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Hamiltonian Methods in Hydrodynamics

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Summary

In the papers [8] and [7], Benjamin proposed a Hamiltonian structure for the Boussinesq model in the two dimensional context. The model describes the motion of an incompressible inviscid fluid with non uniform density, and is used to study internal wave phenomena. The Hamiltonian structure he derived has a peculiarity: in the presence of rigid walls bounding the fluid domain, it breaks down if the density is not constant along them. This singular behavior has been investigated later on by Camassa *et al.* [9]. In that study it is pointed out that the topology of initial conditions can affect the set of conserved quantities of the system. Specifically, the authors proved that for initial conditions with non uniform density along the boundary of the fluid the system retains only some of the conserved quantities it would have when the density is initially constant along the boundary. This behavior, called *topological selection* of conserved quantities, is studied in detail in the special case of a stratified fluid composed of two layers with different constant densities.

The aim of the present work is to investigate the dynamical transition between configurations having different sets of conserved quantities. From a physical point of view, a satisfactory model should allow topological changes of the flow, such as disconnection or re-connection of isopycnals (for example, think about a bubble of air that emerges from the water surface). However, the phenomenon of topological selection of the conserved quantities would seem to suggest that such transitions are not allowed by the Boussinesq equations. We study the same question addressed by Camassa et al [9] in the context of one dimensional shallow water equations with two fluid layers. Specifically we investigate the dynamical interaction of the interface between the two fluids with the upper free surface. This aim is pursued considering a special class of solutions of the model, which provides polynomial field variables of a specific degree. In this setting we prove that the above surfaces can not come in contact, nor can detach, during the time evolution of the system.

The present work is structured as follows: in the first chapter are summarized some basic notions about symmetries, conservation laws and Hamiltonian structures of partial differential equations, to be exploited in the subsequent chapters. The exposition is mainly based on the books of Olver [3] and Krasil'shchik and Vinogradov [5]. In chapter two are resumed the works of Benjamin [8] and Camassa *et al.* [9]. Finally, in chapter three is addressed the study of the shallow water model.

Contents

Chapter 1

Symmetries, Conservation Laws and Hamiltonian Structures

In this section, we introduce some terminology, mathematical tools, and theoretical results from the geometric theory of differential equations, to be used in the subsequent chapters. The material exposed here is mainly from the books of Olver [3], Chapter 2, and Krasil'shchik and Vinogradov [5].

1.1 Geometric Setting for Differential Equations

We consider a system $\mathscr{S}$ of differential equations involving p independent variables $x = (x^1, ..., x^p)$ and q dependent variables $u = (u^1, ..., u^q)$. Let us denote with $X \simeq \mathbb{R}^p$ the space of independent variables and with $U \simeq \mathbb{R}^q$ the space of dependent variables. Given any smooth function $f : X \to U$, it has

$$q\binom{p+k-1}{k}$$

different k-th order partial derivatives. We use the multi-index notation

$$\partial_J f^\alpha(x) = \frac{\partial^k f^\alpha(x)}{\partial x^{j_1} \partial x^{j_2} ... \partial x^{j_k}} \qquad 1 \le \alpha \le q$$

to denote them. In this notation, $J = (j_1, ..., j_k)$ is an *unordered* k-tuple of integers, each ranging from 1 to p, indicating which derivatives are being taken. The *order* of such a multi-index, denoted by $|J| = k$, indicates how many derivatives are being taken. Let U_k represent the space with coordinates u^α_J corresponding to $\alpha = 1, ..., q$ and all multi-indices J of order k, designed so as to represent the k-th order partial derivatives of functions from X to U. Further, set

$$U^{(n)} = U \times U_1 \times ... \times U_n$$

to be the Cartesian product space, whose coordinates represent all the derivatives of functions $u = f(x)$ of all orders up to n. A typical point in $U^{(n)}$ will be denoted by $u^{(n)}$ and has components u_J^α where $\alpha = 1, ..., n$ and J runs over all multi-indices of order from 0 to n.

Given a smooth function $u = f(x)$, there is an induced function $u^{(n)} = \mathrm{pr}^{(n)} f(x)$, called the n-th *prolongation*, or the *n-jet* of f, which is defined by the equations

$$u_J^\alpha = \partial_J f^\alpha(x)$$

The total space $X \times U^{(n)}$, whose coordinates represent the independent variables, the dipendent variables and the derivatives of the dependent variables of order up to n is called the n-th order *jet space* of the underlying space $X \times U$. Similarly, if $M \subset X \times U$ is some open subset, we let

$$M^{(n)} = M \times U_1 \times ... \times U_n$$

denote the associated n-jet space[1].

A system $\mathscr{S}$ of n-th order differential equations in p independent and q dependent variables is given as the zero locus of a smooth map $\Delta : X \times U^{(n)} \to \mathbb{R}^l$ from the jet space $X \times U^{(n)}$ to some l-dimensional Euclidean space:

$$\Delta_\nu(x, u^{(n)}) = 0 \qquad \nu = 1, ..., l$$

Let

$$\mathscr{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset X \times U^{(n)}$$

denote the subvariety of the total jet space $X \times U^{(n)}$ determined by the system of equations $\mathscr{S}$. A smooth *solution* of the given system of differential equations is a smooth function $u = f(x)$ such that

$$\Delta_\nu(x, \mathrm{pr}^{(n)} f(x)) = 0 \qquad \nu = 1, ..., l$$

In other words, a solution is a smooth function $f : X \to U$ such that the graph of its n-th prolongation lies on the subvariety $\mathscr{S}_\Delta$.

1.2 Symmetry Groups

Generally speaking, a *symmetry* of a system of differential equations $\mathscr{S}$ is a Lie group G, acting on some open subset $M \subset X \times U$ in such a way that "G transforms solutions of $\mathscr{S}$ to other solutions of $\mathscr{S}$". To be more precise, we have to explain the way a group acts on a function. The idea is as follows: starting from a function $u = f(x)$, consider its graph $\Gamma_f \subset X \times U$; let $g \in G$ and compute its action $g \cdot \Gamma_f$ on the graph of f; finally, if g is sufficiently close to the identity, by suitably shrinking the domain of definition of f, it would be possible to express the transformed graph $g \cdot \Gamma_f$ as the graph of another

[1] This construction is a greatly simplified version of the theory of jet bundles occurring in differential geometric theory of partial differential equations.

function $\tilde{f}$, that is $g \cdot \Gamma_f = \Gamma_{\tilde{f}}$. So we write $\tilde{f} = g \cdot f$ and refer to $\tilde{f}$ as the *transform* of f by g. Now we can give a rigorous definition of the concept of symmetry group of a system of differential equations.

Definition 1.2.1. Let $\mathscr{S}$ be a system of differential equations. A *symmetry group* of the system $\mathscr{S}$ is a Lie group G acting on an open subset M of the space of independent and dependent variables for the system with the property that if $u = f(x)$ is a solution of $\mathscr{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution for the system.

Suppose that G is a group of transformations acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables. There is an induced action of G on the n-jet space $M^{(n)}$, called the n-th *prolongation* of G and denoted $\mathrm{pr}^{(n)}G$. This action is defined so that it transforms the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. Consider a point $(x, u^{(n)}) \in X \times U^{(n)}$ and a transformation $g \in G$; to explicitly compute the prolonged action

$$(\tilde{x}, \tilde{u}^{(n)}) = \mathrm{pr}^{(n)} g \cdot (x, u^{(n)})$$

of g on the point $(x, u^{(n)})$, choose a function $f : X \to U$ such that $\mathrm{pr}^{(n)} f(x) = u^{(n)}$ (for example the n-th order Taylor polynomial at x having u^α_J as the coefficients). Then, by definition, the components $\tilde{u}^\alpha_J$ of the the transformed point are the values of the corresponding derivatives of $g \cdot f$ at $\tilde{x}$, that is

$$\tilde{u}^{(n)} = \mathrm{pr}^{(n)}(g \cdot f)(\tilde{x})$$

The definition of symmetry group given above is hardly effective if the task is to find symmetry groups of a system of differential equations. Fortunately there exists a connection between symmetries and invariance of the subvariety $\mathscr{S}_\Delta$ under the prolonged action of the symmetry group.

Theorem 1.2.1. *Let M be an open subset of $X \times U$ and suppose $\Delta(x, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M, with corresponding subvariety $\mathscr{S}_\Delta \subset M^{(n)}$. Suppose G is a group of transformations acting on M whose prolongation leaves $\mathscr{S}_\Delta$ invariant. Then G is a symmetry group of the system of differential equations in the sense of Definition 1.2.1.*

With the final task of constructing an algorithm to systematically find symmetry groups of differential equations, we now turn the attention to the infinitesimal generators of a transformation group. First we define the prolongation of a vector field: the idea is to consider the one-parameter group generated by the vector field, get the prolonged action of this last, and finally compute its infinitesimal generator.

Definition 1.2.2. Let $M \subset X \times U$ be open and suppose v in a vector field on M, with corresponding one-parameter group $\exp(\epsilon \mathrm{v})$. The n-th *prolongation* of v, denoted $\mathrm{pr}^{(n)}\mathrm{v}$, will be a vector field on the n-jet space $M^{(n)}$, defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(n)}(\exp(\epsilon \mathrm{v}))$. In other words,

$$\mathrm{pr}^{(n)}(\mathrm{v})\Big|_{(x, u^{(n)})} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\mathrm{pr}^{(n)}(\exp(\epsilon \mathrm{v}))(x, u^{(n)}) \right]$$

for any $(x, u^{(n)}) \in M^{(n)}$.

From the definition above, working in coordinates, it is possible to deduce a general formula, suitable for computations, for the n-th prolongation of a vector field. To express it in a compact way is useful to introduce the notion of total derivative.

Definition 1.2.3. Let $P(x, u^{(n)})$ be a smooth function of x, u and derivatives of u up to order n defined on an open subset $M^{(n)} \subset X \times U^{(n)}$. The *total derivative* of P with respect to x^i is the unique smooth function $D_i P(x, u^{(n+1)})$ defined on $M^{(n+1)}$ and depending on derivatives of u up to order $n+1$, with the property that if $u = f(x)$ is any smooth function

$$(D_i P)(x, \mathrm{pr}^{(n+1)} f(x)) = \frac{\partial}{\partial x^i}[P(x, \mathrm{pr}^{(n)} f(x))]$$

In other words, $D_i P$ is obtained from P by differentiating P with respect x^i while treating all the u^α's and their derivatives as functions of x. Higher order total derivatives are indicated as follows: if $J = (j_1, ..., j_k)$ is a k-th order multi-index, then the J-th order total derivative total derivative is denoted

$$D_J = D_{j_1} D_{j_2} ... D_{j_k}$$

In coordinates, the i-th total derivative of P has the general expression

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,i} \frac{\partial P}{\partial u^\alpha_J}$$

where, for $J = (j_1, ..., j_k)$,

$$u^\alpha_{J,i} = \frac{u^\alpha_J}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} ... \partial x^{j_k}}$$

Theorem 1.2.2. *Let*

$$\mathrm{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of v *is the vector field*

$$\mathrm{pr}^{(n)}\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{q} \sum_{|J| \leq n} \phi^J_\alpha(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_J}$$

defined on the corresponding jet space $M^{(n)}$. The coefficient functions ϕ^J_α of $\mathrm{pr}^{(n)}\mathrm{v}$ *are given by the following formula:*

$$\phi^J_\alpha(x, u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i\right) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}$$

where $u^\alpha_i = \partial u^\alpha / \partial x^i$, and $u^\alpha_{J,i} = \partial u^\alpha_J / \partial x^i$.

To get an infinitesimal criterion for a group G to be a symmetry of a given system of differential equations $\Delta(x, u^{(n)}) = 0$, we have to add a couple of hypothesis on the system of equations itself. The first require the subvariety $\mathscr{S}_\Delta$, determined by $\Delta = 0$, to be a submanifold of the jet space $X \times U^{(n)}$. In order to conclude so, by invoking the regular value theorem, we need to assume that the Jacobian matrix of Δ

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u_J^\alpha} \right)$$

to be of maximal rank at every point of $\mathscr{S}_\Delta$. A system of differential equations satisfying this hypothesis will be called of *maximal rank*. The second hypothesis is called *local solvability* and requires that for each point $(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta$ there exists a smooth solution $u = f(x)$ of the system, defined for x in a neighborhood of x_0, such that

$$u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0)$$

A system of differential equations will be called *nondegenerate* if it is of maximal rank and locally solvable.

Theorem 1.2.3. *Suppose*

$$\Delta_\nu(x, u^{(n)}) = 0 \qquad \nu = 1, ..., l$$

is a nondegenerate system of differential equations. A connected Lie group of transformations G acting on an open subset $M \subset X \times U$ is a symmetry group of the system if and only if

$$\mathrm{pr}^{(n)}\mathrm{v}(\Delta_\nu(x, u^{(n)})) = 0, \qquad \nu = 1, ..., l \qquad \textit{whenever} \qquad \Delta(x, u^{(n)}) = 0$$

for every infinitesimal generator v *of G.*

The above theorem, when combined with the prolongation formula of Theorem 1.2.2, leads to a computational algorithm that permits to find all (connected) groups of symmetry of a nondegenerate system of equations $\Delta(x, u^{(n)}) = 0$. Such algorithm is as follows: consider the generic vector field v defined on an open subset $M \subset X \times U$, expressed in coordinates as

$$\mathrm{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

Now compute its prolongation, as described in Theorem 1.2.2, and require the equation

$$\mathrm{pr}^{(n)}\mathrm{v}(\Delta_\nu(x, u^{(n)})) = 0, \qquad \nu = 1, ..., l$$

to hold. Use the condition $\Delta(x, u^{(n)}) = 0$ in the last equation to eliminate some of the derivatives of the dependent variables. The resulting equations will contain $\xi^i(x, u)$ and $\phi_\alpha(x, u)$, along with their derivatives with respect to x and u, multiplied by combinations of the u_J^α's. Now, since the coefficient functions ξ^i and ϕ_α depend only on x and u, and the resulting equations have to hold for all values of the u_J^α's, it will be obtained a system of partial differential equations in $\xi^i(x, u)$ and $\phi_\alpha(x, u)$ whose solutions leads to all the symmetry groups of the equations at hand (see Example 1.3.1 below).

1.3 Generalized Symmetries

So far we have considered only symmetry groups whose infinitesimal generators

$$\mathrm{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

have coefficient functions ξ^i and ϕ_α depending only on the variables x and u. A generalization is obtained allowing those coefficients to depend also on derivatives of the dependent variables u^α_J. The resulting object is not anymore the infinitesimal generator of a one-parameter group of transformations acting on the space $X \times U$, and its geometric interpretation will be discussed later in this paragraph. To distinguish these new objects from those treated in §1.2, we refer to the latter as *geometric symmetries*, or also *point symmetries*, while refer to the former as *generalized symmetries*.

To proceed rigorously we have to introduce some terminology. We let $\mathscr{A}$ denote the algebra of smooth functions $P(x,u^{(n)})$ depending on x, u and derivatives of u up to some finite, but unspecified, order n. The functions of $\mathscr{A}$ are called *differential functions*. If we do not care as to precisely how much derivatives of u that P depends on, we will write $P[u] = P(x,u^{(n)})$ for P, where the square brackets will serve to remind that P depends on x, u and derivatives of u. We further define $\mathscr{A}^l$ to be the vector space of l-tuples of differential functions, $P[u] = (P_1[u], ..., P_l[u])$, where each $P_j \in \mathscr{A}$.

Definition 1.3.1. A *generalized vector field* will be a (formal) expression of the form

$$\mathrm{v} = \sum_{i=1}^{p} \xi^i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u]\frac{\partial}{\partial u^\alpha}$$

in which ξ^i and ϕ_α are smooth differential functions.

Treating a generalized vector field v as it was an ordinary one, we can define its n-th prolongation just as it is described in Theorem 1.2.2:

$$\mathrm{pr}^{(n)}\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{q}\sum_{|J|\le n} \phi^J_\alpha[u]\frac{\partial}{\partial u^\alpha_J}$$

where the coefficients are determined by the formula

$$\phi^J_\alpha = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u^\alpha_i\right) + \sum_{i=1}^{p}\xi^i u^\alpha_{J,i}$$

with the same notation as before. Since all the prolongations of v have the same general expression for their coefficient functions ϕ^J_α, it is helpful to pass to the "infinite" prolongation, and take care af all the derivatives at once. Specifically, given a generalized vector field v, its *infinite prolongation* (or *prolongation* for short) is the formally infinite sum

$$\mathrm{pr}\,\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{q}\sum_{J} \phi^J_\alpha\frac{\partial}{\partial u^\alpha_J}$$

where the sum now extends over all multi-indices J of order $k \geq 0$. Note that questions on the "convergence" of the former infinite sum never arise because, if $P \in \mathscr{A}$ is any differential function, the evaluation $\mathrm{pr}\,\mathrm{v}(P)$ is composed of finitely many terms, in that $P[u]$ depends only on finitely many derivatives of u.

Definition 1.3.2. Let

$$\Delta_\nu(x, u^{(n)}) = 0 \qquad \nu = 1, .., l$$

be an n-th order system of differential equations. The k-th *prolongation* of this system is the $(n+k)$-th order system of differential equations

$$\Delta^{(k)}(x, u^{(n+k)}) = 0$$

obtained by differentiating the equations in Δ in all possible ways up to order k. In other words, $\Delta^{(k)}$ consists of the equations

$$D_J \Delta_\nu(x, u^{(n+k)}) = 0$$

where $\nu = 1, ..., l$ and J runs over all multi-indices of orders $0 \leq |J| \leq k$.

Definition 1.3.3. A generalized vector field v is a *generalized infinitesimal symmetry* of a system of differential equations

$$\Delta_\nu[u] = \Delta_\nu(x, u^{(n)}) = 0 \qquad \nu = 1, ..., l$$

if and only if

$$\mathrm{pr}\,\mathrm{v}[\Delta_\nu] = 0 \qquad \nu = 1, ..., l$$

for every smooth solution $u = f(x)$.

Remark 1.3.1. For technical reasons, it will be required the systems of differential equations considered to be *totally nondegenerate* unless stated otherwise; namely, they and all their prolongations are of maximal rank and locally solvable.

Among all the generalized vector fields, those in which the coefficients $\xi^i[u]$ of the $\partial/\partial x^i$ are zero play a distinguished role.

Definition 1.3.4. Let $Q[u] = (Q_1[u], ..., Q_q[u]) \in \mathscr{A}^q$ be a q-tuple of differential functions. The generalized vector field

$$\mathrm{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}$$

is called an *evolutionary vector field*, and Q is called its *characteristic*.

According to the prolongation formula of Theorem 1.2.2, the prolongation of an evolutionary vector field takes the particular simple form

$$\mathrm{pr}\,\mathrm{v}_Q = \sum_{\alpha,J} D_J Q_\alpha \frac{\partial}{\partial u^\alpha_J}$$

Any generalized vector field v has an associated *evolutionary representative* v_Q in which the characteristic Q has entries

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \qquad \alpha = 1, ..., q \tag{1.1}$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$. The motivation for considering such associated evolutionary representative is that it determine essentially the same symmetry of the given evolutionary vector field.

Proposition 1.3.1. *A generalized vector field* v *is a symmetry of a system of differential equations if and only if its evolutionary representative* v_Q *is.*

Consider an evolutionary vector field v_Q such that its characteristic Q vanishes on all solutions of the given system Δ. Then also its prolongation pr v vanishes on all solutions, so v_Q is automatically a generalized symmetry of the system. Such symmetries are called *trivial* and we are primarily interested in nontrivial symmetries. A generalized symmetry will be called *trivial* if its evolutionary form is. Moreover two generalized symmetries v and $\tilde{\mathrm{v}}$ are called *equivalent* if their difference $\mathrm{v} - \tilde{\mathrm{v}}$ is a trivial symmetry. This induces an equivalence relation on the space of generalized symmetries of the given system; by a symmetry of the system we will mean a whole equivalence class of generalized symmetries, each differing from the other by a trivial symmetry.

The computation of generalized symmetries of a given system of differential equations proceeds in the same way as the earlier computations of geometric symmetries, but with some added features. To simplify the computations, is desirable to put the symmetry in evolutionary form v_Q. One must then a priori fix the order of derivatives on which the characteristic $Q(x, u^{(m)})$ may depend. Finally one must deal with the occurrence of trivial symmetries; the easiest way to handle these is to eliminate any superfluous derivatives in Q by substitution using the prolongations of the system.

We now turn the attention on the geometric interpretation of the generalized vector fields. As was anticipated before, if v is a generalized vector field, its flow $\exp(\epsilon \mathrm{v})$ can not act geometrically on the underlying domain $M \subset X \times U$ since the coefficients of v depends on derivatives of u, which are also being transformed. Instead, there is a natural action of $\exp(\epsilon \mathrm{v})$ on a space of smooth functions, defined as follows. First replace v by its evolutionary representative v_Q. Then we compute the flow of v_Q as the solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial \epsilon} = Q(x, u^{(m)}) \\ u(x,0) = f(x) \end{cases} \tag{1.2}$$

where Q is the characteristic of v. The solution $u(x, \epsilon)$ will determine the group action[2]:

$$\big(\exp(\epsilon \mathrm{v}_Q) f\big)(x) \equiv u(x, \epsilon)$$

[2]We are not involved here with technicalities about existence and uniqueness of solutions, and assume these to hold in some space of smooth functions.

The basic symmetry property of the one-parameter group $\exp(\epsilon\mathrm{v})$ generated by the evolutionary vector field v is the following.

Theorem 1.3.1. *The evolutionary vector field* $\mathrm{v} = \mathrm{v}_Q$ *is a symmetry of the system of differential equations* Δ *if and only if the corresponding group* $\exp(\epsilon\mathrm{v})$ *transforms solutions of the system to other solutions.*

Remark 1.3.2. The proof of Theorem 1.3.1 rely upon various technical assumptions, not stated here. See Olver [3] §5.1 for details.

As with ordinary vector fields, there is a Lie bracket between generalized vector fields, which arises from their prolongation.

Definition 1.3.5. Let v and w be generalized vector fields. Their *Lie bracket* $[\mathrm{v}, \mathrm{w}]$ is the unique generalized vector field satisfying

$$\mathrm{pr}\,[\mathrm{v}, \mathrm{w}](P) = \mathrm{pr}\,\mathrm{v}\big(\mathrm{pr}\,\mathrm{w}(P)\big) - \mathrm{pr}\,\mathrm{w}\big(\mathrm{pr}\,\mathrm{v}(P)\big)$$

for all differential functions $P \in \mathscr{A}$.

Proposition 1.3.2. *Let* v_Q *and* v_R *be evolutionary vector fields. Then their Lie bracket* $[\mathrm{v}_Q, \mathrm{v}_R] = \mathrm{v}_S$ *is also an evolutionary vector field with characteristic*

$$S = \mathrm{pr}\,\mathrm{v}_Q(R) - \mathrm{pr}\,\mathrm{v}_R(Q)$$

where $\mathrm{pr}\,\mathrm{v}_Q$ *acts component-wise on* $R \in \mathscr{A}$*, with entries* $\mathrm{pr}\,\mathrm{v}_Q(R_k)$*, and conversely. Moreover, if* v *and* w *are generalized vector fields with characteristics* Q *and* R *respectively, then their Lie bracket* $[\mathrm{v}, \mathrm{w}]$ *has characteristic* S *as given by the preceding formula.*

In components, the Lie bracket between two evolutionary vector fields v_Q and v_R has the following expression

$$[\mathrm{v}_Q, \mathrm{v}_R] = \sum_{\alpha,\nu=1}^{q} \sum_J \left(D_J Q_\nu \frac{\partial R_\alpha}{\partial u_J^\nu} - D_J R_\nu \frac{\partial Q_\alpha}{\partial u_J^\nu} \right) \frac{\partial}{\partial u^\alpha}$$

The Lie bracket between generalized vector fields has the usual properties of bilinearity, skew-symmetry and Jacobi identity. This, in light of Definition 1.3.5, makes the space of generalized symmetries of a nondegenerate system of differential equations a Lie algebra.

We now turn the attention to the particular case of *evolution equations*, that is equations of the type

$$\frac{\partial u}{\partial t} = P[u]$$

where $P[u] \in \mathscr{A}^q$ depends on $x \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ and x-derivatives of u only. Consider the evolutionary symmetry v_Q with characteristic $Q[u]$ which, in principle, can depend on time derivatives of u. Substituting according to the equation and its prolongations, is readily seen that the symmetry v_Q must be *equivalent* to one whose characteristic depends only on x, u and x-derivatives of u. For a system of evolution equations we have $\Delta[u] = u_t - P[u]$, so that the condition for a generalized vector field (whose characteristic can be supposed to not depend on time derivatives of u) to be an infinitesimal symmetry of the system reads

$$D_t Q_\nu = \mathrm{pr}\,\mathrm{v}_Q(P_\nu), \qquad \nu = 1, ..., q$$

This condition can be expressed in a more succinct form as the following proposition shows.

Proposition 1.3.3. *An evolutionary vector field* v_Q *is a symmetry of the system of evolution equations* $u_t = P[u]$ *if and only if*

$$\frac{\partial \mathrm{v}_Q}{\partial t} + [\mathrm{v}_P, \mathrm{v}_Q] = 0 \tag{1.3}$$

holds identically in $(x, t, u^{(m)})$. *(Here* $\partial \mathrm{v}_Q / \partial t$ *denotes the evolutionary vector field with characteristic* $\partial Q/\partial t$.*)*

Example 1.3.1. In order to illustrate how the above theory works in practice, we compute the symmetries of the inviscid Hopf equation

$$\Delta := u_t + u u_x = 0$$

The space of independent variables is $X \simeq \mathbb{R}^2$ with coordinates x, t, while the space of dependent variables is $X \simeq \mathbb{R}$ with coordinate u. Suppose for the moment we are interested in geometric symmetries only. That is, we look for infinitesimal generators of the form

$$\mathrm{v} = \alpha(x, t, u)\partial_x + \phi(x, t, u)\partial_u$$

In order to apply Theorem 1.2.3, we have to compute the first prolongation of the above vector field:

$$\mathrm{pr}^{(1)}\mathrm{v} = \mathrm{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t}$$

where

$$\begin{aligned} \phi^x =& D_x\phi - u_x D_x \alpha - u_t D_x \tau = \\ &= \phi_x + \phi_u u_x - (\alpha_x + \alpha_u u_x)u_x - (\tau_x + \tau_u u_x)u_t \end{aligned}$$

$$\begin{aligned} \phi^x =& D_t\phi - u_x D_t \alpha - u_t D_t \tau = \\ &= \phi_t + \phi_u u_t - (\alpha_t + \alpha_u u_t)u_x - (\tau_t + \tau_u u_t)u_t \end{aligned}$$

As prescribed by Theorem 1.2.3, v is a symmetry if and only if $\mathrm{pr}^{(1)}\mathrm{v}(\Delta) = 0$ whenever $\Delta = 0$. This condition leads to the following equation

$$\begin{aligned} \phi u_x + (\phi_x + \phi_u u_x - (\alpha_x + \alpha_u u_x)u_x + (\tau_x + \tau_u u_x)uu_x)u + \\ + \phi_t - \phi_u uu_x - (\alpha_t - \alpha_u uu_x) + (\tau_t - \tau_u uu_x)uu_x = 0 \end{aligned}$$

Since the coefficient functions α, τ, ϕ do not depend on u_x, which can assume arbitrary values, all the coefficients of the above u_x-polynomial must be zero. This leads to the following system of equations, to be solved for α, τ, ϕ:

$$\begin{cases} \phi_x u + \phi_t = 0 \\ \phi - u\alpha_x + u^2\tau_x - \alpha_t + u\tau_t = 0 \\ -u\alpha_u + \tau_u u^2 + \alpha_u u - \tau_u u^2 = 0 \end{cases}$$

Clearly, the third equation is trivially satisfied, so the the system consists of two equations only. From the first, we have

$$\phi(x,t,u) = \varphi(x - ut, u)$$

for some functions φ, ψ. One way to proceed further is to suppose $\tau \equiv 0$. As we see in the following, this does not affect the generality of the solution. By doing so, the second equation gives

$$\alpha(x,t,u) = t\varphi(x - ut, u) + \psi(x - ut, u)$$

and we end with the following general form for the infinitesimal generators

$$\mathrm{v} = (\psi(x - ut, u) + t\varphi(x - ut, u))\partial_x + \varphi(x - ut, u)\partial_u \tag{1.4}$$

To this point we can claim to have found all the geometric symmetries of the system at hand $\Delta = 0$. The natural next step is to compute generalized symmetries. Although we can proceed, at least in principle, like as we did for geometric symmetries, allowing α, τ, ϕ to depend on derivatives of u up to some order, this would lead to very cumbersome computations. Moreover, such approach has the drawback of not easily allowing to classify symmetries: in the above computations is not evident that we can take τ to vanish without limitations on the solution. Both these difficulty are addressed by working with infinitesimal generators in evolutionary form, and using Proposition 1.3.3. Suppose we are looking for generalized symmetries of first order, i.e. with infinitesimal generators of the form

$$\mathrm{v}_Q = Q(x,t,u,u_x)\partial_u$$

As pointed out above in this section, we can always take the characteristic Q to not depend on time-derivatives of u. Let us denote $P = -uu_x$, so that the system $\Delta = 0$ writes in the form of evolution equation:

$$u_t = P$$

Thus, according to equation (1.3), the characteristic Q of a candidate symmetry must satisfy

$$\frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} - u_x^2\frac{\partial Q}{\partial u_x} = -u_x Q$$

The solution to this equation is found via method of characteristics as

$$Q(x,t,u,u_x) = u_x R(x - ut, u, t - \tfrac{1}{u_x})$$

where R is an arbitrary function. We now ask which of the above characteristics correspond to geometric symmetries. For this to be true, Q needs to be of the form

$$Q = \phi(x,t,u) - u_x\alpha(x,t,u) + uu_x\tau(x,t,u)$$

In particular, it has to be affine in u_x. Correspondingly R has to be affine in its third argument, and we get the general expression

$$\begin{aligned} Q(x,t,u,u_x) &= u_x\xi(x - ut, u) + (tu_x - 1)\eta(x - ut, u) \\ &= -\eta(x - ut, u) + u_x(\xi(x - ut, u) + t\eta(x - ut, u)) \end{aligned} \tag{1.5}$$

One vector field having that as characteristic is, for example

$$\mathrm{v} = -(\xi(x-ut,u) + t\eta(x-ut,u))\partial_x - \eta(x-ut,u)\partial_u \tag{1.6}$$

which is precisely (1.4), found with the other method. But now, we can argue why this is the most general geometric symmetry as claimed above. Consider another symmetry

$$\mathrm{v}' = \alpha\partial_x + \tau\partial_t + \phi\partial_u$$

with characteristic $Q' = \phi - \alpha u_x - \tau u_t$. In light of Proposition 1.3.1, it corresponds to the evolutionary symmetry $\mathrm{v}_{Q'}$. But the characteristic (1.5) is uniquely determined, so $\mathrm{v}_{Q'}$ has to be equivalent to v_Q. This means that the difference $Q - Q'$ must vanish on solutions (i.e. whenever $u_t = -uu_x$). From this condition we see that it must hold

$$\begin{cases} \phi = -\eta \\ u\tau - \alpha = \xi + t\eta \end{cases}$$

Thus, for fixed ξ, η, there is quite a lot of freedom in the choice of the coefficients α, τ, but, whatever they are, they determine the *same* symmetry. So it is not restrictive to suppose $\tau = 0$, as in (1.6).

1.4 Conservation Laws

Given a system of differential equations $\Delta[u] = 0$, a *conservation law* is a divergence expression

$$\mathrm{Div}P = 0$$

which vanishes for all solutions $u = f(x)$ of the system. Here

$$P = (P_1[u], ..., P_p[u])$$

is a p-tuple of smooth differential functions of x, u and derivatives of u, and $\mathrm{Div}P = D_1P_1 + ... + D_pP_p$ is the *total divergence*. For the case of a system of evolution equations,

$$u_t = Q[u] \tag{1.7}$$

one of the independent variables is distinguished as the time t, and the remaining variables $x = (x^1, ..., x^p)$ are called spatial variables. In this case a conservation law takes the form

$$D_tT + \mathrm{Div}X = 0 \tag{1.8}$$

in which Div denotes the "spatial" divergence of X with respect to $x^1, ..., x^p$. The *conserved density*, T, and the associated *flux*, $X = (X_1, ..., X_p)$, are functions of x, t, u and the derivatives of u with respect to both x and t.

Now let $\Omega \subset \mathbb{R}^p$ be some spatial domain, and $u = f(x,t)$ a smooth solution of a system of evolution equations (1.7) defined on Ω. Consider the time dependent functional

$$\mathscr{T}[t; f(\,\cdot\,, t)] = \int_\Omega T(x, t, \mathrm{pr}^{(n)} f(x,t))dx \tag{1.9}$$

Taking its time derivative, and using equation (1.8), we get[3]

$$\frac{d}{dt}T[t; f(\,\cdot\,,t)] = \int_\Omega D_t T(x,t,\mathrm{pr}^{(n+1)}f(x,t))dx = -\int_\Omega \mathrm{Div}X(x,t,\mathrm{pr}^{(n+1)}f(x,t))dx$$

Hence, by the divergence theorem, it follows that the time derivative of the quantity (1.9) depends only on the boundary conditions imposed on $\partial\Omega$. In particular, for some classes of solutions[4], the evaluation $\mathscr{T}[t; f(\,\cdot\,,t)]$ does not depend explicitly on t, and is therefore referred to as a *conserved quantity* (or a *constant of the motion*).

The total time derivative $D_t T$ can be made explicit as

$$D_t T = \partial_t T + \sum_{\alpha,J}\frac{\partial T}{\partial u_J^\alpha}\frac{\partial u_J^\alpha}{\partial t} = \partial_t T + \sum_{\alpha,J} D_J Q^\alpha \frac{\partial T}{\partial u_J^\alpha} = \partial_t T + \mathrm{pr}\,\mathrm{v}_Q(T)$$

where we substituted for $\partial u_J^\alpha/\partial t$ according to the system of evolution equations (1.7) and its prolongations. Hence, the condition for T to be a conserved density can be stated as

$$\partial_t T + \mathrm{pr}\,\mathrm{v}_Q(T) + \mathrm{Div}X = 0 \tag{1.10}$$

1.5 Hamiltonian Systems

To each differential function $L \in \mathscr{A}$ we can associate a (time-depending) functional $\mathscr{L}$, called *Lagrangian*, acting on smooth functions $u = f(x,t)$ as

$$\mathscr{L}[t; f(\,\cdot\,,t)] = \int_\Omega L(x,t,\mathrm{pr}^{(n)}f(x,t))dx$$

where $\Omega \subset X$ is some spatial domain. Provided we ignore boundary contributions, a second function $\tilde{\mathscr{L}} \in \mathscr{A}$ will determine the same functional, i.e.

$$\int_\Omega L[u]dx = \int_\Omega \tilde{L}[u]dx$$

if and only if it differs from L by a total divergence. This motivates the introduction of an equivalence relation among differential functions. The idea is to regard as equivalent two differential functions that define the same Lagrangian. Accordingly, given two differential functions L and $\tilde{L}$, we say them to be *equivalent* if their difference is a total divergence:

$$L \sim \tilde{L} \iff L - \tilde{L} = \mathrm{Div}P \tag{1.11}$$

for some p-tuple of differential functions $P \in \mathscr{A}^p$. Hence, the space of Lagrangians, denoted by $\mathscr{F}$, is naturally identified with the quotient of the space $\mathscr{A}$ of differential

[3]Note that if T and X depend on derivatives of u of order up to n, then D_tT and $\mathrm{Div}X$ depend on derivatives up to order $n+1$.

[4]That is, for all solutions $u = f(x,t)$ such that $X(x,t,\mathrm{pr}^{(n)}f(x,t)) \to 0$ as $x \to \partial\Omega$ whenever Ω is bounded, and for $|x| \to \infty$ otherwise.

functions under the equivalence relation above. In other words, $\mathscr{F}$ is isomorphic to the quotient vector space of $\mathscr{A}$ under the subspace of total divergences:

$$\mathscr{F} \simeq {}^{\mathscr{A}}\!\big/_{\mathrm{Div}(\mathscr{A}^p)}$$

The natural projection from $\mathscr{A}$ to $\mathscr{F}$ will be suggestively denoted by an integral sign, so that $\int L dx \in \mathscr{F}$ stands for the functional (or equivalence class) associated to the differential function $L \in \mathscr{A}$.

Remark 1.5.1. In order the above notion of equivalence to make sense, we must ensure the divergence appearing in (1.11) to have zero integral on the given domain. If not stated otherwise, we shall denote with $\mathscr{A}$ the subalgebra of differential functions $P[u] = P(x, t, u^\alpha_J)$ which, together with all their derivatives, vanish when $u^\alpha_J = 0$ for all α, J. Furthermore, we assume the functionals of $\mathscr{F}$ to act on a class of functions $u = f(x, t)$ which vanish sufficiently rapidly near the boundary.

Definition 1.5.1. Let $\mathscr{L}[u]$ be a Lagrangian as above. The *variational derivative* of $\mathscr{L}$ is the unique q-tuple

$$\delta\mathscr{L}[u] = (\delta_1\mathscr{L}[u], ..., \delta_q\mathscr{L}[u])$$

with the property that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathscr{L}[f + \epsilon\eta] = \int_\Omega \delta\mathscr{L}[f(x)] \cdot \eta(x) dx$$

whenever $u = f(x)$ is a smooth function defined on Ω, and $\eta(x) = (\eta^1(x), ..., \eta^q(x))$ is a smooth function with compact support in Ω. The component $\delta_\alpha\mathscr{L} = \delta\mathscr{L}/\delta u^\alpha$ is called the *variational derivative of* $\mathscr{L}$ *with respect to* u^α.

Definition 1.5.2. For $1 \leq \alpha \leq q$, the α-th *Euler operator* is given by

$$\mathsf{E}_\alpha = \sum_J (-1)^{|J|} D_J \frac{\partial}{\partial u^\alpha_J}$$

the sum extending on all multi-indices J of order $|J| \geq 0$. Note that to apply E_α to any differential function $L[u]$ only finitely many terms in the summation are required, since L depends on only finitely many derivatives u^α_J.

Performing an integration by parts, is easily seen that the variational derivative of a give functional $\mathscr{L}[u] = \int L[u] dx$ is found by applying the Euler operator to L:

$$\delta\mathscr{L} = \mathsf{E}(L)$$

where $\mathsf{E}(L) = (\mathsf{E}_1(L), ..., \mathsf{E}_q(L))$.

1.5.1 Poisson structures

There is a one-to-one correspondence between differential functions and (non linear) differential operators acting on $C^\infty(X, U)$: if $F \in \mathscr{A}^r$ is a differential function, its corresponding differential operator $\mathscr{D}_F$ is

$$\mathscr{D}_F(f) = F \circ \mathrm{pr}\, f$$

where pr f stands for the infinite prolongation of f (also called its *infinite jet*). Among all linear differential operators, those which can be expressed in terms of total derivatives play a distinguished role. Following Krasil'shchik and Vinogradov [5], we call them $\mathscr{C}$-*differential operators*. Accordingly, a linear operator $\mathscr{D} : \mathscr{A} \to \mathscr{A}$ is called a $\mathscr{C}$-differential operator of order k if it has the form

$$\mathscr{D} = \sum_{|J|\le k} R_J[u] D_J \tag{1.12}$$

where $a_J \in \mathscr{A}$ are given differential functions. To such $\mathscr{C}$-differential operator is associated its (formal) *adjoint* $\mathscr{D}^* : \mathscr{A} \to \mathscr{A}$, such that

$$P \cdot \mathscr{D}Q \sim Q \cdot \mathscr{D}^* P \tag{1.13}$$

for all differential functions $P, Q \in \mathscr{A}$; it is a $\mathscr{C}$-differential operator too, given by

$$\mathscr{D}^* = \sum_J (-1)^{|J|} D_J \circ R_J[u] \tag{1.14}$$

In the composition appearing in this formula, $R_J[u]$ is understood as the multiplication operator: for any differential function $Q \in \mathscr{A}$,

$$\mathscr{D}^*(Q) = \sum_J (-1)^{|J|} D_J(R_J Q)$$

Definition (1.12) is naturally extended to the multidimensional case: a $\mathscr{C}$-differential operator $\mathscr{D} : \mathscr{A}^n \to \mathscr{A}^m$ is a matrix operator with entries

$$\mathscr{D}^{\alpha\beta} = \sum_{|J|\le k} R_J^{\alpha\beta}[u] D_J \tag{1.15}$$

where $R^{\alpha\beta} \in \mathscr{A}$. Its adjoint is defined accordingly with formula (1.13), as the $\mathscr{C}$-differential operator $\mathscr{D}^* : \mathscr{A}^m \to \mathscr{A}^n$ with entries

$$(\mathscr{D}^*)^{\alpha\beta} = (\mathscr{D}^{\beta\alpha})^* = \sum_J (-1)^{|J|} D_J \circ R_J[u]^{\beta\alpha} \tag{1.16}$$

Let $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$ be a $\mathscr{C}$-differential operator (that may depend on x, u and derivatives of u). Given two Lagrangians $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, we define their *Poisson bracket* as the functional

$$\{\mathscr{P}, \mathscr{Q}\} = \int \mathsf{E}(P) \cdot \mathscr{D}\mathsf{E}(Q) dx = \int \sum_{\alpha,\beta} \mathsf{E}_\alpha(P) \mathscr{D}^{\alpha\beta} \mathsf{E}_\beta(Q) dx \tag{1.17}$$

Definition 1.5.3. A $\mathscr{C}$-differential operator $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$ is called *Hamiltonian* if its Poisson bracket satisfies the conditions of *skew-symmetry*

$$\{\mathscr{P}, \mathscr{Q}\} = -\{\mathscr{Q}, \mathscr{P}\}$$

and the *Jacobi identity*

$$\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} + \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} + \{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} = 0$$

for all functionals $\mathscr{P}, \mathscr{Q}, \mathscr{R} \in \mathscr{F}$.

The above definition is impractical to check if a given operator $\mathscr{D}$ is indeed Hamiltonian. The following result can help checking if the skew-symmetry property is met.

Proposition 1.5.1. *Let $\mathscr{D}$ be a $\mathscr{C}$-differential operator. The Poisson bracket it defines is skew-symmetric if and only if $\mathscr{D}$ is skew-adjoint:*

$$\mathscr{D} = -\mathscr{D}^*$$

Proof. Let $\mathscr{P} = \int P dx$ and $\mathscr{Q} = \int Q dx$ be Lagrangians. Then, by definition (1.13) of the adjoint operator, we have

$$\sum_{\alpha\beta} \mathsf{E}_\alpha(P)\mathscr{D}^{\alpha\beta}\mathsf{E}_\beta(Q) \sim -\sum_{\alpha\beta} \mathsf{E}_\alpha(Q)\mathscr{D}^{\alpha\beta}\mathsf{E}_\beta(P)$$

Integrating this equation, we arrive to

$$\{\mathscr{P}, \mathscr{Q}\} = -\{\mathscr{Q}, \mathscr{P}\}$$

For the converse, assume that skew-symmetry holds. Then we have

$$\int \mathsf{E}(P) \cdot (\mathscr{D} + \mathscr{D}^*)\mathsf{E}(Q) dx = 0$$

for all differential functions $P, Q \in \mathscr{A}$. Hence it follows $\mathscr{D} + \mathscr{D}^* = 0$. $\square$

To check for the Jacobi identity is usually much more laborious. However, also in this case, there is a criterion simplifying the task, but to be stated we will need some further notation. To begin, we define a *vertical multi-vector* as an alternating multi-linear map from $\mathscr{A}^q$ to $\mathscr{A}$; we denote by θ^α_J the "elementary" vertical uni-vector, acting as

$$\langle \theta^\alpha_J; P \rangle = D_J P^\alpha, \qquad P \in \mathscr{A}^q$$

Any general vertical k-vector can be expressed as a finite sum

$$\hat{\Theta} = \sum_{\boldsymbol{\alpha},\boldsymbol{J}} R^{\boldsymbol{\alpha}}_{\boldsymbol{J}}[u] \theta^{\alpha_1}_{J_1} \wedge ... \wedge \theta^{\alpha_1}_{J_1}, \qquad R^{\boldsymbol{\alpha}}_{\boldsymbol{J}}[u] \in \mathscr{A}$$

whose action, on a k-tuple of differential functions, is defined accordingly as

$$\langle \tilde{\Theta}; P_1, ..., P_k \rangle = \sum_{\boldsymbol{\alpha},\boldsymbol{J}} R^{\boldsymbol{\alpha}}_{\boldsymbol{J}} \det(D_{J_i} P^{\alpha_i}_j), \qquad P_j \in \mathscr{A}^q$$

The total derivatives act on vertical k-vectors as Lie derivatives [3], that is

$$D_i(\theta^\alpha_J) = \theta^\alpha_{J,i}$$

where, as above, J, i stands for the augmented multi-index $J \cup \{i\}$.

As it was done for differential functions above, we may introduce an equivalence relation on the space $\hat{\Lambda}_k$ of vertical k-vectors by identifying two of them if their difference is a total divergence: in other words, for $\hat{\Phi}, \hat{\Psi} \in \hat{\Lambda}_k$,

$$\hat{\Phi} \sim \hat{\Psi} \iff \hat{\Phi} - \hat{\Psi} = \sum_{i=1}^p D_i \hat{\Theta}_i, \qquad \hat{\Theta}_i \in \hat{\Lambda}_k$$

Next, we define a *functional k-vector* as an alternating k-linear map from the space of q-tuples of differential functions $\mathscr{A}^q$ to the space of Lagrangians $\mathscr{F}$ [5]. The space Λ^*_k of

[5] See Olver [3] Ch. 7 for a basic exposition, and [6] §1.8 for a more sophisticated one.

functional k-vectors is readily identified with the quotient of the space of vertical k-vectors under the image of total divergence:

$$\Lambda_k^* \simeq \hat{\Lambda}_k \big/ \mathrm{Div}(\hat{\Lambda}_k)^p$$

To a given $\hat{\Theta} \in \hat{\Lambda}_k$, is associated the functional k-vector (or equivalence class)

$$\Theta = \int \hat{\Theta} dx$$

By means of integration by parts, is always possible to reduce any functional k-vector into a *canonical form*. For example, given a functional uni-vector

$$\gamma = \int \sum_{\alpha=1}^{q} R_J^\alpha \theta_J^\alpha dx$$

we can express it as[6]

$$\gamma = \int \sum_{\alpha=1}^{q} R_\alpha \theta^\alpha dx, \qquad R_\alpha = \sum_J (-1)^{|J|} D_J R_J^\alpha$$

Similarly, any functional bi-vector $\Theta \in \Lambda_2^*$ has the canonical form

$$\Theta = \frac{1}{2} \int \Big[\sum_{\alpha,\beta=1}^{q} \theta^\alpha \wedge \mathscr{D}_{\alpha\beta} \theta^\beta \Big] dx \tag{1.18}$$

where $\mathscr{D} = (\mathscr{D}_{\alpha\beta})$ is some skew-adjoint $\mathscr{C}$-differential operator. Finally, let $\mathrm{v}_{\mathscr{D}\theta}$ denote a formal evolutionary vector field whose characteristic is the q-tuple

$$(\mathscr{D}\theta)_\alpha = \sum_{\beta=1}^{q} \mathscr{D}_{\alpha\beta} \theta^\beta$$

of vertical uni-vectors. We define its prolonged action on the space of vertical multi-vectors by setting

$$\mathrm{pr}\, \mathrm{v}_{\mathscr{D}\theta}(\theta_J^\alpha) = 0$$

and extending it to act as a derivation on general vertical multi-vectors.

Theorem 1.5.1. *Let $\mathscr{D}$ be a skew-adjoint $\mathscr{C}$-differential operator and Θ, given by (1.18), its corresponding functional bi-vector. Then $\mathscr{D}$ is Hamiltonian if and only if*

$$\mathrm{pr}\, \mathrm{v}_{\mathscr{D}\theta}(\Theta) = 0$$

[6] For this reason, we may identify the space of functionals uni-vectors Λ_1^* with that of evolutionary vector fields, which in turn is isomorphic with $\mathscr{A}^q$.

Example 1.5.1 (Euler equations)**.** The Euler equations for a two dimensional, incompressible fluid flow can be cast as a single evolution equation,

$$\zeta_t = \zeta_x \psi_y - \zeta_y \psi_x \tag{1.19}$$

involving the vorticity ζ and the stream function ψ; these are related by

$$\zeta = -\Delta\psi$$

Equation (1.19) has the form of a Hamiltonian system,

$$\zeta_t = \mathscr{D}\mathsf{E}(H)$$

for the $\mathscr{C}$-differential operator

$$\mathscr{D} = \zeta_x D_y - \zeta_y D_x \tag{1.20}$$

and the Hamiltonian functional

$$\mathscr{H} = \int H dx = \int \tfrac{1}{2}|\nabla\psi|^2 dx$$

representing the total energy of the fluid. One tricky point here is to recognize that $\mathsf{E}(H) \equiv \mathsf{E}_\zeta(H) = \psi$. This issue is deferred until section §2.1, where it will be addressed in a more general setting. Now we show that the operator (1.20) is indeed Hamiltonian. The skew-adjointness is easily proved: by formula (1.13),

$$\begin{aligned}\mathscr{D}^* P &= -D_y(\zeta_x P) + D_x(\zeta_y P) = -\zeta_{xy}P - \zeta_x D_y(P) + \zeta_{xy}P + \zeta_y D_x(P) = \\ &= -\zeta_x D_y(P) + \zeta_y D_x(P) = -\mathscr{D}P\end{aligned}$$

The functional bi-vector (1.18) corresponding to $\mathscr{D}$ is

$$\Theta = \frac{1}{2}\int \theta \wedge \mathscr{D}\theta dx = \frac{1}{2}\int \theta \wedge (\zeta_x\theta_y - \zeta_y\theta_x) dxdy = \frac{1}{2}\int (\zeta_x \theta\wedge\theta_y - \zeta_y\theta\wedge\theta_x)dxdy$$

Then,

$$\begin{aligned}\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\Theta) &= \frac{1}{2}\int \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\zeta_x\theta\wedge\theta_y - \zeta_y\theta\wedge\theta_x)dxdy = \\ &= \frac{1}{2}\int (\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\zeta_x)\wedge\theta\wedge\theta_y - \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\zeta_y)\wedge\theta\wedge\theta_x)dxdy = \\ &= \frac{1}{2}\int \left(\sum_J \frac{\partial\zeta_x}{\partial\zeta_J} D_J(\mathscr{D}\theta)\wedge\theta\wedge\theta_y - \sum_J \frac{\partial\zeta_y}{\partial\zeta_J}D_J(\mathscr{D}\theta)\wedge\theta\wedge\theta_x\right)dxdy = \\ &= \frac{1}{2}\int (D_x(\mathscr{D}\theta)\wedge\theta\wedge\theta_y - D_y(\mathscr{D}\theta)\wedge\theta\wedge\theta_x)dxdy = \\ &= \frac{1}{2}\int ((\zeta_{xx}\theta_y + \zeta_x\theta_{xy} - \zeta_{xy}\theta_x - \zeta_y\theta_{xx})\wedge\theta\wedge\theta_y + \\ &\qquad - (\zeta_{xy}\theta_y + \zeta_x\theta_{yy} - \zeta_{yy}\theta_x - \zeta_y\theta_{xy})\wedge\theta\wedge\theta_x)dxdy = \\ &= \frac{1}{2}\int (\zeta_x(\theta_{xy}\wedge\theta\wedge\theta_y - \theta_{yy}\wedge\theta\wedge\theta_x) + \\ &\qquad + \zeta_y(\theta_{xy}\wedge\theta\wedge\theta_x - \theta_{xx}\wedge\theta\wedge\theta_y))dxdy\end{aligned}$$

Integrating by parts the second and the fourth terms, we get

$$\int -\zeta_x\theta_{yy}\wedge\theta\wedge\theta_x dxdy = \int(\zeta_{xy}\theta_y\wedge\theta\wedge\theta_x + \zeta_x\theta_y\wedge\theta\wedge\theta_{xy})dxdy$$
$$\int -\zeta_y\theta_{xx}\wedge\theta\wedge\theta_y dxdy = \int(\zeta_{xy}\theta_x\wedge\theta\wedge\theta_y + \zeta_y\theta_x\wedge\theta\wedge\theta_{xy})dxdy$$

Hence, from the skew-symmetry of the wedge product and Theorem 1.5.1, it follows that $\mathscr{D}$ is Hamiltonian.

Dubrovin and Novikov ([15], [16], [17]) developed a simple implementation criterion, to check at once for the skew-symmetry and the Jacobi identity for a particular class of systems. Such results are based on Riemannian geometry; for this reason, the summation convention of repeated indices will be adopted below.

Definition 1.5.4. A (homogeneous) *system of hydrodynamic type* is an equation of the form

$$u^i_t = v^{i\alpha}_j(u)u^j_\alpha, \qquad i = 1, ..., q$$

where $u^j_\alpha \equiv \partial u^j/\partial x^\alpha$. As above, q is the number of dependent variables, while the index $\alpha = 1, ..., d$ runs across the spatial independent variables.

Definition 1.5.5. A *functional of hydrodynamic type* is one of the form

$$\mathscr{H}[u] = \int H(u)dx$$

where $H(u)$ is an ordinary function, i.e. not depending on derivatives of u.

Note that a system of hydrodynamic type is an evolution equation involving spatial derivatives of order up to one.

Definition 1.5.6. A *Poisson bracket of hydrodynamic type* is defined by a $\mathscr{C}$-differential operator $\mathscr{D}$ of the form

$$\mathscr{D}^{ij} = g^{ij\alpha}(u)D_\alpha + b^{ij\alpha}_k(u)u^k_\alpha \tag{1.21}$$

where $g^{ij\alpha}$ and $b^{ij\alpha}_k$ are certain functions not depending on derivatives of u.

According to the previous definitions, a *Hamiltonian systems of hydrodynamic type* has the form

$$u^i_t = \left(g^{ij\alpha}\frac{\partial^2 H}{\partial u^j\partial u^k} + b^{ij\alpha}_k\frac{\partial H}{\partial u^j}\right)u^k_\alpha$$

where $H, g^{ij\alpha}, b^{ij\alpha}_k$ do not depend on derivatives of u. We first focus to the spatially one-dimensional case $d = 1$, omitting the index α. The following proposition clarifies the geometric meaning of the introduced elements.

Proposition 1.5.2. *1. The class of Poisson brackets of hydrodynamic type is invariant with respect to changes of the field variables of the form*

$$u^i \mapsto v^i(u), \qquad i = i, ..., q \tag{1.22}$$

2. *Under these change of variables, the coefficients* $g^{ij}(u)$ *transform as the components of a tensor of type (0,2), that is,*

$$g^{ab}(v) = \frac{\partial v^a}{\partial u^i}\frac{\partial v^b}{\partial u^j} g^{ij}(u(v)), \qquad a, b = 1, ..., q$$

3. *If the matrix* $(g^{ij}(u))$ *is non-degenerate, it defines the quantities* $\Gamma^i_{jk}(u)$ *by the equality*

$$b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u), \qquad i, j, k = 1, ..., q \tag{1.23}$$

Under change of variables (1.22), the quantities $\Gamma^i_{jk}(u)$ *transform as the components of a differential-geometric connection (Christoffel symbols), that is*

$$\Gamma^a_{bc}(v) = \frac{\partial v^a}{\partial u^i}\frac{\partial u^j}{\partial v^b}\frac{\partial u^k}{\partial v^c}\Gamma^i_{jk}(u) + \frac{\partial v^a}{\partial u^i}\frac{\partial^2 u^i}{\partial v^b \partial v^c}$$

Poisson bracket of hydrodynamic type for which $\det(g^{ij}) \neq 0$ are called *non-degenerate*; in what follows we shall only consider non-degenerate brackets. The theorem that follows solves completely the problem of determining if a given Poisson structure $\mathscr{D}^{ij}$ of hydrodynamic type is indeed Hamiltonian.

Theorem 1.5.2. 1. *In order that the bracket (1.21) be skew-symmetric it is necessary and sufficient that the tensor* $g^{ij}(u)$ *be symmetric (i.e., that it define a pseudo-Riemannian metric if* $\det(g^{ij}) \neq 0$*) and the connection* Γ^i_{jk} *be consistent with the metric,* ${g^{ij}}_{;k} = \nabla_k g^{ij} = 0$.

2. *In order that the bracket (1.21) satisfy the Jacobi identity it is necessary and sufficient that the connection* Γ^i_{jk} *have no torsion and the curvature tensor vanish. In this case the connection is defined by the metric* $g^{ij}(u)$ *which can be reduced to constant form. A complete local invariant of the Poisson structure (1.21) is the signature of the pseudo-Riemannian metric* g^{ij}.

Concerning the spatially multidimensional case, $d > 1$, almost all of the results above are verified component-wise for each fixed $\alpha = 1, ..., d$. The main difference is that, in general, there not exist a coordinate system on the space of dependent variables $U \simeq \mathbb{R}^q$ such that all the metrics $g^{ij\alpha}(u)$ are simultaneously constant for $\alpha = 1, ..., d$. We refer to [16] for details on this case.

1.5.2 Hamiltonian vector fields

Having a Poisson structure allows to define the notion of Hamiltonian vector fields, as much as it is done in the context of finite dimensional Hamiltonian dynamics.

Definition 1.5.7. Let $\mathscr{D}$ be a Hamiltonian operator. To each functional $\mathscr{H} = \int H dx \in \mathscr{F}$ there is an evolutionary vector field $\hat{\mathrm{v}}_{\mathscr{H}}$, called the *Hamiltonian vector field* associated with $\mathscr{H}$, which satisfies

$$\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{P}) = \{\mathscr{P}, \mathscr{H}\}$$

for all functionals $\mathscr{P} \in \mathscr{F}$. Here, the action of $\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}$ on the functional $\mathscr{P} = \int P dx$ is defined by formally exchanging the operation $\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}$ with the integral sign:

$$\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{P}) \equiv \int \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}(P) dx$$

The Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{H}}$ has characteristic $\mathscr{D}\delta\mathscr{H} = \mathscr{D}\mathsf{E}(H)$. So to be consistent with our previous notation for evolutionary vector fields, we may write

$$\hat{\mathrm{v}}_{\mathscr{H}} = \mathrm{v}_{\mathscr{D}\mathsf{E}(H)} \tag{1.24}$$

This property follows from an integration by parts:

$$\{\mathscr{P}, \mathscr{H}\} = \int \mathsf{E}(P) \cdot \mathscr{D}\mathsf{E}(H) dx = \int \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\mathsf{E}(H)}(P) dx = \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\mathsf{E}(H)}(\mathscr{P})$$

The Hamiltonian flow corresponding to a functional $\mathscr{H}[u]$ is obtained by exponentiating the corresponding Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{H}}$. According to formula (1.2) for the flow of a generalized vector field, we define the *Hamiltonian system* associated with $\mathscr{H} = \int H dx$ as the system of evolution equations

$$u_t = \mathscr{D}\mathsf{E}(H) \tag{1.25}$$

1.5.3 Symmetries and conservation laws

For a given Hamiltonian system, there are mainly two kinds of conservation laws: the first arises from degeneracies of the Poisson bracket, and is shared with any other system having the given Hamiltonian structure $\mathscr{D}$; the second type of conservation laws arises from symmetries of the particular Hamiltonian functional $\mathscr{H}$ that defines the system.

Definition 1.5.8. Let $\mathscr{D}$ be a $q \times q$ Hamiltonian differential operator. A *distinguished (or Casimir) functional* for $\mathscr{D}$ is a functional $\mathscr{C} \in \mathscr{F}$ satisfying $\mathscr{D}\delta\mathscr{C} = 0$ for all x, u. (If not specified otherwise, any distinguished functional is assumed to not depend on time.)

The Hamiltonian system determined by a distinguished functional is completely trivial: $u_t = 0$. Moreover, from the definition immediately follows that a functional C is distinguished if and only if its Poisson bracket with every other functional vanishes:

$$\{\mathscr{C}, \mathscr{H}\} = 0, \qquad \forall \mathscr{H} \in \mathscr{F}$$

With the notation introduced in this chapter, we can express a conservation law (1.10) as

$$\partial_t T + \mathrm{pr}\,\mathrm{v}_Q(T) \sim 0$$

On the other hand, we have

$$\begin{aligned}
\mathrm{pr}\,\mathrm{v}_Q(T) &= \sum_{\alpha,J} D_J Q^\alpha \frac{\partial T}{\partial u^\alpha_J} = 1\cdot\Big(\sum_J \frac{\partial T}{\partial u^1_J} D_J\Big)(Q^1) + \ldots + 1\cdot\Big(\sum_J \frac{\partial T}{\partial u^q_J} D_J\Big)(Q^q) \sim \\
&\sim Q^1\Big(\sum_J \frac{\partial T}{\partial u^1_J} D_J\Big)^*(1) + \ldots + Q^q\Big(\sum_J \frac{\partial T}{\partial u^q_J} D_J\Big)^*(1) = \\
&= Q^1 \sum_J (-1)^{|J|} D_J\Big(\frac{\partial T}{\partial u^1_J}\Big) + \ldots + Q^q \sum_J (-1)^{|J|} D_J\Big(\frac{\partial T}{\partial u^q_J}\Big) = \\
&= Q^1 \mathsf{E}_1(T) + \ldots + Q^q E_q(T) = Q\cdot \mathsf{E}(T)
\end{aligned}$$

For a Hamiltonian system of evolution equations (1.25), we have $Q = \mathscr{D}\mathsf{E}(H)$, so the condition (1.10) for a differential function T to be a conserved density is equivalent to

$$\partial_t T + \mathsf{E}(T)\cdot\mathscr{D}\mathsf{E}(H) \sim 0$$

This condition is translated on the associated functional $\mathscr{T} = \int T dx$ by means of an integration:

$$\partial_t \mathscr{T} + \{\mathscr{T},\mathscr{H}\} = 0 \tag{1.26}$$

Given a distinguished functional $\mathscr{C} = \int C dx$, is easily verified that the former condition holds. Indeed, $\mathscr{C}$ does not depend explicitly on t, and its Poisson bracket with the Hamiltonian $\mathscr{H}$ vanishes identically by definition. Therefore, each distinguished functional determines a conservation law for the system. As it was anticipated above, this result does not depend on the particular Hamiltonian functional $\mathscr{H}$ defining the system, but only rely on the degeneracy of the Poisson structure $\mathscr{D}$, meaning that $\mathscr{D}$ has non trivial kernel.

As for finite-dimensional Hamiltonian systems, there is an important relation between the Poisson bracket of two functionals and the Lie bracket of their corresponding Hamiltonian vector fields.

Proposition 1.5.3. *Let $\{\,\cdot\,,\,\cdot\,\}$ be a Poisson bracket determined by a differential operator $\mathscr{D}$. Let $\mathscr{P},\mathscr{Q}\in\mathscr{F}$ be functionals, with corresponding Hamiltonian vector fields $\hat{\mathrm{v}}_{\mathscr{P}}$, $\hat{\mathrm{v}}_{\mathscr{Q}}$. Then the following relation holds:*

$$\hat{\mathrm{v}}_{\{\mathscr{P},\mathscr{Q}\}} = [\hat{\mathrm{v}}_{\mathscr{Q}},\hat{\mathrm{v}}_{\mathscr{P}}]$$

Proof. Form Definition (1.5.7) of a Hamiltonian vector field, and using the skew-symmetry and the Jacobi identity of the Poisson bracket, we have

$$\begin{aligned}
\mathrm{pr}\,\hat{\mathrm{v}}_{\{\mathscr{P},\mathscr{Q}\}}(\mathscr{R}) &= \{\mathscr{R},\{\mathscr{P},\mathscr{Q}\}\} = -\{\{\mathscr{P},\mathscr{Q}\},\mathscr{R}\} = \{\{\mathscr{R},\mathscr{P}\},\mathscr{Q}\} + \{\{\mathscr{Q},\mathscr{R}\},\mathscr{P}\} = \\
&= \{\{\mathscr{R},\mathscr{P}\},\mathscr{Q}\} - \{\{\mathscr{R},\mathscr{Q}\},\mathscr{P}\} \equiv \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{Q}}\{\mathscr{R},\mathscr{P}\} - \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{P}}\{\mathscr{R},\mathscr{Q}\} \equiv \\
&\equiv \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{Q}}(\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{P}}(\mathscr{R})) - \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{P}}(\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{Q}}(\mathscr{R})) \equiv \mathrm{pr}\,[\hat{\mathrm{v}}_{\mathscr{Q}},\hat{\mathrm{v}}_{\mathscr{P}}](\mathscr{R})
\end{aligned}$$

for all functionals $\mathscr{R}\in\mathscr{F}$. The last equality follows from Definition (1.3.5) of the Lie bracket. $\square$

Now we are ready for presenting the well known theorem of Noether, relating symmetries and conservation laws.

Theorem 1.5.3 (Noether)**.** *Let $u_t = \mathscr{D}\delta\mathscr{H}$ be a Hamiltonian system of evolution equations. A Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{P}}$ with characteristic $\mathscr{D}\delta\mathscr{P}$, $\mathscr{P} \in \mathscr{F}$, determines a generalized symmetry group of the system if and only if there is an equivalent functional $\tilde{\mathscr{P}} = \mathscr{P} - \mathscr{C}$, differing only from $\mathscr{P}$ by a time-dependent distinguished functional $\mathscr{C}[t;u]$, such that $\tilde{\mathscr{P}}$ determines a conservation law.*

Proof. By a time-dependent distinguished functional we mean a functional $\mathscr{C}[t;u] = \int C(t,x,u^{(n)})dx$, with C depending on t, x, u and x-derivatives of u, and with the property that for each fixed t_0, $\mathscr{C}[t_0;u]$ is a distinguished functional: $\mathscr{D}\delta\mathscr{C} = 0$. By Proposition 1.3.3, the Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{P}}$ is a symmetry for the system $u_t = \mathscr{D}\delta\mathscr{H}$ if and only if

$$\frac{\partial \hat{\mathrm{v}}_{\mathscr{P}}}{\partial t} + [\hat{\mathrm{v}}_{\mathscr{H}}, \hat{\mathrm{v}}_{\mathscr{P}}] = 0$$

Combining this with Proposition 1.5.3, we have

$$\frac{\partial \hat{\mathrm{v}}_{\mathscr{P}}}{\partial t} + \hat{\mathrm{v}}_{\{\mathscr{P},\mathscr{H}\}} = 0$$

which is equivalent to a condition among the characteristics of the involved vector fields:

$$\partial_t \mathscr{D}\delta\mathscr{P} + \mathscr{D}\delta\{\mathscr{P},\mathscr{H}\} = 0$$

If $\mathscr{D}$ does not depend explicitly on time, then it commutes with ∂_t; so we end up with

$$\mathscr{D}\delta\Big(\frac{\partial\mathscr{P}}{\partial t} + \{\mathscr{P},\mathscr{H}\}\Big) = 0$$

This, in turn, is equivalent to

$$\frac{\partial\mathscr{P}}{\partial t} + \{\mathscr{P},\mathscr{H}\} = \tilde{\mathscr{C}}$$

for some time-dependent distinguished functional $\tilde{\mathscr{C}}[t;u] = \int \tilde{C}(t,x,u^{(n)})dx$. Now, define a new distinguished functional $\mathscr{C}$ such that

$$\frac{\partial\mathscr{C}}{\partial t} = \tilde{\mathscr{C}}$$

Substituting in the former equation, we can write

$$\frac{\partial}{\partial t}(\mathscr{P} - \mathscr{C}) + \{\mathscr{P},\mathscr{H}\} = 0$$

that is, since $\mathscr{C}$ is a distinguished functional itself,

$$\frac{\partial}{\partial t}(\mathscr{P} - \mathscr{C}) + \{\mathscr{P} - \mathscr{C},\mathscr{H}\} = 0$$

This is precisely relation (1.26), which says that the functional $\mathscr{P} - \mathscr{C}$ determines a conservation law. $\square$

Example 1.5.2. We now show how the machinery presented in this chapter works concretely, with reference to the Hopf equation already treated in Example 1.3.1:

$$u_t = -uu_x \tag{1.27}$$

First let us note that equation (1.27) has the following Hamiltonian structure:

$$u_t = \mathscr{D}\mathsf{E}(H)$$

where the Poisson structure $\mathscr{D} = D_x$ coincides with the total x-derivative, and the Hamiltonian is $H = -\frac{u^3}{6}$. Equation (1.27) is of hydrodynamic type, in the sense of Definition 1.5.4, so we can use Theorem 1.5.2 to conclude that the Poisson structure $\mathscr{D}$ is actually Hamiltonian (cf. Definition 1.5.3). In Example 1.3.1, we got the general expression (1.5) for the characteristic of a geometric symmetry of equation (1.27), which we rewrite for convenience here

$$Q = -\eta(x-ut,u) + u_x(\xi(x-ut,u) + t\eta(x-ut,u))$$

We want to find conservation laws associated to these symmetries, at least for simple particular cases. For example, if we take $\eta = 0$ and $\xi = u$, then $Q = u_x$. The associated conserved density T is found by solving the equation

$$Q = \mathscr{D}\mathsf{E}(T) \tag{1.28}$$

In general, $T[u]$ can depend on arbitrary x-derivatives of the dependent variable; we search first for solutions which do not depend on derivatives of u. In this case we find

$$T = -\frac{u^2}{2}$$

To write the conservation law for T we have to find the associated flux, i.e. a differential function X such that

$$D_tT + D_xX = 0 \tag{1.29}$$

on solutions of (1.27). We are ensured by the Theorem of Noether (1.5.3) that such X does exist. Note that, for $u_t = -uu_x$,

$$D_tT = -uu_t = u^2u_x = D_x(\tfrac{u^3}{3})$$

Hence we can take $X = -\frac{u^3}{3}$. As another less trivial example, consider $\eta = 1$ and $\xi = 0$, so that $Q = -1 + u_xt$. Also in this case we find a solution of (1.28) which does not depend on derivatives of u:

$$T = xu - t\tfrac{u^2}{2}$$

Arguing as above, we find the associated flux to be

$$X = x\tfrac{u^2}{2} - t\tfrac{u^3}{3}$$

Is important to remark that these conservation laws do not necessarily give rise to constants of the motion. The occurrence of this is tied to the imposed initial conditions. For example, if we take $u(x,0)$ to be of compact support, then the boundary term that arises from the spatial integration of (1.29) vanishes, so that the integral of T is conserved in time.

Chapter 2

The Boussinesq Model

In this section we review the works of Benjamin [8] and Camassa *et al.* [9]. They both concern the Euler equations for a two-dimensional incompressible fluid flow, with variable density. The mathematical apparatus described in Chapter 1 is now required to expose the work of Benjamin: he detects an Hamiltonian structure for the so called Boussinesq model, and uses it to classify conserved quantities, starting form the symmetries of the system. However, the Hamiltonian structure of Benjamin falls apart whenever the fluid density is not constant on the boundaries. This has an impact on the correspondence between symmetries and conservation laws: some quantities can cease to be conserved when the Hamiltonian structure breaks down. This phenomenon is deepened in the work of Camassa *et al.* [9], where it is recognized to be of topological nature.

2.1 Hamiltonian Structure

The Boussinesq model concerns the two-dimensional flow of a incompressible, inviscid fluid with variable density. We let x, y denote the standard Cartesian coordinates on $\mathbb{R}^2$, and u, v the corresponding components of the velocity field of the fluid $\mathbf{u}$. The constrain of incompressibility, $\nabla \cdot \mathbf{u} = 0$, allows the description of the flow by means of the stream function ψ:

$$\begin{cases} u = \psi_y, \\ v = -\psi_x. \end{cases} \tag{2.1}$$

Here and in the following the subscrips x, y denote the partial derivatives $\partial/\partial x, \partial/\partial y$. Following Benjamin [8], we introduce the auxiliary variable

$$\sigma = (\rho v)_x - (\rho u)_y = -(\rho\psi_x)_x - (\rho\psi_y)_y = L_\rho \psi \tag{2.2}$$

where ρ denotes the fluid density. This quantity can be understood as a *density-weighted vorticity*, since the ordinary vorticity is $\zeta = v_x - u_y = -\Delta\psi$. If appropriate boundary conditions are imposed, equation (2.2) admits unique solution ψ for any given σ. Specifically, we will be concerned with the infinite strip $D = \mathbb{R} \times (0, h)$ as the fluid domain; then we assume $\psi = 0$ on the rigid boundaries at $y = 0$ and $y = h$, which make v vanish, and we assume that both ψ and $|\nabla\psi|$ go to zero rapidly enough as $|x| \to +\infty$. Thus, the stream function can be regarded as a linear transformation (depending on ρ) of the vorticity σ:

$$\psi = L_\rho^{-1}\sigma \tag{2.3}$$

Remark 2.1.1. The condition of impermeability on the rigid walls at $y = 0$ and $y = h$ requires that $v = -\psi_x = 0$. Strictly speaking, this only implies the stream function to be constant on the walls. However, to be consistent with the condition

$$\lim_{|x|\to\infty} \psi(x, y, t) = 0, \qquad \forall y \in (0, h), \forall t \in (0, +\infty),$$

we must assume $\psi = 0$ on both the rigid boundaries. Indeed, for any smooth curve γ, with extremes $(x_1, y_1), (x_2, y_2)$, the integral

$$\int_\gamma d\psi = \int_\gamma (udy - vdx)$$

represents the flow of the velocity field across γ. In particular, by letting $y_1 = 0$ and $y_2 = h$, we see that the difference

$$\psi(x, h, t) - \psi(x, 0, t)$$

represents the total volume flow rate across any section of the channel. Since hydrostatic conditions are assumed at the far ends of the domain, it follows that the above difference must be zero.

The equations of conservation of mass and of linear momentum can be written in terms of the variables ρ, σ, ψ as

$$\begin{cases} \rho_t + \partial(\rho, \psi) = 0 \\ \sigma_t + \partial(\sigma, \psi) + \partial(\rho, gy - \frac{|\nabla\psi|^2}{2}) = 0 \end{cases} \tag{2.4}$$

where $\partial(\,\cdot\,,\,\cdot\,)$ denotes the Jacobian derivative $\partial(\,\cdot\,,\,\cdot\,)/\partial(x, y)$. Clearly, the space of independent variables for this problem is $X \simeq \mathbb{R}^3$ with coordinates (x, y, t). The space of dependent variables is $U \simeq \mathbb{R}^2$, with "standard" coordinates (ρ, σ); they are related to the other possible couple of coordinates (ρ, ψ) by formulas (2.2) and (2.3).

The system (2.4) admits the Hamiltonian structure described in the following. Suppose $\mathscr{P} = \int P dx$ and $\mathscr{Q} = \int Q dx$ are functionals as in §1.5. According to our previous definition (1.17), their Poisson bracket is

$$\{\mathscr{P}, \mathscr{Q}\} = \iint_D \mathsf{E}(P) \cdot \mathscr{D}\mathsf{E}(Q) dx dy$$

where $\mathsf{E}(P) = (E_\rho(P), \mathsf{E}_\sigma(P))$ is the Euler operator (cf. §1.5). Now define the differential operator

$$\mathscr{D} = \begin{pmatrix} 0 & -\partial(\rho,\,\cdot\,) \\ -\partial(\rho,\,\cdot\,) & -\partial(\sigma,\,\cdot\,) \end{pmatrix} = \begin{pmatrix} 0 & \rho_y D_x - \rho_x D_y \\ \rho_y D_x - \rho_x D_y & \sigma_y D_x - \sigma_x D_y \end{pmatrix} \tag{2.5}$$

and the Hamiltonian functional

$$\mathscr{H} = \iint_D H dx dy = \iint_D \left(\rho \frac{|\nabla\psi|^2}{2} + gy(\rho - \rho_0)\right) dx dy$$

which represents the total energy of the system. Here $\rho_0 = \rho_0(y)$ is the density of a stable hydrostatic equilibrium reference configuration for the fluid. With this set-up, the system of equations (2.4) can be written in the form of a Hamiltonian system as

$$\omega_t = \mathscr{D}\mathsf{E}(H) \tag{2.6}$$

where $\omega = (\rho, \sigma)^\top$ represent the vector of dependent variables. To see this, is impractical to directly apply Definition 1.5.2 of the Euler operator $\mathsf{E}(H) = (\mathsf{E}_\rho(H), \mathsf{E}_\sigma(H))$, since in the expression for H appears ψ instead of σ. To circumvent this difficulty, we directly work with Definition 1.5.1 of the variational derivative: when $\mathscr{H}$ is considered as a function of ρ, ψ then

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathscr{H}[\rho + \epsilon\xi, \psi + \epsilon\zeta] = \iint_D (\mathsf{E}_\rho(H)\xi + \mathsf{E}_\psi(H)\zeta) dx dy \tag{2.7}$$

and when $\mathscr{H}$ is regarded as a function of ρ, σ we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathscr{H}[\rho + \epsilon\xi, \sigma + \epsilon\eta] = \iint_D (\mathsf{E}_\rho(H)\xi + \mathsf{E}_\sigma(H)\eta) dx dy \tag{2.8}$$

Here ξ, η, ζ are smooth functions of compact support in D, which can not be all independent: (2.7) and (2.8) are understood as the *same* derivative, expressed with respect to different coordinate systems. Indeed, by formula (2.2), we have

$$\sigma + \epsilon\eta = L_{(\rho+\epsilon\xi)}(\psi + \epsilon\zeta)$$

Retaining only the first order terms in ϵ, we arrive to the relation

$$\eta = -(\xi\psi_x + \rho\zeta_x)_x - (\xi\psi_y + \rho\zeta_y)_y = -\nabla\cdot(\rho\nabla\zeta + \xi\nabla\psi) \tag{2.9}$$

Thus, the strategy for computing the Euler derivative of H with respect to the variables ρ, σ is as follows: for σ and η related to the other variables by (2.2) and (2.9), the left hand sides of (2.7) and (2.8) must agree, so equating them we get

$$\iint_D (\mathsf{E}_\rho(H)\xi + \mathsf{E}_\psi(H)\zeta) dx dy = \iint_D (\mathsf{E}_\rho(H)\xi + \mathsf{E}_\sigma(H)\eta) dx dy \tag{2.10}$$

Using Definition 1.5.2 of the Euler operator, we can compute the integral on the left; then, by means of relations (2.2) and (2.9) defining σ and η, we express it in the form of the integral on the right. Finally, by comparison, we get the desired expression for $\mathsf{E}(H) = (\mathsf{E}_\rho(H), \mathsf{E}_\sigma(H))$. We remark that the expression for Euler derivative $\mathsf{E}_\rho(H)$ appearing in both sides of (2.10) is not the same, since the differential functions $H[\rho, \psi]$ and $H[\rho, \sigma]$ have different dependencies on ρ. The Euler derivative of H with respect to ρ and ψ is

$$\begin{aligned} \mathsf{E}_\rho(H) &= \frac{\partial H}{\partial \rho} = \frac{|\nabla\psi|^2}{2} + gy \\ \mathsf{E}_\psi(H) &= -D_x\left(\frac{\partial H}{\partial \psi_x}\right) - D_y\left(\frac{\partial H}{\partial \psi_y}\right) = -(\rho\psi_x)_x - (\rho\psi_y)_y \end{aligned}$$

Integrating by parts the term $\mathsf{E}_\psi(H)\zeta$, and recalling that ζ has compact support in D, we can express the integral on the left in (2.10) as

$$\iint_D \left(\left(\frac{|\nabla\psi|^2}{2} + gy \right)\xi + \rho\nabla\psi\cdot\nabla\zeta \right) dxdy$$

Further we use the relation

$$\rho\nabla\psi\cdot\nabla\zeta = -\xi|\nabla\psi|^2 - \psi\nabla\cdot(\rho\nabla\zeta + \xi\nabla\psi) + \nabla\cdot(\rho\psi\nabla\zeta + \xi\psi\nabla\psi)$$

which can be verified by computation. Taking account of the conditions imposed on ψ, the last term is seen to not contribute to the integral. Further, recalling formula (2.9) relating η to the other variables, we can express equation (2.10) as

$$\iint_D \left(\left(gy - \frac{|\nabla\psi|^2}{2} \right)\xi + \psi\eta \right) dxdy = \iint_D (\mathsf{E}_\rho(H)\xi + \mathsf{E}_\sigma(H)\eta) dxdy$$

From this we conclude that the Euler derivative of $H[\rho,\sigma]$, regarded as a function of ρ and σ, is

$$\mathsf{E}_\rho(H) = gy - \frac{|\nabla\psi|^2}{2}$$
$$\mathsf{E}_\sigma(H) = \psi$$

With this result is readily verified that the system of equations (2.4) can be written in the form (2.6) of a Hamiltonian system. We remark that this fact rely only on the boundary conditions imposed on ψ: in particular it does not depend on the behavior of ρ on the boundary.

To qualify as a true Hamiltonian operator, $\mathscr{D}$ has to satisfy skew-symmetry and Jacobi identity. Proposition 1.5.1 ensures the skew-symmetry holds, the operator $\mathscr{D}$ being skew-adjoint,

$$\mathscr{D}^* = \begin{pmatrix} 0 & -D_x\circ\rho_y + D_y\circ\rho_x \\ -D_x\circ\rho_y + D_y\circ\rho_x & -D_x\circ\sigma_y + D_y\circ\sigma_x \end{pmatrix} = -\mathscr{D}.$$

Unfortunately, the system of equations (2.4) is not of hydrodynamic type in the sense of Definition 1.5.4. Hence, in order to check for the Jacobi identity we have to use the general criterion of Theorem 1.5.1. The functional two-vector associated to the operator (2.5) is

$$\begin{aligned}\Theta &= \frac{1}{2}\iint \sum_{\alpha,\beta=1}^{2} \theta^\alpha \mathscr{D}_{\alpha\beta}\theta^\beta dxdy = \\ &= \frac{1}{2}\iint (\rho_y(\theta^1\wedge\theta^2_x + \theta^2\wedge\theta^1_x) - \rho_x(\theta^1\wedge\theta^2_y + \theta^2\wedge\theta^1_y) + \sigma_y\theta^1\wedge\theta^2_x - \sigma_x\theta^2\wedge\theta^2_y)dxdy\end{aligned}$$

Proceeding as in Example 1.5.1, we compute

$$\begin{aligned}\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\Theta) = \frac{1}{2}\iint (&\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\rho_y)\wedge(\theta^1\wedge\theta^2_x + \theta^2\wedge\theta^1_x) - \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\rho_x)\wedge(\theta^1\wedge\theta^2_y + \theta^2\wedge\theta^1_y) + \\ &+ \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\sigma_y)\wedge\theta^1\wedge\theta^2_x - \mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\sigma_x)\wedge\theta^2\wedge\theta^2_y)dxdy\end{aligned}$$

where

$$
\begin{aligned}
\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\rho_x) &= \rho_{xy}\theta^2_x + \rho_y\theta^2_{xx} - \rho_{xx}\theta^2_y - \rho_x\theta^2_{xy}\\
\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\rho_y) &= \rho_{yy}\theta^2_x + \rho_y\theta^2_{xy} - \rho_{xy}\theta^2_y - \rho_x\theta^2_{yy}\\
\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\sigma_x) &= \rho_{xy}\theta^1_x + \rho_y\theta^1_{xx} - \rho_{xx}\theta^1_y - \rho_x\theta^1_{xy} + \sigma_{xy}\theta^2_x + \sigma_y\theta^2_{xx} - \sigma_{xx}\theta^2_y - \sigma_x\theta^2_{xy}\\
\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\sigma_y) &= \rho_{yy}\theta^1_x + \rho_y\theta^1_{xy} - \rho_{xy}\theta^1_y - \rho_x\theta^1_{yy} + \sigma_{yy}\theta^2_x + \sigma_y\theta^2_{xy} - \sigma_{xy}\theta^2_y - \sigma_x\theta^2_{yy}
\end{aligned}
$$

Due only to the skew-symmetry of the wedge product, the coefficients of the second order derivatives of ρ and σ vanish identically. So we get

$$
\begin{aligned}
\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\Theta) = \frac{1}{2}\iint \Big[&\rho_y(\theta^2_{xy}\wedge\theta^1\wedge\theta^2_x + \theta^2_{xy}\wedge\theta^2\wedge\theta^1_x - \theta^2_{xx}\wedge\theta^1\wedge\theta^2_y +\\
&\qquad - \theta^2_{xx}\wedge\theta^2\wedge\theta^1_y + \theta^1_{xy}\wedge\theta^2\wedge\theta^2_x - \theta^1_{xx}\wedge\theta^2\wedge\theta^2_y)+\\
&+ \rho_x(-\theta^2_{yy}\wedge\theta^1\wedge\theta^2_x - \theta^2_{yy}\wedge\theta^2\wedge\theta^1_x + \theta^2_{xy}\wedge\theta^1\wedge\theta^2_y +\\
&\qquad + \theta^2_{xy}\wedge\theta^2\wedge\theta^1_y - \theta^1_{yy}\wedge\theta^2\wedge\theta^2_x + \theta^1_{xy}\wedge\theta^2\wedge\theta^2_y)+\\
&+ \sigma_y(\theta^2_{xy}\wedge\theta^2\wedge\theta^2_x - \theta^2_{xx}\wedge\theta^2\wedge\theta^2_y)+\\
&+ \sigma_x(-\theta^2_{yy}\wedge\theta^2\wedge\theta^2_x + \theta^2_{xy}\wedge\theta^2\wedge\theta^2_y)\Big]dxdy
\end{aligned}
$$

Now we integrate by parts all the terms with a minus sign, changing the pure second derivatives of the θ^i for the mixed ones. For example,

$$
\iint -\sigma_y\theta^2_{xx}\wedge\theta^2\wedge\theta^2_y dxdy = \iint [\sigma_{xy}\theta^2_x\wedge\theta^2\wedge\theta^2_y + \sigma_y\theta^2_x\wedge\theta^2\wedge\theta^2_{xy}]dxdy
$$

In this way, after some computations, we end up with $\mathrm{pr}\,\mathrm{v}_{\mathscr{D}\theta}(\Theta) = 0$ as required by Theorem 1.5.1.

2.2 Symmetries

The computation of symmetry groups for the system of equations (2.4) proceeds mostly as delineated in §1.3, but with some foresight. The space of independent variables for the system of equations (2.6) is $X \simeq \mathbb{R}^3$ with coordinates x, y, t, whereas the space of dependent variables is designed as $U \simeq \mathbb{R}^2$ with coordinates ρ, σ. The candidate symmetry is a generalized vector field v on $X \times U$ of the form

$$
\mathrm{v} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial \rho} + \gamma_2\frac{\partial}{\partial \sigma} \tag{2.11}
$$

Following Benjamin, we suppose a priori that all the symmetry groups of the system are *projectable*, that is the coefficient functions α, β, τ are allowed to depend only on the independent variables x, y, t. This hypothesis is supposed to be very likely correct by Benjamin, and sensibly reduces the computational load. On the other hand, the coefficients $\gamma_1[\rho, \sigma]$ and $\gamma_2[\rho, \sigma]$ are general differential functions depending on x, y, t, ρ, σ

and derivatives of ρ and σ. As described in §1.3, to the former vector field can be associated an evolutionary representative

$$\mathrm{v}_P = P_1\partial_\rho + P_2\partial_\sigma$$

whose characteristic $P = (P_1, P_2)$ is given by

$$\begin{cases} P_1 = \gamma_1 - \alpha\rho_x - \beta\rho_y - \tau\rho_t \\ P_2 = \gamma_2 - \alpha\sigma_x - \beta\sigma_y - \tau\sigma_t \end{cases}$$

The main obstacle to the direct application of the methods described in section §1.3, for the computation of symmetries, is that (2.6) is not a partial differential equation. In fact, H is expressed in terms of ψ, so that the integral operation $\psi = L_\rho^{-1}\sigma$ is entailed. As explained below, this problem can be tackled by allowing the stream function ψ to be part of the set of dependent variables. So we extend the space of dependent variables U to $U^* \simeq \mathbb{R}^3$ with coordinates ρ, σ, ψ. Any vector field (2.11) on $X \times U$ is extended to a vector field on $X \times U^*$,

$$\mathrm{v}^* = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial \rho} + \gamma_2\frac{\partial}{\partial \sigma} + \Gamma\frac{\partial}{\partial \psi}$$

As above, we consider the corresponding evolutionary representative

$$\mathrm{v}_{P^*} = P_1^*\frac{\partial}{\partial \rho} + P_2^*\frac{\partial}{\partial \sigma} + P_3^*\frac{\partial}{\partial \psi},$$

with characteristic $P^* = (P_1^*, P_2^*, P_3^*)^\top$ given by

$$\begin{cases} P_1^* = \gamma_1 - \alpha\rho_x - \beta\rho_y - \tau\rho_t \\ P_2^* = \gamma_2 - \alpha\sigma_x - \beta\sigma_y - \tau\sigma_t \\ P_3^* = \Gamma - \alpha\psi_x - \beta\psi_y - \tau\psi_t \end{cases} \tag{2.12}$$

Since the three dependent variables are related by (2.2), there is a constraint on the coefficients of a candidate symmetry (2.12). To comprehend this fact, assume for a moment $P_1^*[\rho,\sigma]$ and $P_2^*[\rho,\sigma]$ to be given differential functions. As defined by (1.2), the flow of the evolutionary vector field $P_1^*\partial_\rho + P_2^*\partial_\sigma$ is obtained as the solution of the system of evolution equations

$$\begin{cases} \rho_\epsilon = P_1^*[\rho,\sigma] \\ \sigma_\epsilon = P_2^*[\rho,\sigma] \end{cases}$$

Provided it exists, the solution of this problem is a map $\epsilon \mapsto (\rho,\sigma)$. Combining it with formula (2.3), we are able at least in principle to get a map $\epsilon \mapsto \psi$. The derivative of this map with respect to ϵ will be denoted as $\psi_\epsilon = P_3^*$. Now, taking the ϵ-derivative of equation (2.2), and reorganizing terms, we finally get the following relation between the coefficients of v_{P^*}:

$$P_2^* = -\nabla\cdot(P_1^*\nabla\psi) - \nabla\cdot(\rho\nabla P_3^*) \tag{2.13}$$

We remark that the former can be seen as an equation for the unknown P_3^*, which can be solved for every given pair P_1^*, P_2^*.

In order to apply Proposition 1.3.3, characterizing symmetries, we adopt the following notation. Let $\omega^* = (\rho, \sigma, \psi)^\top$ denote the vector of the three dependent variables; also, denote with Q the characteristic of the Hamiltonian vector field of $\mathscr{H}$, i.e. the right hand side of equation (2.6):

$$Q = (Q_1, Q_2)^\top = \mathscr{D}\mathsf{E}(H)$$

Then define a new differential function $Q^* = (Q_1^*, Q_2^*, Q_3^*)^\top$ so that $Q_1^* = Q_1$, $Q_2^* = Q_2$, and $Q_3^* = \psi_t$. With this notation, we can extend the system of evolution equations (2.6) to a system of equations on the jet space of $X \times U^*$ as

$$\omega_t^* = Q^* \tag{2.14}$$

whose first two lines are exactly those of the system (2.6), and the third line is trivial: $\psi_t = \psi_t$. According to Proposition 1.3.3, the vector field v_{P^*} is an infinitesimal symmetry of the system (2.14) if and only if

$$\frac{\partial \mathrm{v}_{P^*}}{\partial t} = [\mathrm{v}_{P^*}, \mathrm{v}_{Q^*}]$$

holds identically. In light of Proposition 1.3.2, the above condition translates in coordinate form as

$$\frac{\partial P^*}{\partial t} = \sum_{i=1}^{3} \sum_{J} \left(D_J(P_i^*) \frac{\partial Q^*}{\partial {\omega^*}_J^i} - D_J(Q_i^*) \frac{\partial P^*}{\partial {\omega^*}_J^i} \right) \tag{2.15}$$

where ${\omega^*}_J^i$ stands for the J-th derivative of the i-th component of ω^*. This equation, supplemented with (2.2) and (2.13), provides the means to compute the symmetries of the system (2.6), with the same procedure of Example 1.3.1. The symmetry group detected by Benjamin consists of nine one-parameter subgroups having the infinitesimal generators listed below together with their characteristics[1]:

- Time translation:

$$\mathrm{v}_3 = \partial_t \qquad P_3 = -\omega_t$$

- Horizontal translation:

$$\mathrm{v}_4 = \partial_x \qquad P_4 = -\omega_x$$

- Vertical translation:

$$\mathrm{v}_5 = \partial_y \qquad P_5 = -\omega_y$$

- Horizontal Galileian boost:

$$\mathrm{v}_6 = -t\partial_x - \rho_y \partial_\sigma \qquad P_6 = t\omega_x - (0, \rho_y)^\top$$

- Vertical Galileian boost:

$$\mathrm{v}_7 = -t\partial_y - \rho_x \partial_\sigma \qquad P_7 = t\omega_y - (0, \rho_x)^\top$$

[1]The numeration, ranging from 3 to 11, is chosen to highlight the correspondence with conserved quantities to be presented in §2.3

- Gravity-compensated rotation:

$$\mathrm{v}_8 = -(y + \tfrac{1}{2}gt^2)\partial_x + x\partial_y + gt\rho_y\partial_\sigma \qquad P_8 = (y + \tfrac{1}{2}gt^2)\omega_x - x\omega_y + (0, gt\rho_y)^\top$$

- Vertical acceleration:

$$\mathrm{v}_9 = gt^2\partial_y - t\partial_t + (\sigma + 2gt\rho_x)\partial_\sigma \qquad P_9 = t\omega_t - gt^2\omega_y + (0, \sigma + 2gt\rho_x)^\top$$

- Trivial scaling:

$$\mathrm{v}_{10} = \rho\partial_\rho + \sigma\partial_\sigma \qquad P_{10} = \omega$$

- Scaling:

$$\mathrm{v}_{11} = -x\partial_x - y\partial_y - \tfrac{1}{2}t\partial_t + \tfrac{1}{2}\sigma\partial_\sigma \qquad P_{11} = x\omega_x + y\omega_y + \tfrac{1}{2}t\omega_t + (0, \tfrac{1}{2}\sigma)^\top$$

We remark that Benjamin does not prove that there are not other symmetry groups than these. Once the generators are obtained, we can get the one-parameter families of transformed solutions, exponentiating the generators according to (1.2). As an example, for the time translation we have to solve

$$\begin{cases} \rho_\epsilon = -\rho_t \\ \sigma_\epsilon = -\sigma_t \end{cases}$$

By means of the method of characteristics, we get the following solution:

$$\begin{cases} \rho(x, y, t, \epsilon) = \tilde{\rho}(x, y, t - \epsilon) \\ \sigma(x, y, t, \epsilon) = \tilde{\sigma}(x, y, t - \epsilon) \end{cases}$$

Then, by equation (2.2), we recover also the transformed stream function:

$$\psi(x, y, t, \epsilon) = \tilde{\psi}(x, y, t - \epsilon)$$

The set $\tilde{\rho}, \tilde{\sigma}, \tilde{\psi}$ satisfy the system of equations (2.6) for every value of the parameter ϵ. We refer to Benjamin [8] for the complete list of transformed solutions corresponding to the other symmetries.

Let us return on equation (2.15). Only the first two lines are meaningful, the last being identically satisfied. To see this, is sufficient to note that, since $Q_3^* = \psi_t$ and P_3^* does not depend on ρ, σ and their derivatives [8], the third line reduces to

$$\frac{\partial P_3^*}{\partial t} = D_t P_3^* - \sum_J D_J(\psi_t)\frac{\partial P_3^*}{\partial \psi_J}$$

which is precisely the definition of the total time derivative of P_3^*. For this reason, equation (2.15) can be simplified retaining only its first two lines. Introducing the notation

$$[Q, P] = \sum_{i=1}^{3}\sum_J \left(D_J(P_i^*)\frac{\partial Q}{\partial {\omega^*}^i_J} - D_J(Q_i^*)\frac{\partial P}{\partial {\omega^*}^i_J} \right)$$

equation (2.15) can be proficiently written in the compact form

$$\frac{\partial P}{\partial t} = [Q, P] \tag{2.16}$$

Given two divergence-free differential functions $F, G \in \mathscr{A}/\mathrm{Div}(\mathscr{A}^2)$, we define their Poisson bracket as

$$\{F, G\} = \mathsf{E}(F) \cdot \mathscr{D}\mathsf{E}(G) \tag{2.17}$$

Then, by direct computation, can be proved the relation

$$[\mathscr{D}\mathsf{E}(F), \mathscr{D}\mathsf{E}(G)] = \mathscr{D}\mathsf{E}\{F, G\} \tag{2.18}$$

which holds for each pairs of differential functions $F[\omega], G[\omega] \in \mathscr{A}/\mathrm{Div}(\mathscr{A}^2)$. This result is the direct analogue to that of Proposition 1.5.3, concerning Hamiltonian systems of partial differential equations. It will be of pivotal role in establishing a Noetherian correspondence between symmetries and conservation laws.

2.3 Conservation Laws

The Poisson bracket determined by (2.5) admits a wide class of distinguished (Casimir) functionals. They are given by

$$\mathscr{C} = \iint_D C dxdy = \iint_D (\alpha(\rho)\sigma + \beta(\rho))dxdy \tag{2.19}$$

for arbitrary real *functions* α, β. Indeed, it is easily verified that $\mathscr{D}\mathsf{E}(C) = 0$. As pointed out in section §1.5, every Casimir functional determines a conservation law. Moreover, functionals of the form (2.19), not involving derivatives of u, give rise to constants of the motion. To see this is sufficient to take the time derivative of $\mathscr{C}$ evaluated on a solution $\omega(x, t)$ of the system (2.6):

$$\frac{d}{dt}\mathscr{C}[\omega(\,\cdot\,, t)] = \iint_D \mathsf{E}(C) \cdot \omega_t dxdy = \iint_D \mathsf{E}(C) \cdot \mathscr{D}\mathsf{E}(H)dxdy = \{\mathscr{C}, \mathscr{H}\}$$

Hence, from the skew-symmetry of the Jacobi bracket follows that $\mathscr{C}[\omega(\,\cdot\,, t)]$ is constant in time. Two important particular cases included in the class of Casimirs (2.19) are total mass and total vorticity:

$$\mathscr{I}_1 = \iint_D (\rho - \rho_0)dxdy, \qquad \mathscr{I}_2 = \iint_D \sigma dxdy$$

Here $\rho_0 = \rho_0(y)$, as in the definition of the Hamiltonian functional, represents the density of a stable hydrostatic equilibrium reference configuration for the fluid. The convergence of the integral (2.19) is not in question here: if it converges, then is a conserved quantity. However, if ρ_0 was not included in I_1, it would be surely infinite for an unbounded domain.

The other important class of conservation laws is tied to the symmetries of the system. In general, only those symmetries that are Hamiltonian vector fields corresponds to a conserved quantity. Conversely, every conserved quantity gives rise to a symmetry. In Benjamin [8], two theorems are presented which adapt that of Noether in the present context.

Theorem 2.3.1. *Let the differential function $T[\omega] \in \mathscr{A}$ define a conservation law for the system (2.6). Then the two-component differential function*

$$P = \mathscr{D}\mathsf{E}(T)$$

is the characteristic of an infinitesimal generator for a symmetry group of the system.

Proof. We have to verify that the condition (2.16) for P to be a symmetry for the system (2.6) is met for $Q = \mathscr{D}\mathsf{E}(H)$. The condition (1.8) for T to be a conserved density is

$$D_t T \sim 0$$

The total time derivative of T is

$$D_t T = \partial_t T + \sum_{i=1}^{2} \sum_J \frac{\partial T}{\partial \omega_J^i} Q_J^i \sim \partial_t T + \mathsf{E}(T) \cdot \mathscr{D}\mathsf{E}(H)$$

Is easily seen the differential operator (2.5) to be skew adjoint, so,

$$D_t T \sim \partial_t T + \mathsf{E}(H) \cdot \mathscr{D}^* \mathsf{E}(T) \sim \partial_t T - \mathsf{E}(H) \cdot \mathscr{D}\mathsf{E}(T)$$

Thus, adopting the notation introduced in (2.17), we can also write the condition (1.8) as

$$\partial_t T \sim \{H, T\} \tag{2.20}$$

which is intended to hold in the quotient space $\mathscr{A}/\mathrm{Div}(\mathscr{A}^2)$. Now we apply the operator $\mathscr{D}\mathsf{E}$ to both sides of equation (2.20)[2]: since the operations ∂_t and $\mathscr{D}\mathsf{E}$ commute, the left hand side gives

$$\mathscr{D}\mathsf{E}(\partial_t T) = \partial_t \mathscr{D}\mathsf{E}(T) = \partial_t P$$

Taking account of equation (2.18), the right hand side of (2.20) gives

$$\mathscr{D}\mathsf{E}\{H, T\} = [\mathscr{D}\mathsf{E}(H), \mathscr{D}\mathsf{E}(T)] = [Q, P]$$

which completes the proof. □

Theorem 2.3.2. *Let*

$$P = \gamma - \alpha\omega_x - \beta\omega_y - \tau\omega_t$$

be the characteristic of a symmetry for the system (2.6), with $D_y \beta = 0$. Assume the differential function $T[\omega]$ be such that $\mathscr{D}\mathsf{E}(T) = P$, and also satisfy

$$\mathsf{E}_\rho(\partial_t T) = g\beta$$

Then T is a conserved density for the system (2.6).

[2] This annihilates every remaining divergence term, left implicit in (2.20) by $\sim$.

Proof. As in the previous proof, the condition for P to be a symmetry is equivalent to

$$\mathscr{D}\mathsf{E}(\partial_t T - \{H, T\}) = 0$$

from which (2.20) is to be inferred. Substituting for P and $\mathsf{E}(H)$, the thesis is verified by direct calculation. □

Is important to remark that not all the symmetries listed in §2.2 are Hamiltonian vector fields. In particular, none of the scaling symmetries $\mathrm{v}_9, \mathrm{v}_{10}, \mathrm{v}_{11}$ are, nor linear combination of them. Consequently, they do not give rise to any conservation law. To every other symmetry is tied a conserved density, in agreement with Theorem 2.3.2. The list of them is reported below, with numeration corresponding to symmetries in §2.2:

$$\begin{aligned}
T_1 &= \rho - \rho_0 \\
T_2 &= \sigma \\
T_3 &= H \\
T_4 &= y\sigma \\
T_5 &= -x\sigma + gt(\rho - \rho_0) \\
T_6 &= xT_1 - tT_4 \\
T_7 &= (y - \tfrac{1}{2}gt^2)T_1 - tT_5 \\
T_8 &= -\tfrac{1}{2}(x^2 + y^2)\sigma + gtT_6 + \tfrac{1}{2}gt^2 T_4
\end{aligned}$$

The first two conserved densities do not have a counterpart in the list of symmetries of §2.2: indeed they correspond to the trivial symmetry $\mathrm{v}_1 = \mathrm{v}_2 = 0$. It is possible to identify the integral

$$\mathscr{T}_4 = \iint_D y\sigma dxdy$$

as the total horizontal impulse of the system, since it is the generator of the translational symmetry in the horizontal direction: $P_4 = \mathscr{D}\delta\mathscr{T}_4$. Similarly, the integral

$$\mathscr{T}_5 = \iint_D -x\sigma dxdy$$

is recognized to represent the total vertical impulse. We now ask which of the conserved densities listed above give rise to a constant of the motion, i.e. define a functional $\mathscr{T} = \int T dxdy$ which is constant on every solution of the system. The answer is of course tied to the geometry of the domain, and, as before, we limit ourselves to the case of infinite strip $D = \mathbb{R} \times (0, h)$. Since such domain is unbounded, we need to make the assumption of *localized motion* in order to allow the integrals considered to converge: to the conditions on ψ stated above, we add that ρ approaches ρ_0 rapidly enough as $|x| \to +\infty$. Moreover, for reasons to be cleared in §2.4, we shall assume the density ρ to be constant along the rigid boundaries. Then, under these hypothesis, is easy to confirm that the following quantities are constants of any motion:

$$\mathscr{T}_1 = \iint_D (\rho - \rho_0) dxdy \qquad \mathscr{T}_2 = \iint_D \sigma dxdy$$

$$\mathscr{T}_3 = \mathscr{H} = \iint_D H dxdy \qquad \mathscr{T}_4 = \iint_D y\sigma dxdy$$

Indeed, since $\mathscr{T}_1$ and $\mathscr{T}_2$ are particular Casimir functional of the type (2.19), it follows, arguing as above, that they are conserved quantities. Still the skew-symmetry of the Jacobi bracket (2.5) is sufficient to prove the conservation of the Hamiltonian $\mathscr{H}$. Concerning the total horizontal impulse, it can be shown [14] that it satisfy the conservation law

$$(y\sigma)_t + X_x + Y_y = 0 \tag{2.21}$$

for the flux

$$\begin{cases} X = uy\sigma + \frac{1}{2}\rho(u^2 - v^2) + \frac{1}{2}y\rho_y(u^2 + v^2) + gy(\rho - \rho_0) \\ Y = vy\sigma + \rho uv - \frac{1}{2}y\rho_x(u^2 + v^2) \end{cases}$$

When integrated on D, assuming the density to be constant along the boundary, this equation gives the conservation of the total horizontal impulse.

2.4 Topological Selection of Conserved Quantities

We now come to the central point of our discussion. The results of §2.1, concerning the skew-symmetry and the Jacobi identity of (2.5), hold true until the conditions of Remark 1.5.1 are verified. We show below that they are questioned if rigid boundary are present. To explain this issue, let us first consider the simpler case of the Euler equations with constant density (1.19), introduced in Example 1.5.1. Given any two functionals $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, for the Poisson bracket defined by (1.20) we have

$$\begin{aligned} \{\mathscr{P}, \mathscr{Q}\} + \{\mathscr{Q}, \mathscr{P}\} &= \iint_D [D_y(\zeta_x \mathsf{E}(P)\mathsf{E}(Q)) - D_x(\zeta_y \mathsf{E}(P)\mathsf{E}(Q))]dxdy = \\ &= \int_{\partial D} (\mathsf{E}(P)\mathsf{E}(Q)\zeta_y dy + \mathsf{E}(P)\mathsf{E}(Q)\zeta_x dx) = \int_{\partial D} \mathsf{E}(P)\mathsf{E}(Q)d\zeta \end{aligned}$$

If $D = \mathbb{R}^2$, for any function $\zeta = \zeta(x, y, t)$ vanishing sufficiently rapidly as $\sqrt{x^2 + y^2} \to +\infty$, we have that the boundary term on the right hand side is zero, and the Jacobi bracket is actually skew-symmetric as expected. However, when $D = \mathbb{R} \times (0, h)$, we are not legitimate any more to assume a solution $\zeta(x, y, t)$ to vanish near the boundary, while it can still be assumed to vanish for $|x| \to +\infty$. In this case we have [3]

$$\{\mathscr{P}, \mathscr{Q}\} + \{\mathscr{Q}, \mathscr{P}\} = \int_{-\infty}^{+\infty} (PQ\zeta_x)\big|_{y=0}^{y=h} dx,$$

which is evidently non zero in the general case. The boundary term still vanishes for all those solutions providing constant ζ along the boundary.

A similar phenomenon happens for the Boussinesq model (2.6). Given any two differential functions $P[\rho, \sigma], Q[\rho, \sigma] \in \mathscr{A}$, with corresponding functionals $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, we

[3] Hereafter we use the notation $(\,\cdot\,)\big|_{y=0}^{y=h} \equiv (\,\cdot\,)\big|_{y=h} - (\,\cdot\,)\big|_{y=0}$.

have

$$\{\mathscr{P},\mathscr{Q}\}+\{\mathscr{Q},\mathscr{P}\}=\iint_D D_x(\rho_y(\mathsf{E}_\rho(P)\mathsf{E}_\sigma(Q)+\mathsf{E}_\sigma(P)\mathsf{E}_\rho(Q))+\sigma_y\mathsf{E}_\sigma(Q)\mathsf{E}_\sigma(P))dxdy+$$
$$+\iint_D D_y(\rho_x(\mathsf{E}_\rho(P)\mathsf{E}_\sigma(Q)+\mathsf{E}_\sigma(P)\mathsf{E}_\rho(Q))+\sigma_x\mathsf{E}_\sigma(Q)\mathsf{E}_\sigma(P))dxdy$$

In order to make this expression more readable, let us define a couple of new differential functions $\tilde{P},\tilde{Q}\in\mathscr{A}$ as

$$\tilde{P}=\mathsf{E}_\rho(P)\mathsf{E}_\sigma(Q)+\mathsf{E}_\sigma(P)\mathsf{E}_\rho(Q)\qquad \tilde{Q}=\mathsf{E}_\sigma(Q)\mathsf{E}_\sigma(P)$$

Thus we have

$$\{\mathscr{P},\mathscr{Q}\}+\{\mathscr{Q},\mathscr{P}\}=\iint_D (D_x(\rho_y\tilde{P}+\sigma_y\tilde{Q})-D_y(\rho_x\tilde{P}+\sigma_x\tilde{Q}))dxdy$$

According to Stokes theorem, the right hand side can be expressed as a boundary integral,

$$\{\mathscr{P},\mathscr{Q}\}+\{\mathscr{Q},\mathscr{P}\}=\int_{\partial D}((\tilde{P}\rho_x+\tilde{Q}\sigma_x)dx+(\tilde{P}\rho_y+\tilde{Q}\sigma_y)dy)=\int_{\partial D}(\tilde{P}d\rho+\tilde{Q}d\sigma)$$

The boundary term clearly vanishes, if $D=\mathbb{R}^2$, for any solution decaying sufficiently fast at infinity. On the other hand, if $D=\mathbb{R}\times(0,h)$, for non-constant boundary conditions, the lack of antisymmetry of the Poisson bracket amounts to

$$S(\mathscr{P},\mathscr{Q})=\{\mathscr{P},\mathscr{Q}\}+\{\mathscr{Q},\mathscr{P}\}=-\int_{-\infty}^{\infty}\mathsf{E}(P)\cdot\begin{pmatrix}0&\rho_x\\ \rho_x&\sigma_x\end{pmatrix}\mathsf{E}(Q)\Big|_{y=0}^{y=h}dx \tag{2.22}$$

for any two functionals $\mathscr{P}=\int Pdxdy$ and $\mathscr{Q}=\int Qdxdy$. We remark that, despite this antisymmetry defect, the equations of motion (2.4) can still be written in the Hamiltonian form (2.6). As pointed out above, this result only relies on the homogeneous boundary conditions imposed on ψ. On the other hand, the lack of antisymmetry affects the correspondence of symmetries and conserved quantities. As an example, consider the family o Casimirs (2.19): if $\omega(x,t)$ is a solution of the system (2.6), we have

$$\frac{d}{dt}\mathscr{C}[\omega(\,\cdot\,,t)]=\{\mathscr{C},\mathscr{H}\}=-\{\mathscr{H},\mathscr{C}\}+S(\mathscr{C},\mathscr{H})=S(\mathscr{C},\mathscr{H})$$

Thanks to the homogeneous boundary conditions imposed on the stream function, the term $\mathsf{E}_\sigma(H)=\psi$ vanishes in (2.22), so we end with the following expression for the time derivative of $\mathscr{C}$:

$$\frac{d}{dt}\mathscr{C}[\omega(\,\cdot\,,t)]=\frac{1}{2}\int_{-\infty}^{\infty}\alpha(\rho)\rho_x|\nabla\psi|^2\Big|_{y=0}^{y=h}dx$$

which is evidently non zero in general. In particular, by taking $\alpha(\rho)=1$, we recover the time derivative of the total vorticity:

$$\frac{d}{dt}\mathscr{T}_2[\omega(\,\cdot\,,t)]=\frac{1}{2}\int_{-\infty}^{\infty}\rho_x|\nabla\psi|^2\Big|_{y=0}^{y=h}dx$$

The generator of horizontal translations may fail to be conserved too. Indeed, form its conservation law (2.21), we see that

$$\frac{d}{dt}\mathscr{T}_4[\omega(\,\cdot\,,t)] = \frac{h}{2}\int_{-\infty}^{\infty} \rho_x |\nabla\psi|^2\Big|_{y=h}\,dx$$

When constant boundary conditions are imposed on the density ρ, each of $\mathscr{T}_2$ and $\mathscr{T}_4$ are separately conserved, as well as any linear combination of them. So we can consider the one-parameter family of conserved quantities

$$\mathscr{T}_\lambda = \mathscr{T}_4 + \lambda\mathscr{T}_2 \tag{2.23}$$

The results above show that none of the $\mathscr{T}_\lambda$ is in general conserved when ρ is not constant on the boundaries. On the other hand, if ρ is constant on the lower boundary $y = 0$, while being non constant on the upper one, then just one element of the family $\mathscr{T}_\lambda$ keeps to be conserved: it is that corresponding to $\lambda = -h$

$$\mathscr{T}_{-h} = \mathscr{T}_4 - h\mathscr{T}_2 \tag{2.24}$$

The phenomenon just exemplified is called *topological selection* of conserved quantities by Camassa *et al.* [9]. Indeed, it can be related to the topological properties of the density function ρ: assuming it to be a monotone non-increasing function of y, then all its level sets are necessarily connected if ρ is constant on both horizontal boundaries; conversely, for non constant ρ on the upper boundary, some level sets can be disconnected.

This point of view is further explored in the work of Camassa *et al.* [9]. They analyze the particular case of a fluid composed of two layers with different constant densities: the water with density ρ_0 and the air with zero density. Let the air-water interface be described as the graph of a function $y = \eta(x,t)$ (see Figure 2.1). The time evolution of

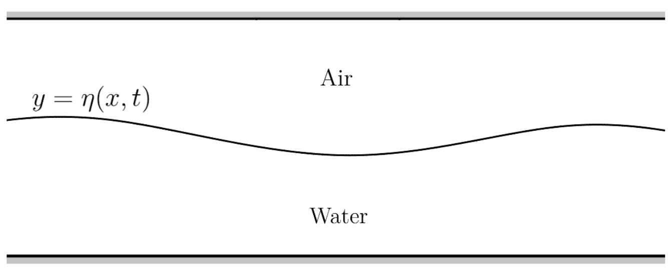


Figure 2.1

the system is described by the incompressible Euler equations: on the air domain they reduce to

$$\nabla p = 0, \qquad \text{for}\ \ \eta(x,t) < y < h \tag{2.25}$$

whereas on the water domain they read

$$\begin{cases} \rho_0(u_t + uu_x + vu_y) = -p_x \\ \rho_0(v_t + uv_x + vv_y) = -p_y - \rho_0 g \\ u_x + v_y = 0 \end{cases} \qquad \text{for } \; 0 < y < \eta(x,t) \tag{2.26}$$

The boundary conditions for this system are assigned by requiring zero vertical velocities at the rigid plates, and continuity of normal velocity and pressure at the fluid interface. We also assume localized initial data, so that u and v tends to zero, and the water surface η approaches a constant asymptotic level sufficiently fast as $|x| \to 0$. Due to this last condition, the difference of the pressures at the far ends of the domain is a constant, irrespective of y and t:

$$\lim_{x\to+\infty} p(x,y,t) - \lim_{x\to-\infty} p(x,y,t) \equiv P_\Delta = \text{const.}$$

From equation (2.25), it follows that, if the air domain is connected, then pressure is constant everywhere on it. In particular, $P_\Delta = 0$ for this case. On the other hand, if the water is partially in contact with the upper plate (see Fig. 2.2), the air domain becomes disconnected, and equation (2.25) now implies the pressure to be constant on each of its connected component, though with different values. In this condition, the far ends pressures do not agree in general: $P_\Delta \neq 0$. The total horizontal momentum, which

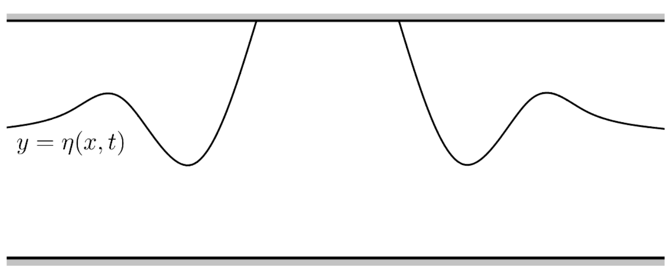


Figure 2.2

coincides with that of the only water, is given by

$$\Pi = \rho_0 \int_{-\infty}^{+\infty} \int_0^{\eta} u dy dx$$

and its time derivative is

$$\dot{\Pi} = -hP_\Delta$$

From this equation, we see that if the air domain is connected, then the total horizontal momentum is a conserved quantity for the system. Conversely, it may fail to be conserved if the air domain is disconnected. For practical reasons, we define the boundary fields

$$u^-(x,t) \equiv \lim_{y\to 0^+} u(x,y,t) \qquad u^+(x,t) \equiv \lim_{y\to h^-} u(x,y,t)$$

and similarly for all other dependent variables in the Euler system. With this notation, we consider the "lower-boundary momentum" as defined by

$$\Pi^- = h\int_{-\infty}^{+\infty} \rho^-(x,t)u^-(x,t)dx$$

It is possible to show that also this quantity is time-invariant. Indeed, from the boundary condition $v^-(x,t) = 0$, and the assumed hydrostatic conditions for $|x| \to \infty$, it follows

$$\dot{\Pi}^- = -h\int_{-\infty}^{+\infty} (\rho_0 u^- u_x^- + p_x^-)dx = -hP_\Delta$$

Hence, as for the total horizontal momentum, Π^- is conserved if the air domain is connected. This allows us to define a one-parameter family of conserved quantities

$$\Pi_\lambda = \Pi + \lambda\Pi^- \tag{2.27}$$

which is the analogue of (2.23) for the continuously stratified fluid. Whenever the air domain is disconnected, neither Π nor Π^- are separately conserved; however, one member of the family (2.27) still is, which corresponds to $\lambda = -1$:

$$\dot{\Pi}_{-1} = \dot{\Pi} - \dot{\Pi}^- = 0 \tag{2.28}$$

This phenomenon is the analog of (2.24) for the continuously stratified fluid, and exemplifies what we called topological selection of conserved quantities: the topological change of the air domain from connected to disconnected results in the collapse of a whole family of conserved quantities (2.27) into the single quantity (2.28).

Chapter 3

The Shallow Water Model

The study carried out by Camassa *et al.* [9], concerning the air-water system, naturally leads to a question: can the time evolution of the system bring the water to come in contact with the upper boundary? Or, that is the same, can the dynamics make the water to detach from a partially wet upper boundary? From an experimental point of view, one expects the answer to be yes. On the other hand, the phenomenon of topological selection of conserved quantities seem to suggest the opposite: if such a transition happens, some quantities could not be conserved any more and vice versa. To address this point in more detail, we consider the simpler model of two-layer shallow water equations:

$$\begin{cases} h_{1t} + (u_1 h_1)_x = 0 \\ h_{2t} + (u_2 h_2)_x = 0 \\ u_{1t} + (\frac{u_1^2}{2} + h_1 + \lambda h_2)_x = 0 \\ u_{2t} + (\frac{u_2^2}{2} + h_1 + h_2)_x = 0 \end{cases} \tag{3.1}$$

The meaning of the variables is as follows: h_1 and h_2 are the thickness of the two fluid layers with constant density ρ_1 and ρ_2 (see Fig. 3.1); the parameter $\lambda = \frac{\rho_2}{\rho_1}$ is the ratio between the two densities, and it is assumed to be in the range $0 < \lambda < 1$; finally, u_1 and u_2 represent the mean velocity in the respective fluid layers. There is one evident

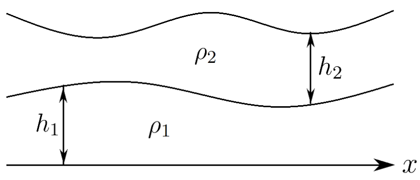


Figure 3.1

difference between this model and the Euler equations for the air-water system of §2.4:

the upper boundary is no more rigid, but it rather coincides with the free surface of the lighter fluid. Despite this, we are still legitimate to ask whether the interface between the two fluids could get in contact with the upper free surface, as a result of the dynamics.

Various mathematical aspects of the two-layer shallow water equations (3.1) are investigated by Ovsyannikov [11]. Of particular relevance is his study of the hyperbolic nature of the system, depending on the flow regime. Let $\mathbf{U} = (h_1, h_2, u_1, u_2)^\top$ denote the vector of dependent variables. Then, the system of equations (3.1) can be written as

$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = 0$$

The characteristic polynomial of the matrix $A(\mathbf{U})$ is

$$P(k) = ((u_1 - k)^2 - h_1)((u_2 - k)^2 - h_2) - \lambda h_1 h_2$$

To qualitatively study the roots of $P(k)$, is convenient to introduce the following auxiliary variables:

$$p = \frac{u_1 - k}{\sqrt{h_1}} \qquad q = \frac{u_2 - k}{\sqrt{h_2}} \tag{3.2}$$

With the new variables, the characteristic equation $P(k) = 0$ takes the form

$$(p^2 - 1)(q^2 - 1) = \lambda \tag{3.3}$$

which represents a fourth order curve in the (p, q) plane having four axes of symmetry (see Fig. 3.2). Also, by eliminating k from equations (3.2), we get the relationship

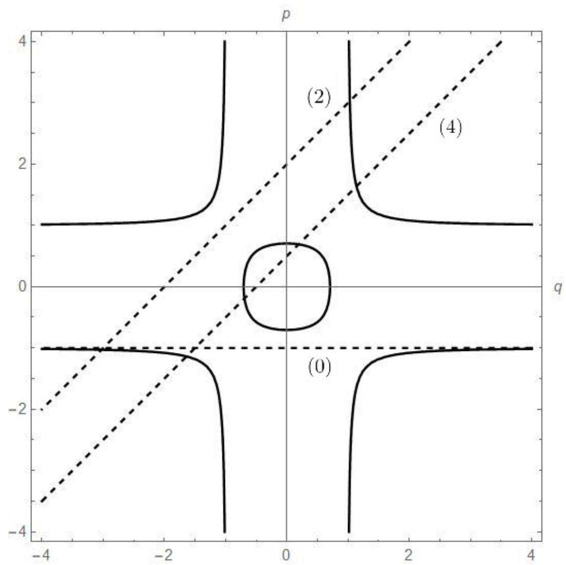


Figure 3.2: The algebraic curve (3.3) for $\lambda = \frac{1}{4}$ (solid) and some of the lines (3.4) (dashed), along with the number of intersection points with the curve (3.3).

$$u_1 - p\sqrt{h_1} = u_2 - q\sqrt{h_2} \tag{3.4}$$

Equation (3.4) describes a line in the (p, q) plane, and its intersections with the curve (3.3) determines the real roots of the characteristic polynomial $P(k)$: if a solution (p, q) is obtained for a given value of $\mathbf{U}$, then one real eigenvalue is found by $k = u_1 - p\sqrt{h_1} = u_2 - q\sqrt{h_2}$. For non zero h_1 and h_2, it follows from Fig. 3.2 that the line (3.4) always has two or four points of intersection with the curve (3.3). Remarkably, if at least one among h_1 and h_2 is zero, then the line (3.4) is parallel to one of the coordinate axes, and the system (3.1) is precluded to be strictly hyperbolic (four distinct real eigenvalues). In particular, there can be no real eigenvalues in this case: for example, the line $p = -1$, corresponding to $h_2 = 0$ and $\sqrt{h_2} = u_2 - u_1$, does not intersect the curve (3.3) for any value $0 < \lambda < 1$. Based on this discussion, it is concluded that the system is of mixed type: its character varies locally according to the solution.

One further important result from the paper [11] is the detection of two conservation laws in addition to those composing the system (3.1), i.e. equations of the form $M_t + N_x = 0$, which can be interpreted as the laws of conservation of momentum and total energy of every single layer of the fluid:

$$\begin{cases} M_1 = 2h_1u_1 + 2\lambda h_2u_2 \\ N_1 = 2h_1u_1^2 + 2\lambda h_2u_2^2 + h_1^2 + 2\lambda h_1h_2 + \lambda h_2^2 \end{cases} \tag{3.5}$$

$$\begin{cases} M_2 = h_1u_1^2 + \lambda h_2u_2^2 + h_1^2 + 2\lambda h_1h_2 + \lambda h_2^2 \\ N_2 = h_1u_1^3 + \lambda h_2u_2^3 + 2h_1^2u_1 + 2\lambda h_1h_2(u_1 + u_2) + 2\lambda h_2^2u_2 \end{cases} \tag{3.6}$$

This observation allows to detect an Hamiltonian structure for the system of equations (3.1). Indeed, taking $H = M_2$ as the Hamiltonian, and defining

$$\mathscr{D} = -\frac{1}{2}\begin{pmatrix} 0 & 0 & D_x & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda}D_x \\ D_x & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda}D_x & 0 & 0 \end{pmatrix}, \tag{3.7}$$

we see the system (3.1) can be written in the form

$$\mathbf{U}_t = \mathscr{D}\mathsf{E}(H). \tag{3.8}$$

where the Euler derivative of H is

$$\mathsf{E}(H) = \begin{bmatrix} \mathsf{E}_{h_1}(H) \\ \mathsf{E}_{h_2}(H) \\ \mathsf{E}_{u_1}(H) \\ \mathsf{E}_{u_2}(H) \end{bmatrix} = \begin{bmatrix} u_1^2 + 2h_1 + 2\lambda h_2 \\ \lambda u_2^2 + 2\lambda h_1 + 2\lambda h_2 \\ 2h_1u_1 \\ 2\lambda h_2u_2 \end{bmatrix}$$

Note that the shallow water equations (3.1) constitute a system of Hydrodynamic type in the sense of Definition 1.5.4. This makes Theorem 1.5.2 available to check for the skew-symmetry and the Jacobi identity. As described in section 1.5.1, to the Poisson structure (3.7) is associated a pseudo-Riemannian metric on the space of dependent variables,

$$g^{ij} = -\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with signature (2,2), as well as the trivial connection, i.e. with vanishing Christoffel symbols. They satisfy all the conditions of Theorem 1.5.2, so it is concluded that the operator (3.7) is Hamiltonian (cf. Definition 1.5.3). Let us note that the system (3.1) can be written in the form (3.8) in an alternative way, that is with Hamiltonian $H = -\frac{M_1}{4}$ and

$$\begin{aligned}\mathscr{D} &= \begin{pmatrix} h_1 D_x + D_x \circ h_1 & 0 & u_1 D_x & 0 \\ 0 & \frac{1}{\lambda}(h_2 D_x + D_x) \circ h_2 & 0 & \frac{1}{\lambda} u_2 D_x \\ D_x \circ u_1 & 0 & 2D_x & 2D_x \\ 0 & \frac{1}{\lambda} D_x \circ u_2 & 2D_x & \frac{2}{\lambda} D_x \end{pmatrix} = \\ &= \begin{pmatrix} 2h_1 & 0 & u_1 & 0 \\ 0 & \frac{2}{\lambda} h_2 & 0 & \frac{1}{\lambda} u_2 \\ u_1 & 0 & 2 & 2 \\ 0 & \frac{1}{\lambda} u_2 & 2 & \frac{2}{\lambda} \end{pmatrix} D_x + \begin{pmatrix} h_{1x} & 0 & 0 & 0 \\ 0 & \frac{h_{2x}}{\lambda} & 0 & 0 \\ u_{1x} & 0 & 0 & 0 \\ 0 & \frac{u_{2x}}{\lambda} & 0 & 0 \end{pmatrix}\end{aligned} \tag{3.9}$$

In spite of this, the metric g^{ij} and the connection Γ^i_{jk}, which the operator (3.9) defines on the space of dependent variables, satisfy the first point of Theorem 1.5.2, but not the second one. Specifically, they are compatible, but the connection Γ^i_{jk} has non vanishing curvature and torsion. Hence the Jacobi bracket defined by (3.9) is skew-symmetric but does not satisfy the Jacobi identity.

In light of the theory developed in §1.5.3, we see that each of the conservation laws (3.1) give rise to the trivial symmetry $\mathrm{v} = 0$. Indeed, any functional of the type

$$\mathscr{C} = \int (\alpha h_1 + \beta h_2 + \gamma u_1 + \delta u_2) dx, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

is clearly a Casimir for the Poisson structure specified by (3.7). On the other hand, for the conservation law (3.5) we have

$$\mathscr{D}\mathsf{E}(M_1) = (-h_{1x}, -h_{2x}, -u_{1x}, -u_{2x})^\top = -\mathbf{U}_x$$

which represents the characteristic of the corresponding symmetry. Based on formula (1.1) for the characteristic of an evolutionary vector field, we see that the conserved density M_1 corresponds to the (geometric) symmetry ∂_x. In other words, the Hamiltonian vector field

$$\mathrm{v}_{\mathscr{D}\mathsf{E}(M_1)} = \partial_x$$

is a symmetry for the system (3.8). It simply stands for the translational invariance of the system (3.1), with respect to the x variable, and justifies the physical interpretation of M_1 as the horizontal linear momentum. By the same line of reasoning, we confirm the Hamiltonian $H = M_2$ to represent the total energy of the system, since it is the generator of the time translational symmetry.

3.1 Polynomial Type Solutions

The system of equations (3.1) admits a special class of solutions, which are polynomial in the spatial variable:

$$\begin{cases} h_1(x,t) = \gamma_1(t)x^2 + \delta_1(t)x + \epsilon_1(t) \\ h_2(x,t) = \gamma_2(t)x^2 + \delta_2(t)x + \epsilon_2(t) \\ u_1(x,t) = \alpha_1(t)x + \beta_1(t) \\ u_2(x,t) = \alpha_2(t)x + \beta_2(t) \end{cases} \tag{3.10}$$

where the coefficient functions $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i,\ i = 1,2$, satisfy the following system of ordinary differential equations

$$\begin{cases} \dot\alpha_1 + \alpha_1^2 + 2\gamma_1 + 2\lambda\gamma_2 = 0 \\ \dot\beta_1 + \alpha_1\beta_1 + \delta_1 + \lambda\delta_2 = 0 \\ \dot\gamma_1 + 3\alpha_1\gamma_1 = 0 \\ \dot\delta_1 + 2\beta_1\gamma_1 + 2\alpha_1\delta_1 = 0 \\ \dot\epsilon_1 + \beta_1\delta_1 + \alpha_1\epsilon_1 = 0 \\ \dot\alpha_2 + \alpha_2^2 + 2\gamma_1 + 2\gamma_2 = 0 \\ \dot\beta_2 + \alpha_2\beta_2 + \delta_1 + \delta_2 = 0 \\ \dot\gamma_2 + 3\alpha_2\gamma_2 = 0 \\ \dot\delta_2 + 2\beta_2\gamma_2 + 2\alpha_2\delta_2 = 0 \\ \dot\epsilon_2 + \beta_2\delta_2 + \alpha_2\epsilon_2 = 0 \end{cases} \tag{3.11}$$

Remark 3.1.1. From a physical point of view, the class of solutions (3.10) has poor global meaning, as the field variables are generically unbounded: it should rather be understood as a local representation.

As mentioned above we are primarily interested in the dynamical interaction between the surfaces bounding the two fluids. To qualitatively study the system of equations (3.11), it is useful to find constants of the motion. One way to do that is to integrate the conservation laws of the system (3.1) on suitable spatial domains, as explained below. Let

$$M_t + N_x = 0 \tag{3.12}$$

be a conservation law for the system (3.1); we search for constants of the motion of the form

$$\int_{a(t)}^{b(t)} M(x,t)dx \tag{3.13}$$

for some appropriate functions $a(t), b(t)$. They are chosen such that the time derivative of (3.13) vanishes. We have

$$\frac{d}{dt}\int_{a(t)}^{b(t)} M dx = \int_{a(t)}^{b(t)} M_t dx + \dot b(t)M(b(t),t) - \dot a(t)M(a(t),t)$$

Substituting according to equation (3.12) and equating to zero, we arrive at the following differential equation for the extremes of integration $a(t), b(t)$:

$$N(a(t), t) - N(b(t), t) + \dot{b}(t)M(b(t), t) - \dot{a}(t)M(a(t), t) = 0 \tag{3.14}$$

Let us denote by

$$\xi = (\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \epsilon_2)^\top \tag{3.15}$$

the vector of dependent variables of the ODE system (3.11). Then, the time derivatives of a and b can be expanded as

$$\dot{a} = \sum_i \frac{\partial a}{\partial \xi_i}\dot{\xi}_i \qquad \dot{b} = \sum_i \frac{\partial b}{\partial \xi_i}\dot{\xi}_i$$

Finally, substituting for $\dot{\xi}$ according to equations (3.11) and inserting in (3.14), we obtain a nonlinear partial differential equation for $a(\xi)$ and $b(\xi)$,

$$N(a(\xi), \xi) - N(b(\xi), \xi) + M(b(\xi), \xi)\sum_i \frac{\partial b}{\partial \xi_i}\dot{\xi}_i - M(a(\xi), \xi)\sum_i \frac{\partial a}{\partial \xi_i}\dot{\xi}_i = 0$$

Finding a solution to such equation can be difficult as much as directly solving system (3.11). In spite of this, for the particular case of the mass conservation laws $(3.1)_1$ and $(3.1)_2$, we are able to explicitly find the functions $a(\xi), b(\xi)$. For example, for the first one we have $M = h_1$ and $N = u_1 h_1$, so equation (3.14) gives

$$u_1(a(t), t)h_1(a(t), t) - u_1(b(t), t)h_1(b(t), t) + \dot{b}(t)h_1(b(t), t) - \dot{a}(t)h_1(a(t), t) = 0$$

Hence, if we choose $a(t)$ and $b(t)$ as the roots of $h_1(x, t)$, then equation (3.14) will be identically satisfied. So we take

$$a = \frac{-\delta_1 - \sqrt{\Delta_1}}{2\gamma_1}, \qquad b = \frac{-\delta_1 + \sqrt{\Delta_1}}{2\gamma_1}$$

where, for convenience, have been defined $\Delta_1 = \delta_1^2 - 4\gamma_1\epsilon_1$ and $\Delta_2 = \delta_2^2 - 4\gamma_2\epsilon_2$. In this case, the integral (3.13) gives

$$\begin{aligned}
\int_{\frac{-\delta_1-\sqrt{\Delta_1}}{2\gamma_1}}^{\frac{-\delta_1+\sqrt{\Delta_1}}{2\gamma_1}} (\gamma_1 x^2 + \delta_1 x + \epsilon_1)dx &= \frac{\delta_1^2\sqrt{\Delta_1}}{4\gamma_1} - \frac{\Delta_1^{3/2}}{12\gamma_1^2} - \frac{\sqrt{\Delta_1}\epsilon_1}{\gamma_1} = \\
&= \frac{\sqrt{\Delta_1}}{\gamma_1}\left(\frac{\delta_1^2}{4\gamma_1} - \frac{\Delta_1}{12\gamma_1} - \epsilon_1\right) = \frac{\sqrt{\Delta_1}}{\gamma_1}\frac{3\delta_1^2 - \Delta_1 - 12\gamma_1\epsilon_1}{12\gamma_1} = \\
&= \frac{\sqrt{\Delta_1}}{\gamma_1}\frac{3\Delta_1 - \Delta_1}{12\gamma_1} = \frac{\Delta_1^{3/2}}{6\gamma_1^2}
\end{aligned}$$

By taking the square of the above integral, we end up with the conserved quantity

$$\kappa_1 = \frac{\Delta_1^3}{\gamma_1^4} = \text{const.} \tag{3.16}$$

We remark that, although for the computations involved it is needed $\Delta_1 > 0$, the final result holds regardless of this. Similarly, we get the second conserved quantity

$$\kappa_2 = \frac{\Delta_2^3}{\gamma_2^4} = \text{const.} \tag{3.17}$$

The existence of these conserved quantities allows to answer the question posed at the beginning of this chapter, at least for the polynomial family of solutions. Indeed, from the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ is clear that the subspaces $\gamma_1 = 0$ and $\gamma_2 = 0$ are invariant for the dynamical system (3.11). Hence the denominator of κ_1 and κ_2 is always strictly positive. This implies the discriminants Δ_1 and Δ_2 to be of constant sign, and that the functions h_1 and h_2 have a constant number of roots during the time evolution of the system. In particular, if at the initial time h_2 is nowhere zero, i.e. the upper free surface, $y = h_1 + h_2$, does not intersect the interface between the two fluids, $y = h_1$, then these two lines never intersect in the future. Also, if a unique point of contact is present, then the two lines keep having a single point of contact for all times, and if the two lines intersect in a couple of distinct points, then they maintain this condition. Similar results hold among the bottom line, $y = 0$, and the interface between the two fluids, $y = h_1$.

As γ_1 and γ_2 are initially zero, i.e. h_1 and h_2 are affine functions of x, the conserved quantities (3.16) and (3.17) break down. In this case we can not rely on them to study the interaction between the fluid interface $y = h_1$ and the upper free surface $y = h_1 + h_2$. However, note that the algebraic varieties

$$V_1 = \{\xi \in \mathbb{R}^{10} : \Delta_1 = 0\} \qquad V_2 = \{\xi \in \mathbb{R}^{10} : \Delta_2 = 0\}$$

are invariant under the flow of the dynamical system (3.11). Indeed it holds

$$\dot{\Delta}_1 = -4\alpha_1\Delta_1 \qquad \dot{\Delta}_2 = -4\alpha_2\Delta_2$$

For non zero γ_1, γ_2, this result leads to the same conclusions above. On the other hand, whenever $\gamma_1 = 0$ and $\gamma_2 = 0$, we have

$$\Delta_1 = |\delta_1| \qquad \Delta_2 = |\delta_2|$$

This implies that if the fluid interface, $y = h_1$ and the upper free surface $y = h_1 + h_2$ are initially parallel, i.e. $\delta_2 = 0$, then they evolve maintaining this condition. Similarly, if they initially intersect in a point, then they keep having one point of intersection for all times. Analogous results holds for the interaction of the interface, $y = h_1$, with the bottom, $y = 0$.

Strictly speaking, these outcomes hold true only for the polynomial class of solutions considered here. However, they lead one to conjecture that the phenomenon of interfaces crossing is not allowed in general by the model (3.1). Indeed, at least for analytical solutions, the zero set of the function h_2 is made of a discrete set of points, near which it is generically approximated by a parabola. An obstruction to this more general statement is the possible occurrence of points near which the Taylor series of h_2 begins with a fourth or greater order term.

Now we further investigate the qualitative behavior of solutions of the polynomial type (3.10). We interpret the system of equations (3.11) as a dynamical system on $\mathbb{R}^{10}$, with coordinates $\xi = (\xi_1, ..., \xi_{10})$ defined by (3.15). As noted above, the subspaces $\gamma_1 = 0$ and $\gamma_2 = 0$ are invariant under the flow of the system. This allows one to study the simple case $\gamma_1 = 0$, $\gamma_2 = 0$, which corresponds to h_1 and h_2 being affine functions of x. In this case, the system (3.11) becomes triangular, and directly solvable. First, we solve for α_1 and α_2 from the equations

$$\begin{cases} \dot{\alpha}_1 + \alpha_1^2 = 0 \\ \dot{\alpha}_2 + \alpha_2^2 = 0 \end{cases}$$

which give

$$\alpha_1(t) = \frac{\bar{\alpha}_1}{\bar{\alpha}_1 t + 1} \qquad \alpha_2(t) = \frac{\bar{\alpha}_2}{\bar{\alpha}_2 t + 1}$$

Hereafter, the overline indicates the initial value of the corresponding variable. We see that as $\bar{\alpha}_1 < 0$ the system have a blow-up at the finite time $t = -\frac{1}{\bar{\alpha}_1}$, and similarly for α_2. Conversely, if $\bar{\alpha}_1 > 0$ and $\bar{\alpha}_2 > 0$, the system evolves towards the steady state $\alpha_1 = \alpha_2 = 0$ for $t \to +\infty$.

Subsequently, we solve for δ_1 and δ_2 from the equations

$$\begin{cases} \dot{\delta}_1 + 2\alpha_1\delta_1 = 0 \\ \dot{\delta}_2 + 2\alpha_2\delta_2 = 0 \end{cases}$$

which give

$$\delta_1(t) = \frac{\bar{\delta}_1}{(\bar{\alpha}_1 t + 1)^2} \qquad \delta_2(t) = \frac{\bar{\delta}_2}{(\bar{\alpha}_2 t + 1)^2}$$

Next, it is possible to integrate the equations

$$\begin{cases} \dot{\beta}_1 + \alpha_1\beta_1 + \delta_1 + \lambda\delta_2 = 0 \\ \dot{\beta}_2 + \alpha_2\beta_2 + \delta_1 + \delta_2 = 0 \end{cases}$$

from which β_1 and β_2 are obtained:

$$\begin{aligned} \beta_1(t) =& (-\bar{\alpha}_2^2(1 + t\bar{\alpha}_2)\bar{\delta}_1 \log(1 + t\bar{\alpha}_1) + \bar{\alpha}_1(\bar{\alpha}_2(\bar{\alpha}_2(1 + t\bar{\alpha}_2)\bar{\beta}_1 + t(\bar{\alpha}_1 - \bar{\alpha}_2)\bar{\delta}_2\lambda) + \\ &- \bar{\alpha}_1(1 + t\bar{\alpha}_2)\bar{\delta}_2\lambda \log(1 + t\bar{\alpha}_2)))/(\bar{\alpha}_1(1 + t\bar{\alpha}_1)\bar{\alpha}_2^2(1 + t\bar{\alpha}_2)) \end{aligned}$$

$$\begin{aligned} \beta_2(t) =& (\bar{\alpha}_1\bar{\alpha}_2(t\bar{\alpha}_1^2\bar{\beta}_2 + t\bar{\alpha}_2\bar{\delta}_1 + \bar{\alpha}_1(\bar{\beta}_2 - t\bar{\delta}_1)) - (1 + t\bar{\alpha}_1)(\bar{\alpha}_2^2\bar{\delta}_1 \log(1 + t\bar{\alpha}_1) + \\ &+ \bar{\alpha}_1^2\bar{\delta}_2 \log(1 + t\bar{\alpha}_2)))/(\bar{\alpha}_1^2(1 + t\bar{\alpha}_1)\bar{\alpha}_2(1 + t\bar{\alpha}_2)) \end{aligned}$$

Finally, from the remaining equations

$$\begin{cases} \dot{\epsilon}_1 + \beta_1\delta_1 + \alpha_1\epsilon_1 = 0 \\ \dot{\epsilon}_2 + \beta_2\delta_2 + \alpha_2\epsilon_2 = 0 \end{cases}$$

we get the last two variables ϵ_1 and ϵ_2:

$$\begin{aligned}\epsilon_1(t) =&(-\bar{\alpha}_2^2\bar{\delta}_1^2\log(1+t\bar{\alpha}_1)+\bar{\alpha}_1(\bar{\alpha}_2(\bar{\alpha}_1\bar{\alpha}_2\bar{\epsilon}_1+t(-\bar{\alpha}_1\bar{\alpha}_2\bar{\beta}_1\bar{\delta}_1+\bar{\alpha}_2\bar{\delta}_1^2+\bar{\alpha}_1^2\bar{\alpha}_2\bar{\epsilon}_1+\bar{\alpha}_1\bar{\delta}_1\bar{\delta}_2\lambda))+\\&-\bar{\alpha}_1\bar{\delta}_1\bar{\delta}_2\lambda\log(1+t\bar{\alpha}_2)))/(\bar{\alpha}_1^2(1+t\bar{\alpha}_1)^2\bar{\alpha}_2^2)\end{aligned}$$

$$\begin{aligned}\epsilon_2(t) =&(-\bar{\alpha}_2^2\bar{\delta}_1\bar{\delta}_2\log(1+t\bar{\alpha}_1)+\bar{\alpha}_1(\bar{\alpha}_2(t\bar{\alpha}_2\bar{\delta}_1\bar{\delta}_2+\bar{\alpha}_1\bar{\alpha}_2\bar{\epsilon}_2+t\bar{\alpha}_1(-\bar{\alpha}_2\bar{\beta}_2\bar{\delta}_2+\bar{\delta}_2^2+\bar{\alpha}_2^2\bar{\epsilon}_2))+\\&-\bar{\alpha}_1\bar{\delta}_2^2\log(1+t\bar{\alpha}_2)))/(\bar{\alpha}_1^2\bar{\alpha}_2^2(1+t\bar{\alpha}_2)^2)\end{aligned}$$

By examining the obtained solution, we see that the overall qualitative behavior is governed by the sign of the initial values $\bar{\alpha}_1$, $\bar{\alpha}_2$. Specifically, for positive $\bar{\alpha}_1$ and $\bar{\alpha}_2$, all the fields decay towards zero for $t \to +\infty$, as Figures 3.3 and 3.4 show. At the opposite side, if one among $\bar{\alpha}_1$ and $\bar{\alpha}_2$ is negative, much of the fields meet a blow-up at finite time (see Fig. 3.5 and Fig. 3.6).

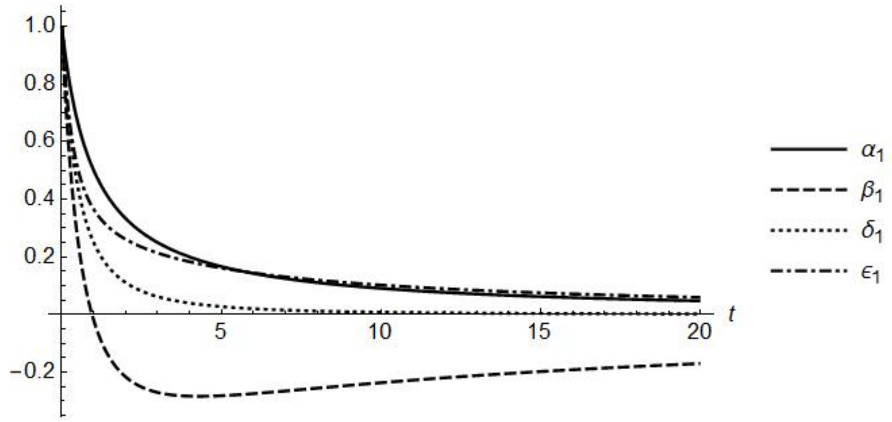


Figure 3.3: $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\beta}_1 = \bar{\beta}_2 = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 1$, $\lambda = 1/2$.

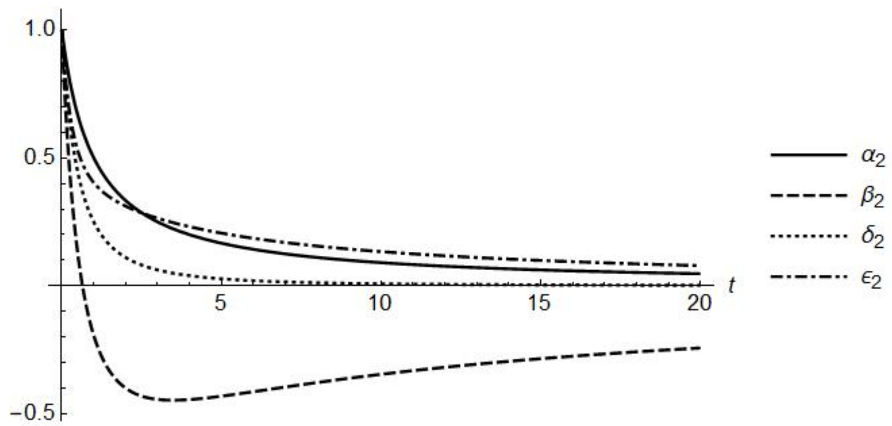


Figure 3.4: $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\beta}_1 = \bar{\beta}_2 = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 1$, $\lambda = 1/2$.

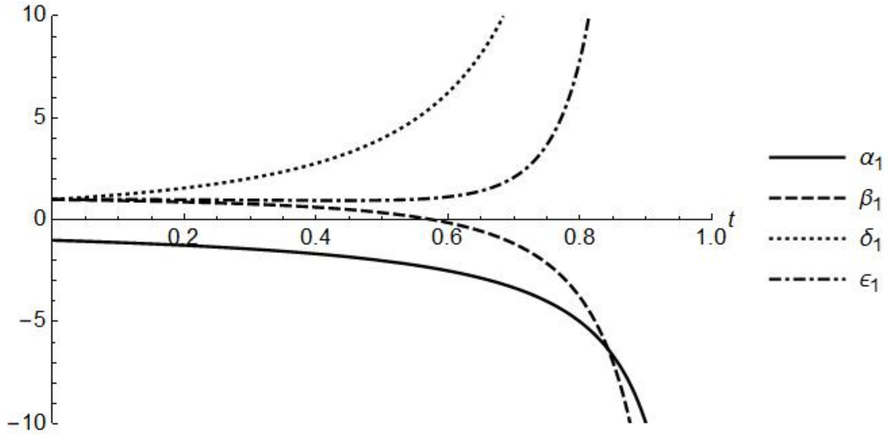


Figure 3.5: $\bar{\alpha}_1 = -1$, $\bar{\alpha}_2 = \bar{\beta}_1 = \bar{\beta}_2 = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 1$, $\lambda = 1/2$.

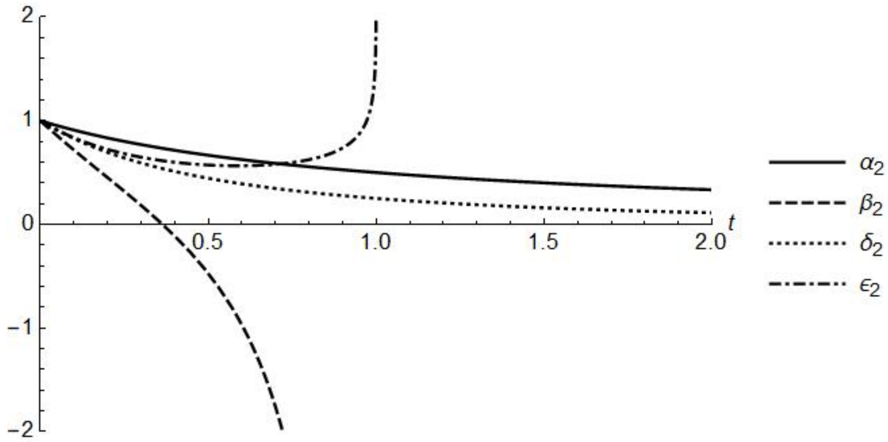


Figure 3.6: $\bar{\alpha}_1 = -1$, $\bar{\alpha}_2 = \bar{\beta}_1 = \bar{\beta}_2 = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 1$, $\lambda = 1/2$.

We now consider the case in which only one of the curvatures, say γ_2, is non zero. Putting $\gamma_1 = 0$, two equations decouple from the system (3.11):

$$\begin{cases} \dot{\alpha}_2 = -\alpha_2^2 - 2\gamma_2 \\ \dot{\gamma}_2 = -3\alpha_2\gamma_2 \end{cases} \tag{3.18}$$

By eliminating the time variable between the two equations, we get

$$\frac{d\alpha}{d\gamma} = \frac{1}{3}\frac{\alpha}{\gamma} + \frac{2}{3}\frac{1}{\alpha}$$

that is,

$$\frac{1}{2}\frac{d\alpha^2}{d\gamma} = \frac{1}{3}\frac{\alpha^2}{\gamma} + \frac{2}{3}$$

By setting $w = \alpha^2$, the equation becomes linear in the new variable. Furthermore, an integrating factor is easily found to be $\gamma^{-2/3}$, so that the equation becomes

$$\frac{d}{d\gamma}(\gamma^{-2/3}w) = \frac{4}{3}\gamma^{-2/3}$$

The general solution to this equation is

$$\kappa = \frac{\alpha_2^2}{\gamma_2^{2/3}} - 4\gamma_2^{1/3} \tag{3.19}$$

where κ is a constant, specified by the initial conditions, i.e. a constant of the motion for the system (3.18). This result allows to draw a phase diagram for the system (3.18) (see Fig. 3.7). There is a single (unstable) equilibrium point at $(\alpha_2, \gamma_2) = (0{,}0)$. The curve corresponding to $\kappa = 0$ (dashed in Fig. 3.7) is a parabola, made of two trajectories plus the equilibrium point. It plays the role of a separatrix line: all the trajectories above it, corresponding to $\kappa < 0$, never tend to the equilibrium point neither in the future nor in the past; instead, the trajectories below it, corresponding to $\kappa > 0$ do the opposite. In particular, the trajectories corresponding to $\gamma_2 < 0$ are homoclinic orbits, traveled in anticlockwise sense. This allows to conclude that for every initial condition with $\bar{\gamma}_2 < 0$, the solution $(\alpha_2(t), \gamma_2(t))$ is bounded for all times. The trajectories corresponding to positive γ_2, are traveled form the right to the the left, so that all of those lying below the separatrix and with $\alpha_2 > 0$ correspond to bounded solutions tending to the equilibrium point for $t \to +\infty$. At the opposite, all of the other trajectories lying in the upper half-plane are unbounded.

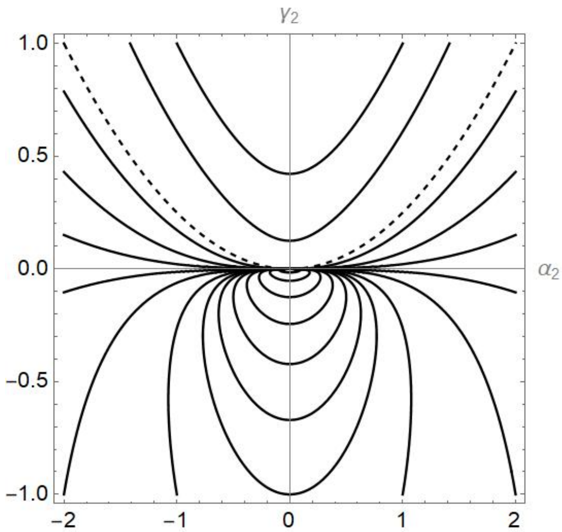


Figure 3.7

Substituting for α_2^2 according to equation (3.19), we get the single first order equation

$$\dot{\gamma}_2^2 = 36\gamma_2^3 + 9\kappa\gamma_2^{8/3}$$

whose solution is obtained in implicit form:

$$t(\gamma_2) = -\frac{\sqrt{\kappa\gamma_2^{8/3} + 4\gamma_2^3}}{\kappa\gamma_2^{5/3}} + \frac{4}{\kappa^{3/2}} \tanh^{-1}\left[\frac{\sqrt{\kappa\gamma_2^{8/3} + 4\gamma_2^3}}{\sqrt{\kappa}\gamma_2^{4/3}}\right] - \bar{C}$$

Here, $\bar{C}$ is a constant, depending on initial conditions, determined by requiring $t(\bar{\gamma}_2) = 0$. This solution allow to conclude that all the trajectories in the upper half plane, except those tending to the equilibrium point, blow up in finite time. For example, if $\bar{\alpha}_2 = 0$ and $\bar{\gamma}_2 > 0$, we have $\kappa = -4\bar{\gamma}^{1/3}$ and $\bar{C} = 0$, so a singularity occurs when t approaches the value

$$\lim_{\gamma_2 \to +\infty} t(\gamma_2) = \frac{\pi}{4\sqrt{\bar{\gamma}_2}}$$

3.2 Further Developments

The general case, corresponding to non vanishing γ_1 and γ_2 is much harder to study. An important simplification comes from noting that the subspace

$$W = \{\xi \in \mathbb{R}^{10} : \beta_1 = \beta_2 = \delta_1 = \delta_2 = 0\}$$

is invariant under the flow of (3.11). Hence, the restriction of (3.11) to W is a well defined dynamical system. Also, the constants of the motion detected above imply that, on W,

$$\epsilon_1 = -\frac{\sqrt[3]{\kappa_1\gamma_1}}{4}, \qquad \epsilon_2 = -\frac{\sqrt[3]{\kappa_2\gamma_2}}{4}.$$

From a physical point of view, solutions on W corresponds to linear velocity fields u_1, u_2, and coaxial parabolas for h_1, h_2:

$$\begin{cases} h_1 = \gamma_1 x^2 - \frac{\sqrt[3]{\kappa_1\gamma_1}}{4} \\ h_2 = \gamma_2 x^2 - \frac{\sqrt[3]{\kappa_2\gamma_2}}{4} \\ u_1 = \alpha_1 x \\ u_2 = \alpha_2 x \end{cases}$$

Hence, the dynamics on W is completely described by the system of equations

$$\begin{cases} \dot{\alpha}_1 = -\alpha_1^2 - 2\gamma_1 - 2\lambda\gamma_2 \\ \dot{\alpha}_2 = -\alpha_2^2 - 2\gamma_1 - 2\gamma_2 \\ \dot{\gamma}_1 = -3\alpha_1\gamma_1 \\ \dot{\gamma}_2 = -3\alpha_2\gamma_2 \end{cases} \tag{3.20}$$

As before, the most promising way of attacking this problem is by searching conserved quantities. Furthermore, as (3.20) descends from the Hamiltonian system of partial differential equations (3.1), one is naturally led to think that (3.20) may be Hamiltonian too. So, we try to obtain a conserved quantity for the system (3.20) starting from the energy conservation law (3.6), with the same procedure used for (3.16) and (3.17). Unfortunately, unlike mass conservation equations, there is no obvious choice of the extremes of integration such as to satisfy equation (3.14) identically. Indeed, is easy to verify that there not exists a couple of points $a(t), b(t)$ such that either $h_1(x,t)$ and $h_2(x,t)$ vanish simultaneously for all times. One possible expedient, is to hypothesize a solution providing piecewise defined h_1 (or h_2) (see Fig. 3.8). Specifically, we allow for a solution[1] of the form

$$h_1(x,t) = \begin{cases} 0, & \text{for } x < a_1(t) \\ \gamma_1 x^2 - \frac{\sqrt[3]{\kappa_1 \gamma_1}}{4}, & \text{for } a_1(t) < x < b_1(t) \\ 0, & \text{for } x > b_1(t) \end{cases}$$

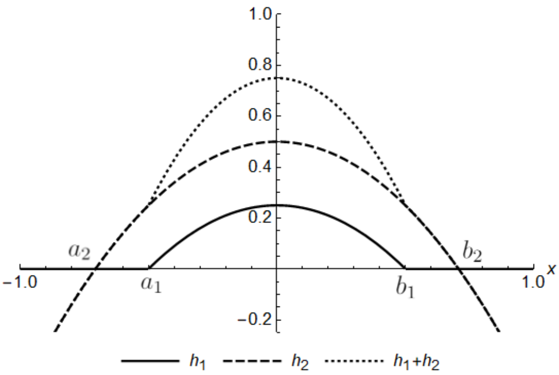
where,

$$a_1 = \frac{\kappa_1^{1/6}}{2\gamma_1^{1/3}}, \qquad b_1 = -\frac{\kappa_1^{1/6}}{2\gamma_1^{1/3}},$$

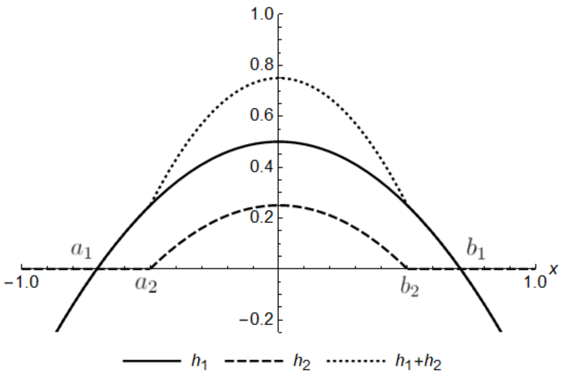
are the roots of the parabolic part of h_1. We denote the roots of h_2 similarly:

$$a_2 = \frac{\kappa_2^{1/6}}{2\gamma_2^{1/3}}, \qquad b_2 = -\frac{\kappa_2^{1/6}}{2\gamma_2^{1/3}}.$$

Therefore, we consider the initial conditions represented in Figure 3.8a, with $\bar{\gamma}_1 = \bar{\gamma}_2 = -1$, $\bar{\epsilon}_1 = 1/4$, $\bar{\epsilon}_2 = 1/2$, which correspond to $\kappa_1 = 1$, $\kappa_2 = 8$, $\bar{a}_1 = -1/2$, $\bar{a}_2 = -1/\sqrt{2}$.



(a) Piecewise defined h_1 and parabolic h_2



(b) Piecewise defined h_2 and parabolic h_1

Figure 3.8

If we put $a(t) = a_2(t)$ and $b(t) = b_2(t)$ in equation (3.14), with $M = M_2$ and $N = N_2$ given by (3.6), it will be identically satisfied, since both h_1 and h_2 are zero. So we can

[1]There is no guaranty that this is indeed a solution of (3.1), but it seems likely, based on the fact that the singular points are located at $h_1 = 0$, where the system is not strictly hyperbolic.

hope the integral

$$\int_{a_2(t)}^{b_2(t)} M_2 dx = \int_{a_2(t)}^{a_1(t)} (\lambda h_2 u_2^2 + \lambda h_2^2) dx + \int_{a_1(t)}^{b_1(t)} M_2 dx + \int_{b_1(t)}^{b_2(t)} (\lambda h_2 u_2^2 + \lambda h_2^2) dx =$$
$$= \frac{\alpha_1^2}{120\gamma_1^{2/3}} - \frac{\gamma_1^{1/3}}{30} + \frac{\lambda\alpha_2^2}{15\sqrt{2}\gamma_2^{2/3}} - \frac{\lambda\gamma_2^{1/3}}{6} - \frac{2\sqrt{2}\lambda\gamma_2^{1/3}}{15} + \frac{\lambda\gamma_2}{60\gamma_1^{2/3}}$$

to be constant in time. Unfortunately, this does not occur:

$$\frac{d}{dt}\int_{a_2(t)}^{b_2(t)} M_2 dx = \frac{\lambda\alpha_2(10\gamma_2 - 8\sqrt{2}\gamma_1 - \frac{3\gamma^{5/3}}{\gamma_1^{2/3}})}{60\gamma_2^{2/3}}$$

Considering a solution with piecewise h_2 and parabolic h_1, as depicted in Figure 3.8b, leads to analogous results. Hence, at present, a possible Hamiltonian structure for the system (3.20) remains undiscovered.

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