{\pi}{2(1-\sigma^2)} \sum_{m=1}^{+\infty} \frac{\beta_m^2}{m^4}\,,$$

where the last equality is obtained by direct computation taking advantage of the limits of the coefficients A and B given in (4.45)–(4.46). Recalling that the energy $\mathbb{E}_T(\psi)$ is defined by (4.32) we see that we have proven (4.56), which is what we wanted.

Since the energy of u^ℓ, defined by (3.3), is equal to $\mathbb{E}_T(u^\ell) = \frac{12(1-\sigma^2)}{Ed^3}\mathbb{E}_T(u)$ and the energy $\mathbb{E}_T(u)$ is decomposed as in (4.54), once we have that (4.56) is true then the only thing left in order to prove (4.43) is to show that $(u_1, f_2)_{L^2}$, $\|u_2\|^2_{H^2_*}$, $(u_1, u_2)_{H^2_*}$ and $(u_2, f)_{L^2}$ are all $o(\ell)$ for $\ell \to 0$. We will treat this asymptotic equalities one at a time.

1. We first prove that $(u_1, f_2)_{L^2} = o(\ell)$ for $\ell \to 0$. Recall that u_1 is defined in (4.50) and f_2 is defined in (4.53) so that by Lemma 4.4.2 their L^2 scalar product is equivalent to the $\ell^2(\mathbb{N})$ scalar product of their Fourier coefficients, resulting in the following

equality

$$
\begin{aligned}
\frac{(u_1, f_2)_{L^2}}{\ell} &= \frac{1}{\ell}\int_\Omega f_2\, u_1 = \\
&= \frac{\pi}{2\ell}\sum_{m,n=1}^{+\infty}\int_{-\ell}^{\ell}\left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right]\times \\
&\quad\times\left[\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)\right] dy \\
&= \pi\sum_{m,n=1}^{+\infty}\frac{m\beta^{\cos}_{m,n}(-1)^n}{\ell(m^2+\omega_n^2)^2}\Big[(m^2+\omega_n^2)A\sinh(\ell m) + \qquad (4.57)\\
&\quad - (m^2-\omega_n^2)B\sinh(\ell m) + (m^2+\omega_n^2)B\ell m\cosh(\ell m)\Big]\,.
\end{aligned}
$$

Notice that in the equation above the coefficients A and B have their implicit index n fixed to zero and not varying from 1 to $+\infty$ as the enclosing sum would imply. Keeping this in mind, we can provide a bound for a generic term of the sum making use of inequalities (4.44), which implies that

$$
|A\cosh(m\ell)|\,,|Bm\ell\sinh(m\ell)| \le \mathcal{C}\frac{|\beta_m|}{m^3}
$$

and in turn

$$
\begin{aligned}
&\Bigg|\frac{m\beta^{\cos}_{m,n}(-1)^n}{\ell(m^2+\omega_n^2)^2}\Big[(m^2+\omega_n^2)A\sinh(\ell m) + \\
&\qquad - (m^2-\omega_n^2)B\sinh(\ell m) + (m^2+\omega_n^2)B\ell m\cosh(\ell m)\Big]\Bigg| \\
&\le \frac{m\left|\beta^{\cos}_{m,n}\right|}{\ell(m^2+\omega_n^2)}\Big[\,|A\sinh(\ell m)| + |B\sinh(\ell m)| + |B\ell m\cosh(\ell m)|\,\Big] \\
&\le \mathcal{C}\frac{\left|\beta^{\cos}_{m,n}\right|}{\ell(m^2+\omega_n^2)}\frac{|\beta_m|}{m^2} \\
&\le \ell\mathcal{C}\left|\beta^{\cos}_{m,n}\right|\frac{|\beta_m|}{(mn)^2} \qquad (4.58)\\
&\le \mathcal{C}\left|\beta^{\cos}_{m,n}\right|\frac{|\beta_m|}{(mn)^2}\,,
\end{aligned}
$$

where step (4.58) is justified since we consider $\ell \in (0,1)$ so that it holds $\ell(m^2+\omega_n^2) \ge \ell\omega_n^2 = (\pi n)^2/\ell$. The obtained bounding sequence, when summed over m ans n ranging over the positive naturals, gives rise to a convergent series as it coincides with the scalar product of the sequence $\left\{\left|\beta^{\cos}_{m,n}\right|\right\}_{m,n=1}^{+\infty} \in \ell^2(\mathbb{N}^2)$ with the tensor product of the two sequences $\{|\beta_m|\}_{m=1}^{+\infty}$ and $\{1/(mn)^2\}_{n=1}^{+\infty}$, which both lie in $\ell^2(\mathbb{N})$.

This proves, according to Lemma 4.4.4, that we can compute the value of the limit $\lim_{\ell\to 0} (u_1, f_2)_{L^2}/\ell$ as the limit of the series (4.57), taken term by term. Reusing inequality (4.58), it is easy to see that this limit is vanishing so that indeed $(u_1, f_2)_{L^2} = o(\ell)$.

2. Next we show that $\|u_2\|^2_{H^2_*} = o(\ell)$ for $\ell \to 0$. After having separated u into the two terms u_1 and u_2, we further split the latter into two new terms u_3 and u_4 which we define as

$$
\begin{aligned}
u_3(x,y) &= \sum_{m,n=1}^{+\infty} \frac{\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)}{(m^2+\omega_n^2)^2}\sin(mx) \\
u_4(x,y) &= \sum_{m,n=1}^{+\infty} \left[A\cosh(my) + Bmy\sinh(my) + \right. \\
&\quad \left. + C\sinh(my) + Dmy\cosh(my)\right]\sin(mx)\,.
\end{aligned}
$$

By triangular inequality follows that to have that, for $\ell \to 0$, it holds $\|u_2\|^2_{H^2_*} = \|u_3 + u_4\|^2_{H^2_*} = o(\ell)$ therefore it is sufficient to provide that $\|u_3\|^2_{H^2_*}$ and $\|u_3\|^2_{H^2_*}$ are individually $o(\ell)$.

Consider u_3. It is easy to check that its second order partial derivatives are given by

$$
\begin{aligned}
u_{3,xx}(x,y) &= -\sum_{m,n=1}^{+\infty} m^2 \frac{\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)}{(m^2+\omega_n^2)^2}\sin(mx) \\
u_{3,xy}(x,y) &= \sum_{m,n=1}^{+\infty} m\omega_n \frac{\beta^{\sin}_{m,n}\cos(\omega_n y) - \beta^{\cos}_{m,n}\sin(\omega_n y)}{(m^2+\omega_n^2)^2}\cos(mx) \\
u_{3,yy}(x,y) &= -\sum_{m,n=1}^{+\infty} \omega_n^2 \frac{\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)}{(m^2+\omega_n^2)^2}\sin(mx)\,.
\end{aligned}
$$

Consequently we can express the laplacian Δu_3 as

$$
\Delta u_3(x,y) = -\sum_{m,n=1}^{+\infty} \frac{\beta^{\cos}_{m,n}\cos(\omega_n y) + \beta^{\sin}_{m,n}\sin(\omega_n y)}{(m^2+\omega_n^2)}\sin(mx)\,.
$$

Thanks to the orthogonality of the two dimensional Fourier base (4.26) the $L^2(\Omega)$ scalar product of the above series of functions can be rewritten as an $\ell^2(\mathbb{N}^2)$ scalar product, resulting in a 2-D equivalent of Lemma (4.4.2), which justifies the following computations:

$$\int_\Omega (\Delta u_3)^2 = \frac{\pi\ell}{2} \sum_{m,n=1}^{+\infty} \frac{\left|\beta^{\cos}_{m,n}\right|^2 + \left|\beta^{\sin}_{m,n}\right|^2}{(m^2+\omega_n^2)^2} \tag{4.59}$$

$$\int_\Omega (u_{3,xy})^2 = \frac{\pi\ell}{2} \sum_{m,n=1}^{+\infty} (m\omega_n)^2 \frac{\left|\beta^{\cos}_{m,n}\right|^2 + \left|\beta^{\sin}_{m,n}\right|^2}{(m^2+\omega_n^2)^4}$$

$$\int_\Omega u_{3,xx}\, u_{3,yy} = \frac{\pi\ell}{2} \sum_{m,n=1}^{+\infty} (m\omega_n)^2 \frac{\left|\beta^{\cos}_{m,n}\right|^2 + \left|\beta^{\sin}_{m,n}\right|^2}{(m^2+\omega_n^2)^4} = \int_\Omega (u_{3,xy})^2 .$$

From the fact that these last two integrals give the same result we deduce that the $H^2_*(\Omega)$ norm of u_3 coincides with the $L^2(\Omega)$ norm of its laplacian (4.59). To show that this value is $o(\ell)$ we note that for every value of $n \geq 1$ it holds $(m^2+\omega_n^2)^2 \geq \omega_n^2 = (\pi n)^2/\ell^2 \geq \ell^{-2}$, so that we have

$$\begin{aligned}\|u_3\|^2_{H^2_*} &= \frac{\pi\ell}{2} \sum_{m,n=1}^{+\infty} \frac{\left|\beta^{\cos}_{m,n}\right|^2 + \left|\beta^{\sin}_{m,n}\right|^2}{(m^2+\omega_n^2)^2} \\ &\leq \frac{\pi\ell^3}{2} \sum_{m,n=1}^{+\infty} \left[\left|\beta^{\cos}_{m,n}\right|^2 + \left|\beta^{\sin}_{m,n}\right|^2\right] \\ &\leq \mathcal{C}\ell^3 = o(\ell)\,, \qquad \text{for } \ell \to 0\,,\end{aligned}$$

where the fact that the last sum is convergent is due to the fact that both $\beta^{\cos}_{m,n}$ and $\beta^{\sin}_{m,n}$ define sequences lying in $\ell^2(\mathbb{N}^2)$.

The proof that $\|u_2\|^2_{H^2_*} = o(\ell)$ for $\ell \to 0$ can be carried out in a similar way, by first applying inequalities (4.44) to provide a bound to its derivatives.

3. The last thing we need to prove is that the scaler products $(u_1, u_2)_{H^2_*}$ and $(u_2, f)_{L^2}$ are both $o(\ell)$ for $\ell \to 0$. Since we already have the following asymptotic inequalities

$$\|u_1\|_{H^2_*} = O(\sqrt{\ell})\,, \quad \|u_2\|_{H^2_*} = o(\sqrt{\ell})\,, \quad \|f\|_{L^2} = O(\sqrt{\ell})\,, \qquad \text{for } \ell \to 0\,,$$

we can obtain the result as a consequence of Schwartz inequality combined with the inequality $\|v\|_{L^2} \leq \mathcal{C}\,\|v\|_{H^2_*}$ obtained in Section 4.1and valid for any function $v \in H^2_*(\Omega)$ and where constant $\mathcal{C}$ is fixed, as shown below

$$\left|(u_1,u_2)_{H^2_*}\right| \leq \|u_1\|_{H^2_*}\,\|u_2\|_{H^2_*} = O(\sqrt{\ell})o(\sqrt{\ell}) = o(\ell)$$

$$|(u_2,f)_{L^2}| \leq \|u_2\|_{L^2}\,\|f\|_{L^2} \leq \mathcal{C}\,\|u_2\|_{H^2_*}\,\|f\|_{L^2} = o(\sqrt{\ell})O(\sqrt{\ell}) = o(\ell)$$

This completes the proof of Theorem 4.4.3.

4.5 Proof of Theorem 4.3.4

As we mentioned, the only remaining step after all the results shown in Section 4.3.4 is to prove the convergence of the series (4.29) in the space $H^2_*(\Omega)$.

We do this by considering separately the two series

$$u_1(x,y) = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\left[\frac{\beta^{\sin}_{m,n}\sin(\omega_n y) + \beta^{\cos}_{m,n}\cos(\omega_n y)}{(\omega_n^2+m^2)^2}\right]\sin(mx) \tag{4.60}$$

and

$$u_2 = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}[A\cosh(my) + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)]\sin(mx)\,, \tag{4.61}$$

whose sum gives back (4.29).

We begin by analyzing the first series (4.60). As noted in Section 4.1, for a function $v \in H^2_*(\Omega)$ the norm $\|v\|_{H^2_*}$ is equivalent to $\|v\|_{H^2_0} = \left\|D^2 v\right\|_{L^2}$, so that we may check individually the convergence in $L^2(\Omega)$ of the series of the second order partial derivatives of (4.60), taken term by term. Consider the derivative $u_{1,xx} = \frac{\partial^2}{\partial x^2}u_1$ which evaluates to

$$u_{1,xx} = -\sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}\left[\frac{m^2\beta^{\sin}_{m,n}}{(\omega_n^2+m^2)^2}\sin(\omega_n y) + \frac{m^2\beta^{\cos}_{m,n}}{(\omega_n^2+m^2)^2}\cos(\omega_n y)\right]\sin(mx)\,. \tag{4.62}$$

This can bee seen as an infinite linear combination of members of the orthogonal basis (4.26) of $L^2(\Omega)$. Thus we can deduce that the series is convergent by noticing that the coefficients $\frac{m^2\beta^{\sin}_{m,n}}{(\omega_n^2+m^2)^2}$ and $\frac{m^2\beta^{\cos}_{m,n}}{(\omega_n^2+m^2)^2}$ both lay in $\ell^2(\mathbb{N}^2)$, which is easily derived from the fact that so do the sequences $\{\beta^{\sin}_{m,n}\}_{m,n\in\mathbb{N}}$ and $\{\beta^{\cos}_{m,n}\}_{m,n\in\mathbb{N}}$ and by the bound $\frac{m^2}{(\omega_n^2+m^2)^2} \le 1$. The derivatives $u_{1,xy}$ and $u_{1,yy}$ can be treated in the same way since they both yield 2-dimensional trigonometric series similar to (4.62).

We move on to the second series (4.61). As in the above case, we report explicitly only the proof that the series for the term by term second order partial derivative $u_{2,xx} = \frac{\partial^2}{\partial x^2}u_2$ converges in $L^2(\Omega)$, since the other derivatives are treated analogously. By differentiation, the form of $u_{2,xx}$ is the following

$$\begin{aligned} u_{2,xx} = -\sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty} m^2\Big[&A\cosh(my) + Bmy\sinh(my) + \\ &+C\sinh(my) + Dmy\cosh(my)\Big]\sin(mx)\,. \end{aligned} \tag{4.63}$$

For each value of m and y consider the coefficient $k_m(y)$ to $\sin(mx)$ in the above sum, defined as

$$k_m(y) = -\sum_{n=0}^{+\infty} m^2\Big[A\cosh(my) + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)\Big]\,,$$

so that the $u_{2,xx}$ can be rewritten as the following Fourier series

$$u_{2,xx} = \sum_{m=1}^{+\infty} k_m(y)\sin(mx) \tag{4.64}$$

Applying the inequalities (4.44) we can bound each coefficient $k_m(y)$ with a constant K_m not depending on y, specifically we have the following

$$\begin{aligned} |k_m(y)| \leq \sum_{n=0}^{+\infty}\left| m^2\Big[A\cosh(my) + Bmy\sinh(my) + C\sinh(my) + Dmy\cosh(my)\Big]\right| \\ \leq \sum_{n=0}^{+\infty} \mathcal{C}\frac{\left|\beta_{m,n}^{\cos}\right| m + \left|\beta_{m,n}^{\sin}\right|\omega_n}{m^2+\omega_n^2} = K_m\,, \end{aligned} \tag{4.65}$$

for every y in $(-\ell,\ell)$ and for a suitably large constant $\mathcal{C} > 0$ depending only on σ. In order to prove our thesis it is now sufficient to show that every K_m is finite and that the sequence $\{K_m\}_{m=1}^{+\infty}$ lies in $\ell^2(\mathbb{N})$, so that the Fourier series (4.64) converges in $L^2(\Omega)$. To do this we compute the following two sums

$$\sum_{n=0}^{\infty}\left(\frac{m}{m^2+\omega_n^2}\right)^2 = \frac{\ell^2m^2\text{csch}^2(\ell m) + \ell m\coth(\ell m) + 2}{4m^2}$$

$$\sum_{n=0}^{\infty}\left(\frac{\omega_n}{m^2+\omega_n^2}\right)^2 = -\frac{\ell^3\left(\ell m\text{csch}^2(\ell m) - \coth(\ell m)\right)}{4\pi^2 m}$$

and notice that for $m \to +\infty$ the two results are $O(1/m)$ and therefore both admit a finite maximum with respect to m, which we will denote as M.

Then, applying Cauchy-Schwarz inequality to the $\ell^2(\mathbb{N})$ scalar product (4.65) we obtain

$$\begin{aligned} K_m &\leq \left(\sum_{n=0}^{+\infty}\left|\beta_{m,n}^{\cos}\right|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{n=0}^{+\infty}\left|\frac{m}{m^2+\omega_n^2}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{+\infty}\left|\beta_{m,n}^{\sin}\right|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{n=0}^{+\infty}\left|\frac{\omega_n}{m^2+\omega_n^2}\right|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{M}\left(\sum_{n=0}^{+\infty}\left|\beta_{m,n}^{\cos}\right|^2\right)^{\frac{1}{2}} + \sqrt{M}\left(\sum_{n=0}^{+\infty}\left|\beta_{m,n}^{\sin}\right|^2\right)^{\frac{1}{2}} \\ &\leq \mathcal{C}\left[\sum_{n=0}^{+\infty}\left|\beta_{m,n}^{\cos}\right|^2 + \left|\beta_{m,n}^{\sin}\right|^2\right]^{\frac{1}{2}}. \end{aligned}$$

From this we conclude that the sequence K_m lies in $\ell^2(\mathbb{N})$ since is norm is bounded by the one of the double sequence $\beta_{m,n}^{\cos} + \beta_{m,n}^{\sin}$, which in turn lies in $\ell^2(\mathbb{N}^2)$.

Thus the series (4.63) of the term by term derivatives $u_{2,xx}$ converges in $L^2(\Omega)$ as it is bounded by the convergent Fourier series

$$\sum_{m=1}^{+\infty} K_m\sin(mx)\,.$$

One can apply similar considerations to show that the same holds for the other components of the hessian matrix of u_2, namely $u_{2,xy}$ and $u_{2,yy}$.

Combining the results obtained for u_1 and u_2, we see that the series (4.29) converges in $H^2_*(\Omega)$, which concludes the proof. $\square$

In order to justify the assumption requiring the Fourier coefficients of f not to depend on ℓ, we provide an example which satisfies this assumption. This should also clarify how keeping fixed the function's Fourier coefficients ensures that the function's overall shape is preserved through an appropriate rescaling.

Specifically, it is useful to normalize the y coordinate as $z = \pi y/\ell$ so that if a point (x, y) ranges over the domain $\Omega = (0, \pi) \times (-\ell, \ell)$ then its normalized coordinates (x, z) range over the fixed domain $\tilde{\Omega} = (0, \pi) \times (-\pi, \pi)$.

Now fix a load function $\tilde{f} = \tilde{f}(x, z)$defined on the normalized domain $\tilde{\Omega}$ and, for every value of $0 < \ell < 1$ define $f^\ell : \omega \to \mathbb{R}$ by the equation

$$f^\ell(x, y) = \tilde{f}(x, z) \tag{4.66}$$

for all $(x, y) \in \Omega$ and $(x, z) \in \tilde{\Omega}$ such that $z = \pi y/\ell$. In this way we obtain, for every considered value of the parameter ℓ, a function f^ℓ having the appropriate domain Ω and all these functions share the same shape as the original function $\tilde{f}$, since they are obtained by $\tilde{f}$ through a rescaling along the y direction.

When one expands function f into its double Fourier series as in (4.27) it can be shown that due to the vertical scaling that we applied in definition (4.66) the Fourier coefficients $\beta^{\cos}_{m,n}$ and $\beta^{\sin}_{m,n}$, for $m \geq 1$ and $n \geq 0$, do not depend on ℓ. To see this observe that as ℓ decreases function f is stretched vertically, while the harmonic base functions $\sin(mx)\cos(\omega_n y)$ and $\sin(mx)\sin(\omega_n y)$ are dilated in exactly the same manner due to the fact that the frequencies ω_n are by definition equal to $\pi n/\ell$. More formally, the Fourier sum of f for ach value of ℓ can be derived from the one of $\tilde{f}$, which is

$$\tilde{f}(x, z) = \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \left[\beta^{\sin}_{m,n} \sin(mx) \sin(nz) + \beta^{\cos}_{m,n} \sin(mx) \cos(nz)\right].$$

Chapter 5

Oscillating Modes

In this section, we consider the eigenvalue problem

$$\begin{cases} \Delta^2 w = \lambda w & \text{in } \Omega \\ w(0,y) = w_{xx}(0,y) = w(\pi,y) = w_{xx}(\pi,y) & \text{for } y \in (-\ell,\ell) \\ w_{yy}(x,\pm\ell) + \sigma w_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ w_{yyy}(x,\pm\ell) + (2-\sigma) w_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{5.1}$$

As detailed in Section 4.1 the natural space where to study this problem is the space $H^2_*(\Omega)$ endowed with the scalar product (4.2).

Constructing a sequence of eigenvalue λ_k and the corresponding eigenfunctions w_k is instrumental to develop the analysis of nonlinear and dynamic models, such as that introduced in section 3.2. To this aim, we can restate problem (5.1) in a variational setting as we did for problem (3.8). The resulting weak formulation states that a non-trivial function $w \in H^2_*(\Omega)$ is an eigenfunction of (5.1) if it solves

$$\int_\Omega \left[\Delta w \Delta v + (1-\sigma)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}) - \lambda w v\right] dx\, dy = 0, \forall v \in H^2_*(\Omega)\,. \tag{5.2}$$

In order to characterize solutions to (5.2) we show how the general theory discussed in Section 2.3 applies in this setting and proceed to explicitly compute the eigenfunctions and eigenvalues using similar techniques to those applied in Section 4.3. Applying the results of the theorem, properties of the explicit eigenfunctions are analysed, both to see how they conform to the expectations and to make further observations which, though always valid for the simple laplacian operator Δ, are not true in general when polyharmonic operators Δ^m, $m \geq 2$ are involved such as the positivity of the first eigenfunction.

5.1 Eigenvalue Theorem

In this section we state and report the proof to Theorem 5.1.1, a result covering the theoretical properties and explicit formulation of the eigenvalues and eigenfunctions of the eigenvalue problem (5.2).

Theorem 5.1.1. [8, Theorem 3.1] *Assume* (3.2)*. Then the set of eigenvalues of* (5.1) *may be ordered in an increasing sequence of strictly positive numbers diverging to* $+\infty$ *and any eigenfunction belongs to* $C^\infty(\overline{\Omega})$*; the set of eigenfunctions of* (5.1) *is a complete system in* $H^2_*(\Omega)$*. Moreover:*
(i) for any $m \geq 1$*, there exists a unique eigenvalue* $\lambda = \mu_{m,1} \in ((1-\sigma^2)m^4, m^4)$ *with corresponding eigenfunction*

$$\left[[\mu_{m,1}^{1/2} - (1-\sigma)m^2] \frac{\cosh\left(y\sqrt{m^2+\mu_{m,1}^{1/2}}\right)}{\cosh\left(\ell\sqrt{m^2+\mu_{m,1}^{1/2}}\right)} + \right.$$
$$\left. + [\mu_{m,1}^{1/2} + (1-\sigma)m^2] \frac{\cosh\left(y\sqrt{m^2-\mu_{m,1}^{1/2}}\right)}{\cosh\left(\ell\sqrt{m^2-\mu_{m,1}^{1/2}}\right)} \right] \sin(mx)\,;$$

(ii) for any $m \geq 1$ *and any* $k \geq 2$ *there exists a unique eigenvalue* $\lambda = \mu_{m,k} > m^4$ *satisfying* $\left(m^2 + \frac{\pi^2}{\ell^2}\left(k-\frac{3}{2}\right)^2\right)^2 < \mu_{m,k} < \left(m^2 + \frac{\pi^2}{\ell^2}(k-1)^2\right)^2$ *and with corresponding eigenfunction*

$$\left[[\mu_{m,k}^{1/2} - (1-\sigma)m^2] \frac{\cosh\left(y\sqrt{\mu_{m,k}^{1/2}+m^2}\right)}{\cosh\left(\ell\sqrt{\mu_{m,k}^{1/2}+m^2}\right)} + \right.$$
$$\left. + [\mu_{m,k}^{1/2} + (1-\sigma)m^2] \frac{\cos\left(y\sqrt{\mu_{m,k}^{1/2}-m^2}\right)}{\cos\left(\ell\sqrt{\mu_{m,k}^{1/2}-m^2}\right)} \right] \sin(mx)\,;$$

(iii) for any $n \geq 1$ *and any* $j \geq 2$ *there exists a unique eigenvalue* $\lambda = \nu_{n,j} > n^4$ *with corresponding eigenfunctions*

$$\left[[\nu_{n,j}^{1/2} - (1-\sigma)n^2] \frac{\sinh\left(y\sqrt{\nu_{n,j}^{1/2}+n^2}\right)}{\sinh\left(\ell\sqrt{\nu_{n,j}^{1/2}+n^2}\right)} + \right.$$
$$\left. + [\nu_{n,j}^{1/2} + (1-\sigma)n^2] \frac{\sin\left(y\sqrt{\nu_{n,j}^{1/2}-n^2}\right)}{\sin\left(\ell\sqrt{\nu_{n,j}^{1/2}-n^2}\right)} \right] \sin(nx)\,;$$

(iv) for any $n \geq 1$ *satisfying* $\ell n\sqrt{2}\,\coth(\ell n\sqrt{2}) > \left(\frac{2-\sigma}{\sigma}\right)^2$ *there exists a unique eigenvalue* $\lambda = \nu_{n,1} \in (\mu_{n,1}, n^4)$ *with corresponding eigenfunction*

$$\left[[\nu_{n,1}^{1/2} - (1-\sigma)n^2] \frac{\sinh\left(y\sqrt{n^2+\nu_{n,1}^{1/2}}\right)}{\sinh\left(\ell\sqrt{n^2+\nu_{n,1}^{1/2}}\right)} + \right.$$
$$\left. + [\nu_{n,1}^{1/2} + (1-\sigma)n^2] \frac{\sinh\left(y\sqrt{n^2-\nu_{n,1}^{1/2}}\right)}{\sinh\left(\ell\sqrt{n^2-\nu_{n,1}^{1/2}}\right)} \right] \sin(nx)\,.$$

Finally, if the unique positive solution $s > 0$ *of the equation*

$$\tanh(\sqrt{2}s\ell) = \left(\frac{\sigma}{2-\sigma}\right)^2 \sqrt{2}s\ell \tag{5.3}$$

is not an integer, then the only eigenvalues and eigenfunctions are the ones given in $(i) - (iv)$*.*

Condition (5.3) has probability 0 to occur in general plates; if it occurs, there is an additional eigenvalue and eigenfunction, see [8]. The eigenvalues are solutions of explicit equations.

Proposition 5.1.2. *Let*

$$\begin{aligned}
\Phi^m(\lambda,\ell) :=& \sqrt{m^2-\lambda^{1/2}}\big(\lambda^{1/2}+(1-\sigma)m^2\big)^2\tanh(\ell\sqrt{m^2-\lambda^{1/2}})+\\
&-\sqrt{m^2+\lambda^{1/2}}\big(\lambda^{1/2}-(1-\sigma)m^2\big)^2\tanh(\ell\sqrt{m^2+\lambda^{1/2}}),\\
\Upsilon^m(\lambda,\ell) :=& \sqrt{\lambda^{1/2}-m^2}\big(\lambda^{1/2}+(1-\sigma)m^2\big)^2\tan(\ell\sqrt{\lambda^{1/2}-m^2})+\\
&+\sqrt{\lambda^{1/2}+m^2}\big(\lambda^{1/2}-(1-\sigma)m^2\big)^2\tanh(\ell\sqrt{\lambda^{1/2}+m^2}),\\
\Psi^n(\lambda,\ell) :=& \sqrt{\lambda^{1/2}-n^2}\big(\lambda^{1/2}+(1-\sigma)n^2\big)^2\tanh(\ell\sqrt{\lambda^{1/2}+n^2})+\\
&-\sqrt{\lambda^{1/2}+n^2}\big(\lambda^{1/2}-(1-\sigma)n^2\big)^2\tan(\ell\sqrt{\lambda^{1/2}-n^2}),\\
\Gamma^n(\lambda,\ell) :=& \sqrt{n^2-\lambda^{1/2}}\big(\lambda^{1/2}+(1-\sigma)n^2\big)^2\tanh(\ell\sqrt{\lambda^{1/2}+n^2})+\\
&+\sqrt{\lambda^{1/2}+n^2}\big(\lambda^{1/2}-(1-\sigma)n^2\big)^2\tanh(\ell\sqrt{\lambda^{1/2}-n^2}).
\end{aligned}$$

Then:
(i) *the eigenvalue* $\lambda = \mu_{m,1}$ *is the unique value* $\lambda \in ((1-\sigma^2)m^4, m^4)$ *such that* $\Phi^m(\lambda,\ell) = 0$*;*
(ii) *the eigenvalues* $\lambda = \mu_{m,k}$ *(*$k \geq 2$*) are the solutions* $\lambda > m^4$ *of the equation* $\Upsilon^m(\lambda,\ell) = 0$*;*
(iii) *the eigenvalues* $\lambda = \nu_{n,j}$ *(*$j \geq 2$*) are the solutions* $\lambda > n^4$ *of the equation* $\Psi^n(\lambda,\ell) = 0$*;*
(iv) *the eigenvalue* $\lambda = \nu_{n,1}$ *is the unique value* $\lambda \in ((1-\sigma^2)n^4, n^4)$ *such that* $\Gamma^n(\lambda,\ell) = 0$*.*

The eigenfunctions in $(i) - (ii)$ are even with respect to y whereas the eigenfunctions in $(iii) - (iv)$ are odd. We call **longitudinal eigenfunctions** the eigenfunctions of the kind $(i) - (ii)$ and **torsional eigenfunctions** the eigenfunctions of the kind $(iii) - (iv)$. Since ℓ is small, the former are quite similar to $c_m \sin(mx)$ whereas the latter are similar to $c_n y \sin(nx)$.

In the sequel, we consider realistic values of σ and ℓ, as in some actual bridges; we take

$$\sigma = 0.2\,, \qquad \ell = \frac{\pi}{150}, \tag{5.4}$$

but very similar results are obtained for values of σ and ℓ close to (5.4). This choice of ℓ models the case where the main span of the bridge is 1 kilometer long and the width 2ℓ is about 13 meters. These values are taken from the original Tacoma Narrows Bridge, see [4, 5]. We denote by

$$\bar{\mu}_{m,k} \text{ and } \bar{\nu}_{n,j} \text{ the eigenvalues of (5.1) given in Theorem 5.1.1 and Proposition 5.1.2 when (5.4) holds.} \tag{5.5}$$

Then, from Theorem 5.1.1, we infer that

$$0.96m^4 < \bar{\mu}_{m,1} < m^4\,,$$
$$(m^2+75^2(2k-3)^2)^2 < \bar{\mu}_{m,k} < (m^2+150^2(k-1)^2)^2 \quad \forall k \geq 2$$

for all integer m. Furthermore, a direct inspection yields that

$$\bar{\nu}_{n,1} \quad \text{does not exist for } 1 \leq n \leq 2734\,. \tag{5.6}$$

In Table 5.1 we collect some numerical values of $\bar{\mu}_{m,k}$ and $\bar{\nu}_{n,j}$ as defined in (5.5).

$\bar{\mu}_{1,1}$	$\bar{\mu}_{2,1}$	$\bar{\mu}_{3,1}$	$\bar{\mu}_{4,1}$	$\bar{\mu}_{5,1}$	$\bar{\mu}_{6,1}$	$\bar{\mu}_{7,1}$
0.96	15.36	77.77	245.8	600.14	1244.6	2306.05

$\bar{\mu}_{8,1}$	$\bar{\mu}_{9,1}$	$\bar{\mu}_{10,1}$	$\bar{\mu}_{11,1}$	$\bar{\mu}_{12,1}$	$\bar{\mu}_{13,1}$	$\bar{\mu}_{14,1}$
3934.57	6303.42	9609.09	14071.4	19933.4	27461.6	36946

$\bar{\nu}_{1,2}$	$\bar{\nu}_{2,2}$	$\bar{\nu}_{3,2}$	$\bar{\nu}_{4,2}$	$\bar{\nu}_{5,2}$
10943.63	43785.82	98560.47	175324.1	274155.8

$\bar{\mu}_{1,2}$	$\bar{\mu}_{2,2}$	$\bar{\mu}_{3,2}$	$\bar{\mu}_{4,2}$	$\bar{\mu}_{5,2}$	$\bar{\mu}_{6,2}$	$\bar{\mu}_{7,2}$	$\times$
1.626	1.628	1.63	1.634	1.638	1.643	1.649	10^8

$\bar{\mu}_{8,2}$	$\bar{\mu}_{9,2}$	$\bar{\mu}_{10,2}$	$\bar{\mu}_{11,2}$	$\bar{\mu}_{12,2}$	$\bar{\mu}_{13,2}$	$\bar{\mu}_{14,2}$	$\times$
1.657	1.665	1.674	1.684	1.695	1.707	1.72	10^8

$\bar{\nu}_{1,3}$	$\bar{\nu}_{2,3}$	$\bar{\nu}_{3,3}$	$\bar{\nu}_{4,3}$	$\bar{\nu}_{5,3}$
$1.2356\cdot 10^9$	$1.2359\cdot 10^9$	$1.2365\cdot 10^9$	$1.2372\cdot 10^9$	$1.2382\cdot 10^9$

Table 5.1. Numerical values of some eigenvalues of problem (5.1) when (5.4) holds.

These results are fairly precise and reliable. The «exact» value of these eigenvalues will be important in the following sections. Here we just point out that

$$\bar{\mu}_{1,1} < \cdots < \bar{\mu}_{10,1} < \bar{\nu}_{1,2} < \bar{\mu}_{11,1} < \cdots < \bar{\mu}_{14,1} < \bar{\nu}_{2,2}, \tag{5.7}$$

$$\bar{\nu}_{n,2} < \bar{\mu}_{m,2} < \bar{\nu}_{n,3} \qquad \text{for all } m = 1,...,14 \text{ and } n = 1,...,5\,. \tag{5.8}$$

In fact, we considered all the $k = 1,2,3,4$, and $j = 2,3,4,5$, for $\bar{\mu}_{m,k}$ and $\bar{\nu}_{n,j}$ with $m \leq 14$ and $n \leq 5$. Let us briefly summarize what we observed numerically.

- The map $m \mapsto \bar{\mu}_{m,1}$ is strictly increasing and $0.96 < \bar{\mu}_{m,1} < 36946.004$ for $m = 1,...,14$.
- The map $m \mapsto \bar{\mu}_{m,2}$ is strictly increasing and $1.62 \cdot 10^8 < \bar{\mu}_{m,2} < 1.721 \cdot 10^8$ for $m = 1,...,14$.
- The map $m \mapsto \bar{\mu}_{m,3}$ is strictly increasing and $4.74 \cdot 10^9 < \bar{\mu}_{m,3} < 4.786 \cdot 10^9$ for $m = 1,...,14$.
- The map $m \mapsto \bar{\mu}_{m,4}$ is strictly increasing and $2.895 \cdot 10^{10} < \bar{\mu}_{m,4} < 2.904 \cdot 10^{10}$ for $m = 1,...,14$.
- The map $n \mapsto \bar{\nu}_{n,2}$ is strictly increasing and $10943.6 < \bar{\nu}_{n,2} < 274155.9$ for $n = 1,...,5$.
- The map $n \mapsto \bar{\nu}_{n,3}$ is strictly increasing and $1.235 \cdot 10^9 < \bar{\nu}_{n,3} < 1.239 \cdot 10^9$ for $n = 1,...,5$.

• The map $n \mapsto \bar{\nu}_{n,4}$ is strictly increasing and $1.297 \cdot 10^{10} < \bar{\nu}_{n,4} < 1.299 \cdot 10^{10}$ for $n = 1, ...,5$.
• The map $n \mapsto \bar{\nu}_{n,5}$ is strictly increasing and $5.648 \cdot 10^{10} < \bar{\nu}_{n,5} < 5.65 \cdot 10^{10}$ for $n = 1, ...,5$.

In terms of the frequencies (the square roots of the eigenvalues) the above observations show that

$$\text{the smallest frequencies of the normal modes are those listed in Table 5.1.} \tag{5.9}$$

These facts explain why we mainly restricted our attention to the eigenvalues in Table 5.1 (i.e. $k = 1{,}2$ and $j = 2{,}3$). Moreover, the eigenvalues $\bar{\mu}_{m,2}$ are much bigger than the eigenvalues $\bar{\mu}_{m,1}$, and this translates in larger frequencies. This means that a bigger amount of energy is needed in order to trigger the normal modes associated with $\bar{\mu}_{m,2}$, so that it is quite unlikely to observe them. The same remark holds also for $\bar{\nu}_{n,2}, \bar{\nu}_{n,3}$.

Note that the restrictions $m \leq 14$ and $n \leq 5$ are not just motivated by the lack of space in this paper but also by the behaviour in actual bridges; at the collapsed Tacoma Narrows Bridge the longitudinal oscillations appeared with at most ten nodes and the torsional oscillation appeared with one node, see.

Finally, by (5.6) we know that the torsional eigenvalues $\bar{\nu}_{n,1}$ do not exist for $n \leq 2734$, while for $n \geq 2735$ the frequencies are very large.

By Lemma 4.1.1 the bilinear form (4.2) is continuous an coercive; we can then apply Theorem 2.3.8 to show that the eigenvalues of (5.1) may be ordered in an increasing sequence of strictly positive numbers diverging to $+\infty$ and that the corresponding eigenfunctions form a complete system in $H^2_*(\Omega)$.

In order to show that the hypothesis of the Theorem are satisfied, we fix the order of derivation to 2 and put $V(\Omega) = H^2_*(\Omega)$ and $\mathcal{Q}[u, \phi] = (u, \phi)_{H^2_*}$, by an appropriate choice of the choice of the coefficients $A_{\alpha\beta}$, and proceed as follows.

1. *The embedding of $V(\Omega) \subset L^2(\Omega)$ is continuous.* This follow by composition of the continuous inclusion $V(\Omega) = H^2_*(\Omega) \hookrightarrow H^2(\Omega)$ and the inclusion $H^2(\Omega) \hookrightarrow L^2(\Omega)$, whose compactness is given by Theorem 2.1.2.

2. *Gårding's inequality* (2.30) *holds.* This is a straightforward consequence of Lemma 4.1.1. Indeed putting $a = b = 1 - \sigma$ we have

$$\begin{aligned} a\,\|u\|^2_{H^m} &= (1-\sigma)\left(\left\||D^2u|\right\|_{L^2} + \|u\|_{L^2}\right) \\ &\leq \left\||D^2u|\right\|_{H^2_*} + (1-\sigma)\,\|u\|_{L^2} = \mathcal{Q}[u,u] + b\,\|u\|_{L^2} \end{aligned}$$

3. *There exists a constant $c > 0$ such that* (2.31) *is satisfied.* This again follows from Lemma 4.1.1: fixing the constant c to $1 + \sigma$ we have

$$\mathcal{Q}[u,u] = \|u\|_{H^2_*} \leq (1-\sigma)\left\||D^2u|\right\|_{L^2} \leq c\,\|u\|_{H^2}$$

Note that the positivity of the eigenvalues is due to the fact that $\lambda_m > a - b = 0$ since we have taken $a = b = 1 - \sigma$.

The eigenfunctions are smooth in $\overline{\Omega}$: this may be obtained by making an odd extension as in Lemma 4.2.4 and with a bootstrap argument. Specifically we have that every eigenfunction w_m is a solution of the problem (4.4) where we set the load function f to $\lambda_m w_m$. Then if w_m lies in $H^k(\Omega)$ for some $k \in \mathbb{N}$, then by Lemma 4.2.4 we obtain that $w_m \in H^{k+4}(\Omega)$ and thus it is of class $C^{k+2}(\overline{\Omega})$. Proceeding by induction over k, starting from the base case $k = 2$ since $w_m \in H^2(\Omega)$ is known, it can be shown that the eigenfunction admits continuous derivatives of arbitrarily high order so that $w_k \in C^\infty(\overline{\Omega})$. This proves the first part of 5.1.1.

Take an eigenfunction w of (5.1) and consider its Fourier expansion with respect to the variable x:

$$w(x,y) = \sum_{m=1}^{+\infty} h_m(y)\sin(mx) \qquad \text{for } (x,y) \in (0,\pi) \times (-\ell,\ell)\,. \tag{5.10}$$

Since $w \in C^\infty$, the Fourier coefficients $h_m = h_m(y)$ are smooth functions and solve the ordinary differential equation

$$h''''(y) - 2m^2 h''_m(y) + (m^4 - \lambda)h_m(y) = 0 \tag{5.11}$$

for some $\lambda > 0$. The eigenfunction w in (5.10) satisfies (3.6), while by imposing (3.7) we obtain the boundary conditions on h_m

$$h''_m(\pm\ell) - \sigma m^2 h_m(\pm\ell) = 0\,, \qquad h'''_m(\pm\ell) + (\sigma - 2)m^2 h'_m(\pm\ell) = 0\,. \tag{5.12}$$

Put $\mu = \sqrt{\lambda} > 0$ and consider the characteristic equation $\alpha^4 - 2m^2\alpha^2 + m^4 - \mu^2 = 0$ related to (5.11). By solving the algebraic equation we find

$$\alpha^2 = m^2 \pm \mu\,. \tag{5.13}$$

Three cases have to be distinguished.

- **The case** $0 < \mu < m^2$. By (5.13) we infer

$$\alpha = \pm\beta \text{ or } \alpha = \pm\gamma \qquad \text{with} \qquad \sqrt{m^2 - \mu} =: \gamma < \beta := \sqrt{m^2 + \mu}\,. \tag{5.14}$$

Hence, possible nontrivial solutions of (5.11)–(5.12) have the form

$$h_m(y) = a\cosh(\beta y) + b\sinh(\beta y) + c\cosh(\gamma y) + d\sinh(\gamma y) \qquad (a,b,c,d \in \mathbb{R})\,. \tag{5.15}$$

By computing the derivatives of h_m and imposing the conditions (5.12) we find the two systems

$$\begin{cases} (\beta^2 - m^2\sigma)\cosh(\beta\ell)a + (\gamma^2 - m^2\sigma)\cosh(\gamma\ell)c = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\sinh(\beta\ell)a + (\gamma^3 - m^2(2-\sigma)\gamma)\sinh(\gamma\ell)c = 0\,, \end{cases} \tag{5.16}$$

$$\begin{cases} (\beta^2 - m^2\sigma)\sinh(\beta\ell)a + (\gamma^2 - m^2\sigma)\sinh(\gamma\ell)c = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\cosh(\beta\ell)a + (\gamma^3 - m^2(2-\sigma)\gamma)\cosh(\gamma\ell)c = 0\,. \end{cases} \tag{5.17}$$

There exists a nontrivial solution h_m of (5.11) of the form (5.15) if and only if there exists a nontrivial solution of the two systems (5.16). The first system in (5.16) admits a nontrivial solution (a, c) if and only if

$$(\beta^2 - m^2\sigma)(\gamma^3 - m^2(2-\sigma)\gamma)\cosh(\beta\ell)\sinh(\gamma\ell)$$
$$= (\gamma^2 - m^2\sigma)(\beta^3 - m^2(2-\sigma)\beta)\sinh(\beta\ell)\cosh(\beta\ell)\,.$$

By (5.14), this is equivalent to

$$\frac{\gamma}{(\gamma^2 - m^2\sigma)^2}\tanh(\ell\gamma) = \frac{\beta}{(\beta^2 - m^2\sigma)^2}\tanh(\ell\beta)\,. \tag{5.18}$$

Recalling that both β and γ depend on μ, we prove

Lemma 5.1.3. *Assume (3.2). For any $m \geq 1$ there exists an unique $\mu = \mu_m \in (0, m^2)$ such that (5.18) holds; moreover we lso have $\mu_m \in ((1-\sigma)m^2, m^2)$.*

Proof. Consider the function $\eta_m(t) := \frac{t}{(t^2 - m^2\sigma)^2} \cdot \tanh(\ell t)$ for any $t \in [0, +\infty) \setminus \{\sqrt{\sigma}m\}$. Then

$$\eta_m'(t) = \frac{(-3t^2 - m^2\sigma)\sinh(\ell t)\cosh(\ell t) + \ell t(t^2 - m^2\sigma)}{(t^2 - m^2\sigma)^3\cosh^2(\ell t)} \quad \forall t \in [0, +\infty) \setminus \{\sqrt{\sigma}m\}\,.$$

For any $t > \sqrt{\sigma}m$ we have

$$\eta_m'(t) < \frac{-3t^2\sinh(\ell t)\cosh(\ell t) + \ell t^3}{(t^2 - m^2\sigma)^3\cosh^2(\ell t)} < \frac{2\ell t^3}{(t^2 - m^2\sigma)^3\cosh^2(\ell t)} < 0\,.$$

This shows that η_m is decreasing in $(\sqrt{\sigma}, +\infty)$ and, if $\beta > \gamma > \sqrt{\sigma}m$ then $\eta_m(\beta) < \eta_m(\gamma)$ so that (5.18) cannot hold. We have proved that if γ and β satisfy (5.18) then necessarily $\gamma \in [0, \sqrt{\sigma}m)$.

Since $\beta = \sqrt{2m^2 - \gamma^2}$, identity (5.18) is equivalent to

$$\frac{\frac{2m^2-\gamma^2}{(}\gamma^2 - m^2\sigma^2)^2}{[(2-\sigma)m^2 - \gamma^2]^2}\tanh(\ell\sqrt{2m^2 - \gamma^2}) = \gamma\tanh(\ell\gamma)\,. \tag{5.19}$$

Then we define

$$g_m(t) := \frac{\sqrt{2m^2 - \gamma^2}(\gamma^2 - m^2\sigma)^2}{[(2-\sigma)m^2 - \gamma^2]^2}\tanh(\ell\sqrt{2m^2 - t^2}) \qquad \forall t \in [0, \sqrt{\sigma}m]\,.$$

The function $t \mapsto [m^2\sigma - t^2]/[(2-\sigma)m^2 - t^2]$ i nonnegative and decreasing and hence so is its square. It then follows that g_m is decreasing in $[0, \sqrt{\sigma}m]$ and $g_m(\sqrt{\sigma}m) = 0$. On the other hand, the map $t \mapsto t\tanh(\ell t)$ i increasing in $[0, \sqrt{\sigma}m]$ and vanishes at $t = 0$. This proves that there exists a unique $\gamma_m \in (0, \sqrt{\sigma}m)$ satisfying (5.19). The statements of the lemma now follow by putting $\mu_m = m^2 - \gamma_m^2$. ☐

In the next result we prove that the sequence $\{\mu_m\}$ found in Lemma 5.1.3 is increasing.

Lemma 5.1.4. *Assume* (3.2)*. For any* $m \geq 1$*, let* μ_m *be as in Lemma 5.1.3. Then* $\mu_m < \mu_{m+1}$ *for all* $m \geq 1$*.*

Proof. By (5.14), the equation (5.18) reduces to

$$\Phi(m,\mu) := \sqrt{\frac{m^2-\mu}{m^2+\mu}}\left(\frac{m+(1-\sigma)m^2}{\mu-(1-\sigma)m^2}\right)^2 \frac{\tanh(\ell\sqrt{m^2-\mu})}{\tanh(\ell\sqrt{m^2+\mu})} = 1\,. \tag{5.20}$$

We consider Φ as a function defined in the region o the plane $\{(m,\mu)\in\mathbb{R};(1-\sigma)m^2 < \mu < m^2\}$. In this region, the three maps

$$(m,\mu)\mapsto\sqrt{\frac{m^2-\mu}{m^2+\mu}},\ (m,\mu)\mapsto\left(\frac{\mu+(1-\sigma)m^2}{\mu-(1-\sigma)m^2}\right)^2,\ (m,\mu)\mapsto\frac{\tanh(\ell\sqrt{m^2-\mu})}{\tanh(\ell\sqrt{m^2+\mu})},$$

are all positive, strictly increasing with respect to m, and strictly decreasing with respect to μ. Therefore, the function $m\mapsto\mu_m$, implicitly defined by $\Phi(m,\mu_m)=1$, is strictly increasing. □

Similarly, the second system in (5.16) has nontrivial solutions (b,d) if and only if

$$\begin{aligned}&(\beta^2-m^2\sigma)(\gamma^3-m^2(2-\sigma)\gamma)\sinh(\beta\ell)\cosh(\gamma\ell)\\ &=(\gamma^2-m^2\sigma)(\beta^3-m^2(2-\sigma)\beta)\cosh(\beta\ell)\sinh(\gamma\ell)\,.\end{aligned}$$

By (5.14), this is equivalent to

$$\frac{\beta}{(\beta^2-m^2\sigma)^2}\coth(\ell\beta) = \frac{\gamma}{(\gamma^2-m^2\sigma)^2}\coth(\ell\gamma)\,. \tag{5.21}$$

Recalling that both β and γ depend on μ, we prove

Lemma 5.1.5. *Assume* (3.2)*, Then there exists a unique* $\mu=\mu^m\in(0,m^2)$ *satisfying* (5.21) *if and only if*

$$\ell m\sqrt{2}\coth(\ell m\sqrt{2}) > \left(\frac{2-\sigma}{\sigma}\right)^2\,. \tag{5.22}$$

Moreover in such a case we have $\mu^m\in((1-\sigma)m^2,m^2)$*.*

Proof. The function $\eta_m(t) := \frac{t}{(t^2-m^2\sigma)^2}\cdot\cot(\ell t)$ is strictly decreasing for $t\in(\sqrt{\sigma}m,+\infty)$ because it is the product of two positive and strictly decreasing functions. In particular, if $\beta>\gamma>\sqrt{\sigma}m$ then $\eta_m(\beta)<\eta_m(\gamma)$ so that (5.21) cannot hold. This proves that if γ and β satisfy (5.21) then necessarily $\gamma\in(0,\sqrt{\sigma}m)$.

By (5.14) identity (5.21) is equivalent to

$$\frac{\sqrt{2m^2-\gamma^2}(\gamma^2-m^2\sigma)^2}{[(2-\sigma)m^2-\gamma^2]^2}\coth(\ell\sqrt{2m^2-\gamma^2}) = \gamma\coth(\ell\gamma)\,. \tag{5.23}$$

Then we define

$$g_m(t) = \frac{\sqrt{2m^2-t^2}(m^2\sigma-t^2)2}{[(2-\sigma)m^2-t^2]^2}\coth(\ell\sqrt{2m^2-t^2}) \qquad \forall t\in[0,\sqrt{\sigma}m]\,. \tag{5.24}$$

We have

$$g'_m(t) = \frac{\ell t(m^2\sigma - t^2)^2}{[(2-\sigma)m^2 - t^2]^2 \sinh^2(\ell\sqrt{2m^2-t^2})} - t\coth(\ell\sqrt{2m^2-t^2})\times \tag{5.25}$$
$$\times\,(m^2\sigma - t^2)\frac{8(1-\sigma)m^2(2m^2-t^2) + (m^2\sigma - t^2)[(2-\sigma)m^2 - t^2]}{\sqrt{2m^2-t^2}[(2-\sigma)m^2-t^2]^3}$$
$$< [\ell\sqrt{2m^2-t^2} - \sinh(\ell\sqrt{2m^2-t^2})\cosh(\ell\sqrt{2m^2-t^2})]\times$$
$$\times\,\frac{t(m^2\sigma-t^2)^2[(2-\sigma)m^2-t^2]}{\sqrt{2m^2-t^2}[(2-\sigma)m^2-t^2]^3\sinh^2(\ell\sqrt{2m^2-t^2})}$$

which is negative for anu $t \in (0, \sqrt{\sigma}m)$. Therefore g_m is decreasing in $(0, \sqrt{\sigma}m)$ with $g_m(0) = \sqrt{2}m\left(\frac{\sigma}{2-\sigma}\right)^2 \coth(\ell m\sqrt{2})$ and $g_m(\sqrt{\sigma}m) = 0$. On the other hand, the map $t \mapsto t\coth(\ell t)$ is increasing in $(0, \sqrt{\sigma}m)$ and tends to $1/\ell$ as $t \to 0^+$. This proves that there exists a unique $\gamma^m \in (0, \sqrt{\sigma}m)$ satisfying (5.23) if and only if (5.22) holds. The proof of the lemma now follows by putting $\mu^m = m^2 - (\gamma^m)^2$. □

Note also that (5.22) holds if and only if m is large enough, that is,

$$\exists m_\sigma \geq 1 \text{ such that (5.22) holds if and only if } m \geq m_\sigma\,. \tag{5.26}$$

In particular, if $\ell\sqrt{2}\coth(\ell\sqrt{2}) > \left(\frac{\sigma}{2-\sigma}\right)^2$ then $m_\sigma = 1$. We now prove that the sequence $\{\mu^m\}$, found in Lemma 5.1.5, is increasing.

Lemma 5.1.6. *Assume (3.2). For any $m \geq 1$, let μ^m as in the statement of Lemma 5.1.5. Then $\mu^n < \mu^{m+1}$ for any $m \geq m_\sigma$, see (5.26).*

Proof. Let $m \geq m_\sigma$; by Lemma 5.1.5 we know that $\mu^m < m^2$ and $\mu^{m+1} > (1-\sigma)(m+1)^2$. Therefore, we may restrict our attention to the case where $(1-\sigma)(m^1)^2 < m^2$ and μ^m, $\mu^{m+1} \in ((1-\sigma)(m+1)^2, m^2)$ since otherwise the statement follows immediately. For

$$(m,\mu) \in A := \{(m,\mu) \in \mathbb{R}^2; m \geq m_\sigma, (1-\sigma)(m+1)^2 < \mu < m^2\}\,,$$

consider the functions

$$\Gamma(m,\mu) := \frac{\sqrt{\mu+m^2}[\mu-(1-\sigma)m^2]^2}{[\mu+(1-\sigma)m^2]^2}\coth(\ell\sqrt{\mu+m^2})\,,$$
$$K(m,\mu) := \sqrt{m^2-\mu}\coth(\ell\sqrt{m^2-\mu})\,.$$

On the interval $\mu < s < \frac{\mu}{1-\sigma}$, both the positive maps

$$s \mapsto \frac{\sqrt{\mu+s}[\mu-(1-\sigma)s]^2}{[\mu+(\sigma)s]^2} \quad \text{and} \quad s \mapsto \coth(\ell\sqrt{\mu+s})$$

have strictly negative derivatives. Moreover, if g_m is as in (5.24), then we have that $\Gamma(m,\mu) = g_m(\sqrt{m^2-\mu})$ and (5.25) proves that $\mu \mapsto \Gamma(m,\mu)$ has strictly positive derivative. Summarizing,

$$\frac{\partial\Gamma}{\partial m}(m,\mu) < 0 \quad \text{and} \quad \frac{\partial\Gamma}{\partial\mu}(m,\mu) > 0 \quad \forall (m,\mu) \in A\,. \tag{5.27}$$

It is also straightforward to verify that

$$\frac{\partial K}{\partial m}(m,\mu) > 0 \quad \text{and} \quad \frac{\partial K}{\partial \mu}(m,\mu) < 0 \quad \forall (m,\mu) \in A\,. \tag{5.28}$$

Finally, put

$$\Psi(m,\mu) := \frac{K(m,\mu)}{\Gamma(m,\mu)} \quad \forall (m,\mu) \in A\,. \tag{5.29}$$

The function $m \mapsto \mu^m$ is implicitly defined by $\Psi(m,\mu^m) = 1$, see (5.21) and (5.14). By (5.27)–(5.28) we infer

$$\frac{\partial \Psi}{\partial m}(m,\mu) > 0 \quad \text{and} \quad \frac{\partial \Psi}{\partial \mu}(m,\mu) < 0 \quad \forall (m,\mu) \in A\,.$$

This proves that the map $m \mapsto \mu^m$ is increasing. □

5.1.1 Positivity of the first eigenfunction

This section is dedicated to a consequence of Theorem 5.1.1, which shows how deriving the explicit formulation of the eigenfunctions might also prove useful in analyzing theoretical properties of (5.2).

It is important to notice a special feature of the first eigenfunction of (5.2), i.e., its positivity. From Theorem 5.1.1 we derive the explicit form of the first eigenfunction of problem (5.2) which reads

$$u_{1,1}(x,y) = \left[[\mu_{1,1}^{1/2} - (1-\sigma)] \frac{\cosh\left(y\sqrt{1+\mu_{1,1}^{1/2}}\right)}{\cosh\left(\ell\sqrt{1+\mu_{1,1}^{1/2}}\right)} + \right. \\ \left. + [\mu_{1,1}^{1/2} + (1-\sigma)] \frac{\cosh\left(y\sqrt{1-\mu_{1,1}^{1/2}}\right)}{\cosh\left(\ell\sqrt{1-\mu_{1,1}^{1/2}}\right)} \right] \sin(x)\,,$$

where its corresponding eigenvalue $\lambda = \mu_{1,1}$ is the unique value $\lambda \in (1-\sigma^2,1)$ such that $\Phi^1(\lambda,\ell) = 0$, having defined $\Phi^1(\lambda,\ell)$ as

$$\Phi^1(\lambda,\ell) := +\sqrt{1-\lambda^{1/2}}\big(\lambda^{1/2} + (1-\sigma)\big)^2 \tanh\left(\ell\sqrt{1-\lambda^{1/2}}\right) + \\ -\sqrt{1+\lambda^{1/2}}\big(\lambda^{1/2} - (1-\sigma)\big)^2 \tanh\left(\ell\sqrt{1+\lambda^{1/2}}\right).$$

From (3.2), it follows that both $\mu_{1,1}^{1/2} - (1-\sigma)$ and $\mu_{1,1}^{1/2} + (1+\sigma)$ are positive, so that the eigenfunction $u_{1,1}$ is convex with respect to the variable y for all $x \in (0,\pi)$. Since the first eigenfunction is also symmetric with regard to the x axis, that is $u_{1,1}(x,-y) = u_{1,1}(x,y)$, we obtain that for all $x \in (0,\pi)$ it holds

$$\min_{y\in(-\ell,\ell)} u_{1,1}(x,y) = u(x,0) \geq 0\,,$$

so that $u_{1,1}$ is positive on Ω. s The importance of this fact resides in its relation to the positivity preserving property, which is a crucial property in the study of elliptic partial differential equations whose definition is reported below.

Definition 5.1.7. [9, Definition 3.1] Consider a well-posed general polyharmonic problem on a domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega\,, \\ B_j(x; D)u = 0 \text{ for } j = 1, \ldots, m & \text{on } \partial\Omega\,, \end{cases} \tag{5.30}$$

as described in Section 2.2. We say that (5.30) satisfies the positivity preserving property when the following implication holds for u and f satisfying (5.30):

$$f \geq 0 \implies u \geq 0\,,$$

i.e., a non-negative forcing term results in a non-negative solution u of (5.30).

One might prove using a dual cone decomposition technique, see for example [9, Theorem 3.7], that if problem (5.30) satisfies the positivity preserving property then the corresponding first eigenfunction is of one sign, so that a positive first eigenfunction constitutes a necessary conditions for the positivity preserving property to hold, indeed the positivity of the first eigenfunction is sometimes considered a "weaker form" of the positivity preserving property.

It is well known that the positivity preserving property holds for second order elliptic operators under Dirichlet boundary conditions, as a consequence of the maximum principle. For higher order polyharmonic operators the positivity preserving property has been proven only for certain domains and boundary conditions, while counter examples have been proposed to show that this property does not hold in general. One of this examples is closely related to our setting and for this reason it is worth mentioning. Consider a square domain with Dirichlet boundary conditions. Coffman [6] showed that the eigenfunction of Δ^2 relative to the first eigenvalue oscillates infinitely often when (x, y) tends to one of the corners. For this reason the result of Theorem 5.1.1 is particularly significant, since it identifies a class of problems which still exhibit a precursor of the positivity preserving property.

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