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Analysis of composite structures with peridynamics

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Summary

Abstract

Numerical prediction of crack growth is an important problem in computational mechanics. The difficulty in this problem arises from the basic incompatibility of cracks with the partial differential equations that are used in the classical theory of solid mechanics. The spatial derivatives needed for these partial differential equations to make sense do not exist on a crack tip or surface. Any numerical method derived from these equations inherits this difficulty in modeling cracks. In spite of the progress that has been made in developing meshfree methods aimed at modelling fracture such meshfree techniques generally require some special method of evaluating the spatial derivatives on each crack surface.

They also require supplemental relations that govern the initiation of cracks, as well as their growth velocity and direction. These relations must be applied along each crack tip, leading to inherent complexity of the method, particularly when multiple cracks occur and interact in three dimensions. It is also possible to construct constitutive models that lead asymptotically to localization in a continuum but these techniques do not entirely avoid the need for special treatment of a crack after it forms. As an attempt at improving this situation, a theory of solid mechanics has been proposed that does not require spatial derivatives to be evaluated within a body. This theory, known as the peridynamic theory, instead uses integral equations. The objective is to reformulate the basic mathematical description of solid mechanics in such a way that the identical equations hold either on or off of a discontinuity such as a crack.

In particular in this thesis I wanted to highlight the potential of PDDO (Peridynimac differential operator) that I used to describe the RZT (Refined zig zag theory). In the first chapter I briefly introduced the concept of peridynamics while in the second and third chapter I went into more detail regarding differential operators and applications of peridynamic respectively. In the fourth chapter I introduced the Refined Zig Zag theory instead in the fifth I applied PDDO to the RZT.

Chapter 1: What is peridynamics?

The Peridynamic theory was originally introduced for the solution of deformation field equations without any structural idealizations [8]. It satisfies all the fundamental balance laws of classical (local) continuum mechanics; however, it is different in the sense that it is a nonlocal continuum theory and it introduces an internal length parameter into the field equations.
The advantage of peridinamics with respect to traditional methods lies in its characteristic of making use of integral equations, which replace the differential equations ones present in the continuum theory which are the main causes of resolution difficulties in the presence of fracture phenomena. The use of integrals, unlike derivatives, presents no problems when discontinuities are present in the integration domain.
In peridynamics the body is composed of material points and each material point can interact with other points within a finite distance called *horizon*.

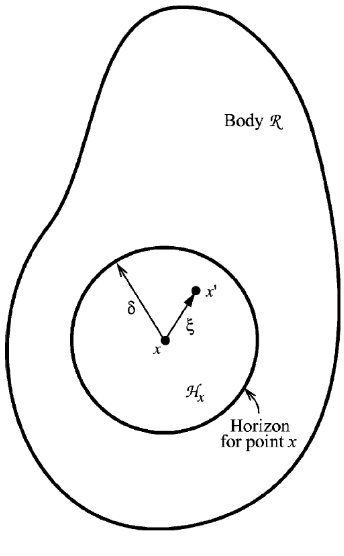


Fig 1.1 The point x of the body interacts directly with each point belonging to the region of influence of horizon δ by means of bonds (ξ). [8]

There is a force of interaction that relates two material points, not necessarily adjacent that lie within the region that has the horizon as its

radius and a point as its center called *main*; this force is called *peridynamic force* or *pairwise force*. In the bond based theory the interaction that connects two peridinamic points within the horizon is called *bond.*
The peridynamic theory may be thought of as a continuum version of molecular dynamics. The acceleration of any particle at x in the reference configuration at time t is found from [8]:

$$\rho\ddot{u}(x,t) = \int_{H_x} f(u(x',t) - u(x,t), x' - x)dV_{x'} + b(x,t) \ ,$$

where H_x is a neighborhood of *x*, *u* is the displacement vector field, , *q* is mass density in the reference configuration, *b* is a prescribed body force density field and *f* is a *pairwise force function* whose value is the force vector (per unit volume squared) that the particle *x'* exerts on the particle *x*. Now we can denote the relative position of two particles in the reference configuration by ξ:

$$\xi = x' - x$$

And their relative displacement is:

$$\eta = u(x',t) - u(x,t)$$

It is important to note that $\eta + \xi$ is the *current* relative position vector between two particles.
The direct physical interaction between the particles at x' and x is called a bond and in the special case of an elastic interaction to be defined below, a spring. The concept of a bond that extends over a finite distance is a fundamental difference between the peridynamic theory and the classical heory, which is based on the idea of contact forces (interactions between particles that are in direct contact with each other). It is convenient to

assume that for a given material there is a positive number δ, called the horizon, such that

$$|\,\xi\,| > \delta \Rightarrow f(\eta,\xi) = 0 \quad \forall\eta.$$

In other words, the particle x cannot 'see' beyond this horizon. For the remainder of this discussion, H_x will denote the spherical neighbourhood of x in R with radius δ (Fig. 1).

The pairwise force function f is required to have the following properties:

$$f(-\eta,-\xi) = -f(\eta,\xi) \qquad \forall\eta,\xi$$

Which assures conservation of linear momentum, and

$$(\xi+\eta)\,X\,f(\eta,\xi) = 0 \quad \forall\eta,\xi$$

this assures conservation of angular momentum. The last equation means that the force vector between these particles is parallel to their current relative position vector.
A material is called *microelastic* if the pairwise force function is derivable from a scalar *micropotential w*:

$$f(\eta,\xi) = \frac{\partial w}{\partial\eta}(\eta,\xi) \qquad \forall\eta,\xi$$

The micropotential is the energy in a single bond and has dimensions of energy per unit volume squared. The energy per unit volume in the body at a given point (i.e., the local strain energy density) is therefore found from

$$W = \frac{1}{2}\int_{H_x} w(\eta,\xi)dV_\xi$$

There is the factor of 1/2 because each endpoint of a bond ''owns'' only half the energy in the bond. If a body is composed of a microelastic material, work done on it by external forces is stored in recoverable form in much the same way as in the classical theory of elasticity. Furthermore, it can be shown that the micropotential depends on the relative displacement vector η only through the scalar distance between the deformed points. Thus, there is a scalar-valued function $\widehat{w}$ such that

$$\widehat{w}(y, \xi) = w(\eta, \xi) \qquad \forall \eta, \xi \quad y = |\eta + \xi|.$$

Therefore, the interaction between any two points in a microelastic material may be thought of as an elastic (and possibly nonlinear) spring. The spring properties may depend on the separation vector ξ in the reference configuration.

Anisotropy may be included in the microelastic response through this dependence on the direction of ξ [8].

$$\widehat{w}(y, \xi) = \beta^2 \widehat{w}(y, |\xi|) + (1 - \beta^2)\widehat{w}(y, |\xi|), \beta = g\, \xi/|\xi|,$$

Which explicity supplies the dependence of the bond energy on the bond direction.

Combining the equations and differentiating the latter with respect to the components of η leads to

$$f(\eta, \xi) = \frac{\xi + \eta}{|\xi + \eta|} f(|\xi + \eta|, \xi) \quad \forall \eta, \xi$$

Where f is the scalar-valued function difined by

$$f(y,\xi) = \frac{\partial \widehat{w}}{\partial y}(y,\xi) \qquad \forall y,\eta,$$

And we can write

$$\widehat{w}(y,-\xi) = \widehat{w}(y,\xi)$$

Which will henceforth be assumed.
The previous equation [8], together with the equation of motion, essentially contain the peridynamic model for a nonlinear microelastic material. It is interesting to note that the issue of how to treat rigid rotation does non arise in this formulation because y is invariant under rotation of the body. Similarly, objectivity of a constitutive model is not an issue in this approach.
A linearized version of the theory for a microelastic material takes the form

$$f(\eta,\xi) = C(\xi)\eta \qquad \forall \eta,\xi$$

Where C is the material's *micromodulus* function, whose value is a second order tensor given by

$$C(\xi) = \frac{\partial \hat{f}}{\partial \eta}(0,\xi) \quad \forall \xi.$$

This function inherits the following requirement:

$$C(-\xi) = C(\xi) \quad \forall \xi.$$

Chapter 2: Peridynamic Differential Operator

The peridynamic differential operator [2] uses the concept of Peridynamic interactions and it is based on the orthogonality property of the Peridynamic functions [2]. It renew the nonlocal interactions at a point by considering its association with the other points into an arbitrary domain of interaction. The Peridynamic differentiation recovers the local differentiation as this interaction domain approaches zero. It converts the local form of differentiation to its nonlocal PD form. It is simply a bridge between differentiation and integration. Therefore, the PDDO enables numerical differentiation through integration.

The PDDO enables the computational solution of complex differential equations and evaluation of derivatives of smooth or scattered data in the presence of jump discontinuities or singularities. It provides the solution to linear and nonlinear PDEs in a unified manner regardless of their intrinsic behavior and presence of a singularity without any derivative reduction process and special treatment. It does not have any limitations on the order of the partial derivatives of the spatial variables and temporal variable. This may become significant if temporal nonlocality space-time nonlocality is of concern.

For the approximation of zeroth-order derivative (function itself), the PDDO, RK, and G-RK are all equivalent. The PDDO and G-RK are also equivalent for the approximation of first-order derivatives. Pertinent to the zeroth- and first-order derivatives, the reproducing conditions and the correction functions of the RK and G-RK are the same as the orthogonality conditions and the PD functions of the PDDO, respectively. Otherwise, there exists no correspondence when approximating the higher-order derivatives.

The PDDO employs the concept of PD interactions and the PD functions without performing any differentiation. It employs neither a kernel function nor reproducibility conditions for different orders of derivatives. It enables accurate determination of any arbitrary order of partial derivatives of the spatial and temporal functions. The PD functions for the derivatives are determined directly by making them orthogonal to each term in the Taylor series expansion . Both the lower- and higher-order derivatives influence each other while determining the PD functions in the presence of a nonsymmetric family. The PDDO is free of the requirement of symmetric kernels. This feature removes the necessity of ghost points near the boundary. Therefore, it is not a special case of RK or G-RK operators.

The derivation of the PDDO is explained by considering a function f(x) with a single variable, x. According to the PD concept, the variation of the field f . f(x) at point x is influenced by its interaction with the other points, x0 in the domain. As shown in the figure below, the spacing between these two points is $\xi = x' - x$.

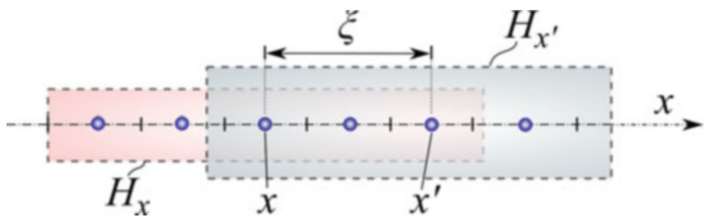


Fig. 2.1 Interaction between material points x' and x. [2]

Each point in the domain occupies an infinitesimally small entity (time or length), dl. Also, each point x has its own family members, and it only interacts with points in its own family, H_x.

Similarly, point x' is influenced by the variation of points in its own family, $H_{x'}$. Furthermore, the size of each family can be different.

The degree of interaction between the points is specified by a nondimensional influence (weight) function, $w(x' - x)$, which can be different for each point. The location of a point with respect to its family shape may not necessarily be symmetric. If symmetric, the size of each family is established by a characteristic parameter (length), δ, referred to as the "horizon." Also, the points within a distance, δ of x, are called the family of x, H_x.

The peridynamics differential operator can be constructed by considering the TSE of a scalar field $f(x') = f(x + \xi)$ as [2]

$$f(x+\xi)=\sum_{k=0}^{N}\frac{1}{n!}\xi^{n}\frac{\mathrm{d}^{\mathrm{n}}f(x)}{dx^{n}}+\mathrm{R(N,x)}$$

where $\xi = x' - x$, with R(N,x) representing the remainder. Assuming the contribution of the remainder is negligibly small and multiplying each term in this expression by the peridynamic functions, $g_N^p(\xi)$ with (p=0,1, …,N) and integrating over the family of point x defined as $H_x = \{x' \in [\mathrm{a}, \mathrm{b}]\}$ result in [2]

$$\int_{H_x} f(x+\xi)g_N^p(\xi)d\xi = f(x)\int_{H_x} g_N^p(\xi)d\xi + \frac{\partial f(x)}{\partial x}\int_{H_x} \xi\, g_N^p(\xi)\, d\xi + \\ + \frac{\partial^2 f(x)}{\partial x^2}\int_{H_x} \frac{1}{2!}\,\xi^2\, g_N^p(\xi)\, d\xi + \cdots \\ + \frac{\partial^N f(x)}{\partial x^N}\int_{H_x} \frac{1}{N!}\,\xi^N\, g_N^p(\xi)\, d\xi + R(N,x)$$

For a point, x, symmetrically located in its family, the horizon, δ, defines the extent of its family as $H_x = \{x' \in [\mathrm{a} = -\delta, \mathrm{b} = \delta]\}$.
The orthogonality property of PD functions, $g_N^p(\xi)$, can be written as

$$\int_{H_x} \frac{1}{n!}\,\xi^n\, g_N^p(\xi)\, d\xi = \delta_{np}\ \ with\ \ (n,p = 0,1,\dots,N)$$

In which δ_{np} represents the Kronecker symbol. Invoking these orthogonality conditions results in the explicit form of the peridynamic expressions for the derivatives as

$$\frac{d^p f(x)}{dx^p} \int_{H_x} f(x+\xi) g_N^p(\xi) d\xi$$

in which p denotes the order of differentiation. Although not a limitation, the peridynamic functions can be constructed as polynomials in the form

$$g_N^p(\xi) = \sum_{q=0}^{N} \mathrm{a}_q^{\mathrm{p}} \mathrm{w}_q(\xi)\, \xi^{\mathrm{q}}$$

Where $\mathrm{w}_{\mathrm{q}}(\xi)$ is the weight functions associated with each term ξ^{q} in the polynomial expansion.

Depending on the nature of the nonlocality, the weight function representing the degree of interaction may be the same or different for each term in the TSE. With this representation, the orthogonality property of the peridynamic functions leads to [2]

$$\sum_{q=0}^{N} \mathrm{A}_{nq}\ a_q^p = b_n^p$$

In which

$$A_{nq} = \int_{H_x} w_q\ (\xi)\ \xi^{n+q}\ d\xi$$

And

$$b_n^p = n!\ \delta_{np}.$$

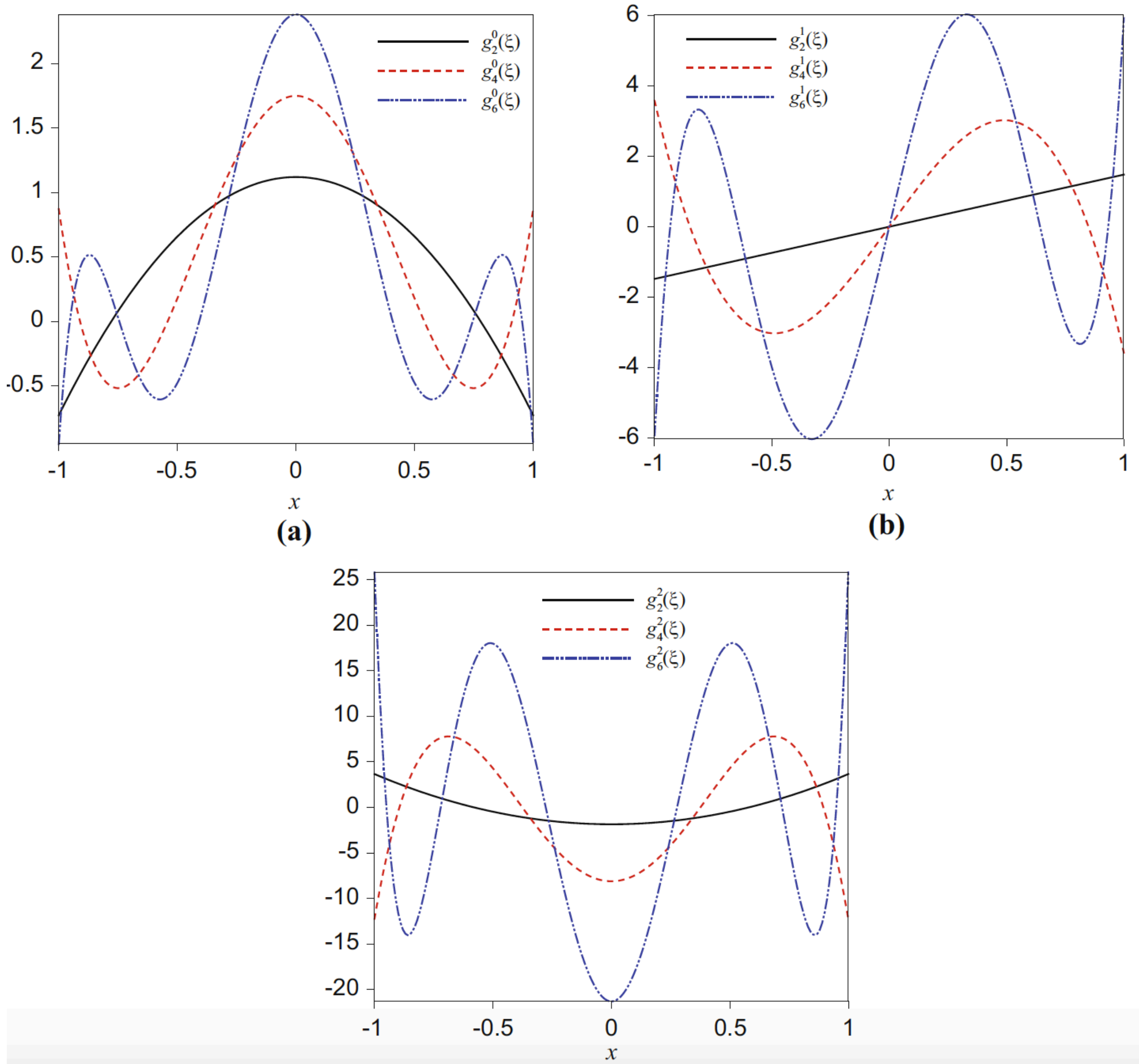
The unknown coefficients, a_q^p , can be determined from the solution of the first equation. It is worth noting that n is not necessarily equal to p with $n \geq p$, and the nonlocal PD differentiation approaches its local value when $n \to \infty$.
The PDDO recovers the local differentiation as the family size, H_x, decreases or the number of terms in the functions, $g_N^p(\xi)$, increases. Thus, the degree of nonlocality reduces with decreasing family size and with increasing number of terms in the TSE. The condition number of the coefficient (shape) matrix, A_{nq}, becomes poor, and round-off error may become significant for higher-order derivatives such as $n \geq 10$. Therefore, it may be advantageous to normalize the range of integration with respect to the domain of interaction $H_x = \{x' \in [\mathrm{a}, \mathrm{b}]\}$ and employ a preconditioning method prior to solving for the unknown coefficients. In general, the round-off error can be avoided by increasing the family size for higher-order derivatives.
Also, it is important to use the optimum family size to achieve convergence and
sufficient accuracy within a practical amount of computational time.
The normalization can be achieved by introducing a new viariable as $\bar{\xi} = 2\left[\xi - \frac{b+a}{2}\right]/(b-a)$ with $\xi = \left[(b-a)\bar{\xi} + (b+a)\right]/2$. The expressions for the derivatives and PD functions become [2]

$$\frac{d^p f(x)}{dx^p} = \frac{b-a}{2}\int_{-1}^{1} f\left(x + \frac{1}{2}(b-a)\bar{\xi} + (b-a)\right) g_N^p(\xi) d\bar{\xi}$$

And

$$g_N^p\left(\frac{1}{2}(b-a)\bar{\xi}(b+a)\right)=\sum_{q=0}^{N}\mathrm{a}_q^{\mathrm{p}}\mathrm{w}_q\left(\frac{1}{2}(b-a)\bar{\xi}+(b+a)\right)\bar{\xi}^q$$

For a point, x, symmetrically located in its family which is normalized over the horizon $H_x = \{x' \in [-1,1]\}$ with a uniform grid spacing and $\mathrm{w}_q(\xi) = 1$, the PD functions, $g_n^p(\xi)$, for different values of $0 \leq p \leq 2$ and $2 \leq n \leq 6$ are shown in the figure below



In a M-dimensional space, the TSE of a scalar field $f(x') = f(x+\xi)$ with many variables can be expressed as

$$f(x+\xi) =$$

$$= \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \dots \sum_{n_N=0}^{N-n_1 \dots n_{N-1}} \frac{1}{n_1!\, n_2! \dots n_N!} \xi_1^{n_1} \xi_2^{2} \dots \xi_M^{n_N} \frac{\partial^{n_1+n_2+\dots+n_N} f(x)}{\partial x_1^{n_1}\, \partial x_2^{n_2} \dots \partial x_M^{n_N}}$$

$$+R(N,x)$$

Where $\xi = x' - x$ with $R(N,x)$ representing the remainder. Assuming the contribution of the remainder is negligibly small and invoking the property of the orthogonal function, $g_N^{p_1 p_2 \dots p_N}(\xi)$, result in the peridynamic nonlocal expression for the partial derivatives of any order as [2]

$$\frac{\partial^{p_1+p_2+\dots+p_N} f(x)}{\partial x_1^{p_1}\, \partial x_2^{p_2} \dots \partial x_M^{p_N}} = \int_{H_x} f(x+\xi)\; g_N^{p_1 p_2 \dots p_N}(\xi) dx_1 dx_2 \dots dx_M$$

In which p_i denotes the order of differentiation with respect to variable x_i with $i = 1, \dots M$. The PD functions $g_N^{p_1 p_2 \dots p_N}(\xi)$ has the orthogonality property of

$$\frac{1}{n_1!\, n_2! \dots n_N!} \int_{H_x} \xi_1^{n_1} \xi_2^{n_2} \dots \xi_M^{n_M}\;\; g_N^{p_1 p_2 \dots p_N}(\xi) dx_1 dx_2 \dots dx_M$$
$$= \delta_{n_1 p_1} \delta_{n_2 p_{2\dots}} \delta_{n_{N-1} p_{N-1}} \delta_{n_N p_N}$$

In which $n_i = 0, \dots N$. They can be constructed as

$$g_N^{p_1 p_2 \dots p_N}(\xi) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \dots \sum_{n_N=0}^{N-n_1 \dots n_{N-1}} a_{q_1 q_2 \dots q_N}^{p_1 p_2 \dots p_N} \, w_{q1 q_2 \dots q_N}(|\xi|) \, \xi_1^{q_1} \xi_2^{q_2} \dots \xi_M^{q_N}$$

Where $w_{q1 q_2 \dots q_N}(|\xi|)$ is the weight function associated with each term $\xi_1^{q_1} \xi_2^{q_2} \dots \xi_M^{q_N}$
In the polynomial expansion. Depending on the nature of the nonlocality, the weight function representing the degree of interaction may be the same or different for each term in the TSE.
The unknown coefficients, $a_{q_1 q_2 \dots q_N}^{p_1 p_2 \dots p_N}$, can be determined from the solution of

$$\sum_{q_1=0}^{N} \sum_{q_2=0}^{N-q_1} \dots \sum_{q_N=0}^{N-q_1 \dots q_{N-1}} A_{(n_1 n_2 \dots n_N)(q_1 q_2 \dots q_N)} \, a_{q_1 q_2 \dots q_N}^{p_1 p_2 \dots p_N} = b_{n_1 n_2 \dots n_N}^{p_1 p_2 \dots p_N}$$

In which $q_i = 0, \dots, N$. The coefficient matrix is constructed as

$$A_{(n_1 n_2 \dots n_N)(q_1 q_2 \dots q_N)} = \int_{H_x} w_{q_1 q_2 \dots q_N}(|\xi|) \, \xi_1^{n_1+q_1} \xi_2^{n_2+q_2} \dots \xi_M^{n_N+q_N} dx_1 dx_2 \dots dx_M$$

And

$$b_{n_1 n_2 \dots n_N}^{p_1 p_2 \dots p_N} = n_1!\, n_2! \dots n_M!\, \delta_{n_1 p_1} \delta_{n_2 p_2} \dots \delta_{n_N p_N}.$$

The peridynamic differential operators [2] recovers the local differentiation as the size of family H_x decreases or the number of terms in the functions $g_N^{p_1 p_2 \dots p_N}(\xi)$ increases. It requires the computation of the coefficients, $a_{q1 q_2 \dots q_N}^{p_1 p_2 \dots p_N}$, and the condition number of the coefficient (shape) matrix, $A_{(n_1 n_2 \dots n_N)(q_1 q_2 \dots q_N)}$, may become poor for higher-order derivatives.
Therefore, when computing higher-order derivatives, the family size needs to be adjusted accordingly. If it is too small, then round-off errors dominate, and if it is too large, then the results deviate from local values.
The coefficients of the PD functions can be determined without any difficulty.
Although it is not a limitation, the weight functions $w_{q_1 q_2 q_3}(\xi)$ can be replaced with $w_n(|\xi|)$ for simplification based on the order of differentiation.

Chapter 3: Kinematics of Peridynamic, equations of motion and applications

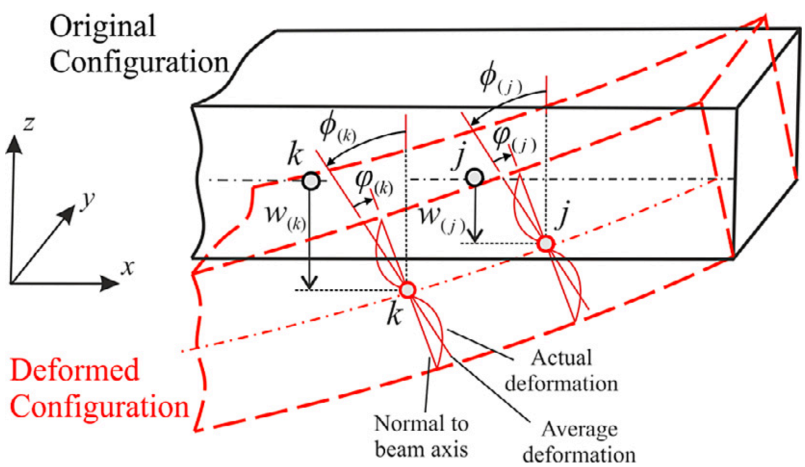
Peridynamic equations can be used for describe the behaviour, for example, of the Timoshenko beam or the Kirchhoff's plate. This chapter will be briefly shown how with the kinematics of Peridynamic can be represent the displacement in the Timoshenko beam and Kirchhoff's plate respectively.

3.1 Beam kinematics

As shown in the figure below [8] we can express the trasverse shear angles ($\phi_{(j)}$ and $\phi_{(k)}$) of material points j and k:

$$\phi_{(j)} = \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \phi_{(j)} sgn\left(x_{(j)} - x_{(k)}\right) \right)$$

$$\phi_{(k)} = \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \phi_{(k)} sgn\left(x_{(j)} - x_{(k)}\right) \right)$$



In which $w_{(j)}$, $\phi_{(j)}$ and $w_{(k)}$, $\phi_{(k)}$ represent the out-of-plane deflection and rotation of material points j and k, respectively. The distance between the material points is specified as $\xi_{(j)(k)} = |x_{(j)} - x_{(k)}|$.

If k is the point of interest, the transverse shear angle, $\phi_{(k)(j)}$ can be defined [8]:

$$\phi_{(k)(j)} = \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \frac{\phi_{(j)} + \phi_{(k)}}{2} sgn\left(x_{(j)} - x_{(k)}\right)\right)$$

The curvature between the material points j and k can be defined as

$$k_{(k)(j)} = \left(\frac{\phi_{(j)} - \phi_{(k)}}{\xi_{(j)(k)}}\right)$$

If j is the point of interest the interaction become:

$$\phi_{(j)(k)} = \left(\frac{w_{(k)} - w_{(j)}}{\xi_{(j)(k)}} - \frac{\phi_{(k)} + \phi_{(j)}}{2} sgn\left(x_{(j)} - x_{(k)}\right)\right)$$

Or

$$\phi_{(j)(k)} = -\phi_{(k)(j)}$$

And

$$k_{(k)(j)} = \left(\frac{\phi_{(k)} - \phi_{(j)}}{\xi_{(j)(k)}}\right)$$

Or

$$k_{(k)(j)} = -k_{(k)(j)}$$

3.2 Beam equations of motion

Focusing on the beam equations of motion we can write the total kinetic energy as:

$$T = \frac{1}{2}\sum_{k=1}^{\infty} \rho[\dot{w}_{(k)}^2 + \frac{I}{A}(\dot{\phi}_{(k)}^2)]\, V_{(k)}$$

Where $V_{(k)}$ is the infinitesimally incremental volume of material point k. The parameter ρ represent the mass density, I is the moment of inertia, A is the cross sectional area of the beam.
The total potential energy of the system is [8]:

$$U = \sum_{k=1}^{\infty}\left\{\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2}\left[\widetilde{w}_{(k)(j)}\left(k_{(k)(j)}\right) + \widetilde{w}_{(j)(k)}\left(k_{(j)(k)}\right)\right]V_{(j)} - \tilde{b}_{(k)}\phi_{(k)}\right\}V_{(k)} +$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2}\left[\widehat{w}_{(k)(j)}\left(\phi_{(k)(j)}\right) + \widehat{w}_{(j)(k)}\left(\phi_{(j)(k)}\right)\right]V_{(j)} - \hat{b}_{(k)}w_{(k)}\right\}V_{(k)}$$

Where $\tilde{b}_{(k)}$ is the body moment and $\hat{b}_{(k)}$ represent the body force of the material point k.
Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{w}_{(k)}} - \frac{\partial L}{\partial w_{(k)}} = 0$$

And

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_{(k)}} - \frac{\partial L}{\partial \phi_{(k)}} = 0$$

Applying the Lagrangian equation L = T – U we can write the following equations of motion [8]:

$$\rho \ddot{w}_{(k)} + \sum_{j=1}^{\infty} \frac{1}{2}\left[\xi_{(j)(k)} \hat{f}_{(k)(j)} \frac{\partial \phi_{(k)(j)}}{\partial\left(w_{(k)}\right)} + \xi_{(j)(k)} \hat{f}_{(j)(k)} \frac{\partial \phi_{(j)(k)}}{\partial\left(w_{(k)}\right)}\right] V_{(j)} - \hat{b}_{(k)} = 0$$

And

$$\frac{\rho I}{A}\ddot{\phi}_{(k)} + \sum_{j=1}^{\infty} \frac{1}{2}\xi_{(j)(k)}\left[\tilde{f}_{(k)(j)} \frac{\partial\left(k_{(k)(j)}\right)}{\partial\left(\phi_{(k)}\right)} + \tilde{f}_{(j)(k)} \frac{\partial\left(k_{(j)(k)}\right)}{\partial\left(\phi_{(k)}\right)}\right] V_{(j)}$$

$$+ \sum_{j=1}^{\infty} \frac{1}{2}\xi_{(j)(k)}\left[\hat{f}_{(k)(j)} \frac{\partial\left(k_{(k)(j)}\right)}{\partial\left(\phi_{(k)}\right)} + \hat{f}_{(k)(j)} \frac{\partial\left(k_{(j)(k)}\right)}{\partial\left(\phi_{(k)}\right)}\right] V_{(j)} - \tilde{b}_{(k)} = 0$$

In which

$$\hat{f}_{(k)(j)} = \frac{1}{\xi_{(j)(k)}} \frac{\partial \widehat{w}_{(k)(j)}\left(\phi_{(k)(j)}\right)}{\partial \phi_{(k)(j)}}, \quad \hat{f}_{(j)(k)} = \frac{1}{\xi_{(j)(k)}} \frac{\partial \widehat{w}_{(j)(k)}\left(\phi_{(j)(k)}\right)}{\partial \phi_{(j)(k)}}$$

$$\tilde{f}_{(k)(j)} = \frac{1}{\xi_{(j)(k)}} \frac{\partial \widetilde{w}_{(k)(j)}\left(k_{(k)(j)}\right)}{\partial k_{(k)(j)}}, \quad \tilde{f}_{(j)(k)} = \frac{1}{\xi_{(j)(k)}} \frac{\partial \widetilde{w}_{(j)(k)}\left(k_{(j)(k)}\right)}{\partial k_{(j)(k)}}$$

That represent the peridynamic interaction forces between material points j and k. For a linear material behaviour:

$$\hat{f}_{(k)(j)} = c_s \phi_{(k)(j)}, \quad \hat{f}_{(j)(k)} = c_s \phi_{(j)(k)}$$

$$\tilde{f}_{(k)(j)} = c_b k_{(k)(j)}, \quad \tilde{f}_{(j)(k)} = c_b k_{(j)(k)}$$

Where c_s and c_b are the peridynamic material parameters.
Finally, substituting the peridynamic forces, we can write the peridynamic equations [8]:

$$\rho \ddot{w}_{(k)} = c_s \sum_{j=1}^{\infty} \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \frac{\phi_{(j)} + \phi_{(k)}}{2} sgn\left(x_{(j)} - x_{(k)}\right) \right) V_{(j)} + \hat{b}_{(k)}$$

And

$$\frac{\rho I}{A} \ddot{\phi}_{(k)} = c_b \sum_{j=1}^{\infty} \frac{\phi_{(j)} - \phi_{(k)}}{\xi_{(j)(k)}} V_{(j)} + \frac{1}{2} c_s \sum_{j=1}^{\infty} \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} sgn\left(x_{(j)} - x_{(k)}\right) - \frac{\phi_{(j)} + \phi_{(k)}}{2} \right) \xi_{(j)(k)} V_{(j)} + \tilde{b}_{(k)}$$

In an integral form:

$$\rho \ddot{w}_{(k)} = c_s \int_H \left(\frac{w(x',t) - w(x,t)}{\xi} - \frac{\phi(x',t) - \phi(x,t)}{2} sgn(x' - x) \right) dV' + \hat{b}(x,t)$$

and

$$\frac{\rho I}{A} \ddot{\phi}(x,t) = \int_H c_b \left(\frac{\phi(x',t) - \phi(x,t)}{\xi} \right) dV' + \int_H \left(c_s \frac{1}{2} \left(\frac{w(x',t) - w(x,t)}{\xi} sgn(x' - x) - \frac{\phi(x',t) + \phi(x,t)}{2} \right) \xi \right) dV' + \tilde{b}(x,t)$$

In which

$$c_s = \frac{2kG}{A\delta^2}, \quad c_b = \frac{2EI}{\delta^2 A^2} + \frac{1}{4} \frac{kG}{A}$$

Where k for example is 5/6 for rectangular cross sections and represent the shear correction factor.

3.3 Example: Timoshenko beam under pure bending and transverse force loading

Considering a static case, the derivatives of the time are equal to zero. For compare the peridynamic solution with the analytical solution has been considered:

- L=1 the length of the beam
- $\delta = 3.015\Delta x$ (δ is the horizon and Δx is the grid spacing)
- $\Delta x = 0.01$ (distance between material points)
- $A = 0.1 \times 0.1\, m^2$ (cross sectional area)

- $E = 200\ GPa$ (Young's modulus)
- $\tilde{b} = 3.33 \times 10^9\ N/m^2$ (body load for bending corresponding to an applied moment of $M = 3.33 \times 10^5\ Nm$)
- $\hat{b} = 5 \times 10^9 N/m^3$ (body load for transverse loading corresponding to an applied load of $P = 5 \times 10^5\ N$)

The transverse displacement and the rotation for the analytical solution the following equations have been considered:
For transverse loading the vertical displament is:

$$w = \frac{Px}{kGA} + \frac{P}{2EI}\left(Lx^2 - \frac{x^3}{3}\right)$$

And the rotation is:

$$\phi = \frac{P(2Lx - x^2)}{2EI}$$

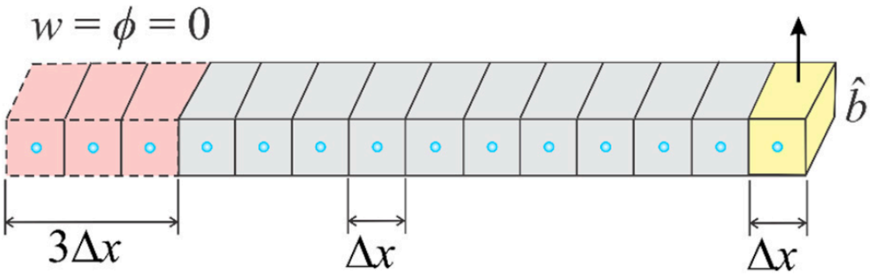
Regarding the pure bending the transversal displacement is given by:

$$w = \frac{Mx^2}{2EI}$$

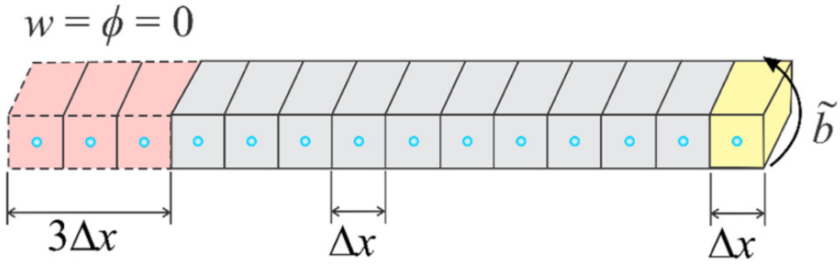
And the rotation is:

$$\phi = \frac{Mx}{EI}$$

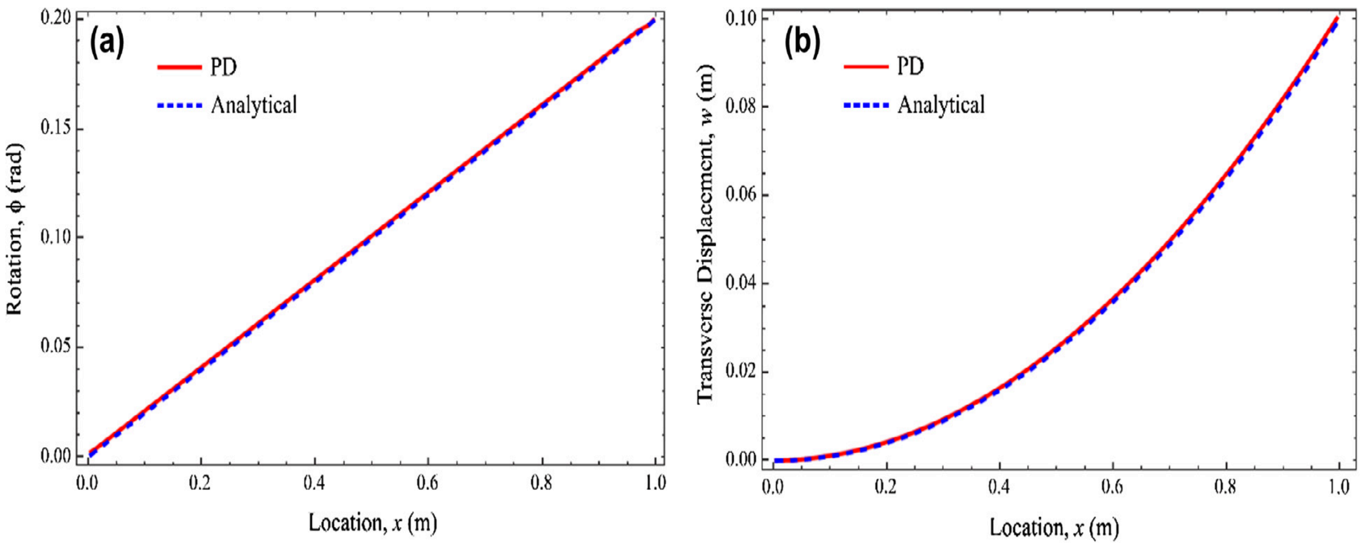
In the figures below [8] is shown the discretization,the applied load and the boundary conditions.
Considering the applied load:



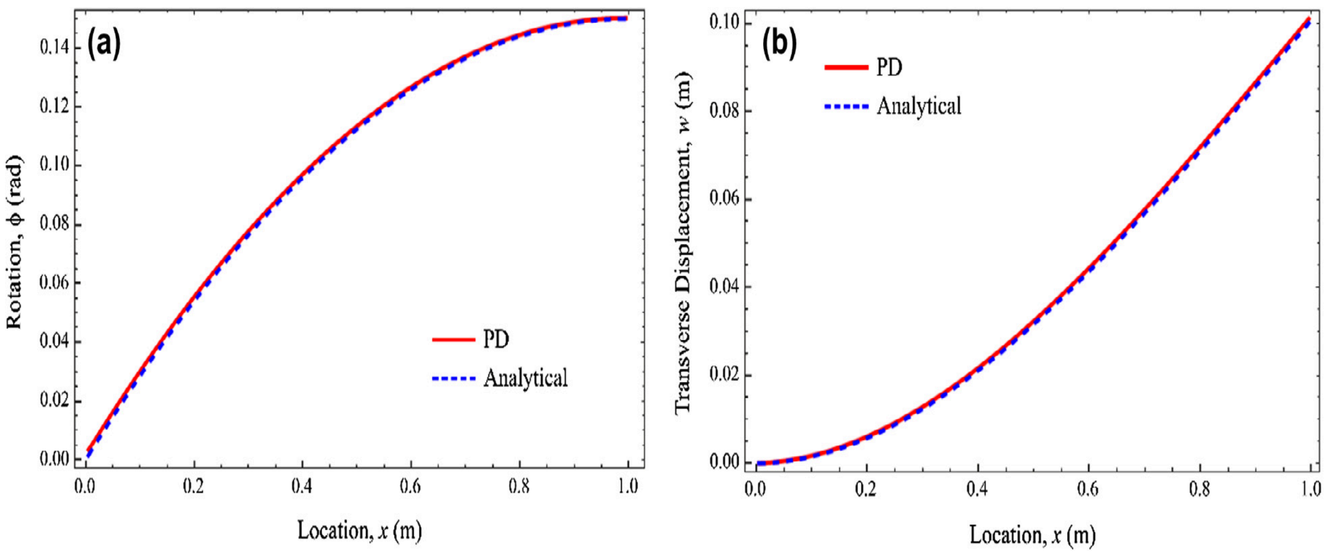
Considering the pure banding:



The variation of rotation and transverse displacement along the Timoshenko beam under pure bending is shown in the figure below [8]:

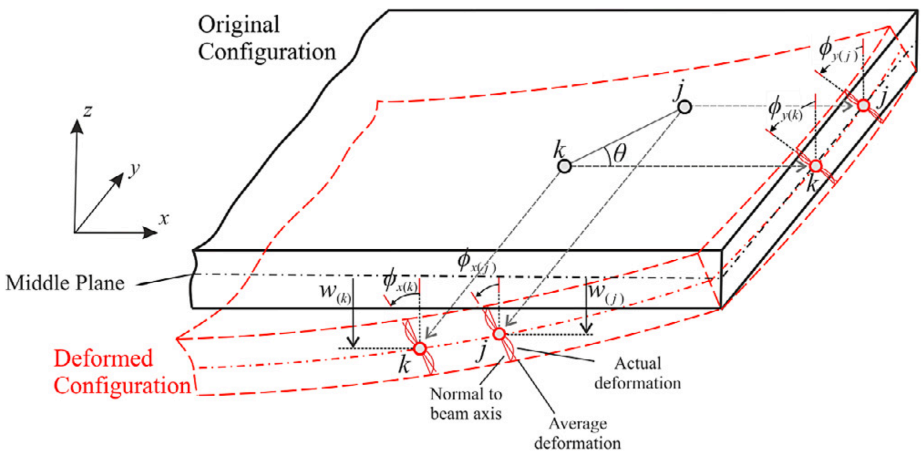


The variation of rotation and transverse displacement along the Timoshenko beam under transverse force loading is shown in the figure below:



3.4 Plate kinematics

Regarding the plate, as illustrated in the figure below [8], $\phi_{(k)}$ and $\phi_{(j)}$ represent the rotations between the material points j and k.



If k is the point of interest the curvature $k_{(k)(j)}$ is given by:

$$k_{(k)(j)} = \left(\frac{\phi_{(j)} - \phi_{(k)}}{\xi_{(j)(k)}}\right)$$

Throught coordinate transformation we can write the rotation and curvature as:

$$\phi_{(j)} = \phi_{x(j)} cos\theta + \phi_{y(j)} sin\theta$$

$$\phi_{(k)} = \phi_{x(k)} + \phi_{y(k)} sin\theta$$

And

$$k_{(k)(j)} = \left(\frac{\phi_{x(j)} - \phi_{x(k)}}{x_{(j)} - x_{(k)}}\right)\cos^2\theta + \left(\frac{\phi_{y(j)} - \phi_{y(k)}}{y_{(j)} - y_{(k)}}\right)$$

In which $x_{(j)} - x_{(k)} = \xi_{(j)(k)} cos\theta$ and $y_{(j)} - y_{(k)} = \xi_{(j)(k)} sin\theta$. Obviously $\xi_{(j)(k)}$ is the distance between the material points j and k.
The trasverse shear angles of the points j and k can be expressed as:

$$\phi_{(j)} = \theta_{(k)(j)} - \phi_{(j)}$$

$$\phi_{(k)} = \theta_{(k)(j)} - \phi_{(k)}$$

Considering the material point k as the point of interest we can write the transverse shear angle $\phi_{(k)(j)}$ as:

$$\phi_{(k)(j)} = \frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \frac{\phi_{(j)} + \phi_{(k)}}{2}$$

If j is the point of interest:

$$\phi_{(j)(k)} = \frac{w_{(k)} - w_{(j)}}{\xi_{(j)(k)}} - \frac{\phi_{(k)} + \phi_{(j)}}{2}$$

And

$$k_{(j)(k)} = \left(\frac{\phi_{(k)} - \phi_{(j)}}{\xi_{(j)(k)}}\right)$$

It is important to note that $\phi_{(j)(k)} = -\phi_{(k)(j)}$ and $k_{(j)(k)} = -k_{(k)(j)}$.

The PD equations of motion at material point k can be derived by applying the principle of virtual work

$$\delta \int_{t_0}^{t_1} (T - U)\, dt = 0$$

Where U is the potential energie in the beam or plate and T is the total kinetic energie. Solving the Lagrange equation, the principle is satisfied:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \left(\frac{\partial L}{\partial q_k}\right) = 0$$

Where the vector q_k includes the independent field variables (out of plane deflection and rotations), and the Lagrangian L is defined as:

$$L = T - U$$

3.5 Plate equations of motion

Under shear deformation and bending the total kinetic energy is:

$$T = \frac{1}{2}\rho \sum_{k=1}^{\infty} \left[\dot{u}_{(k)}^2 + \dot{v}_{(k)}^2 + \dot{w}_{(k)}^2\right] V_{(k)}$$

Integrating throught the thickness and expressing displacements component in terms of rotation $(u_{(k)} = -z\,\phi_{x(k)}$ and $v_{(k)} = -z\,\phi_{y(k)})$:

$$T = \frac{1}{2}\rho \sum_{k=1}^{\infty} \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\dot{w}_{(k)}^2 + z^2 \dot{\phi}_{x(k)}^2 + z^2 \dot{\phi}_{y(k)}^2\right] dz \right) A_{(k)}$$

Or

$$T = \frac{1}{2}\rho \sum_{k=1}^{\infty} \left(\dot{w}_{(k)}^2 + \frac{h^2}{12} \dot{\phi}_{x(k)}^2 + \frac{h^2}{12} \dot{\phi}_{y(k)}^2 \right) A_{(k)}$$

Where h is the thickness of the plate and $A_{(k)}$ is the infinitesimally small incremental area of each material point.
Summing the micropotentials between material points ($\widetilde{w}_{(k)(j)}(k_{(k)(j)})$ and $\widehat{w}_{(k)(j)}(\phi_{(k)(j)})$), the total potential energy of the plate can be written as [8]:

$$U = \sum_{k=1}^{\infty} \left\{ \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2} \left[\widetilde{w}_{(k)(j)}\left(k_{(k)(j)}\right) + \widetilde{w}_{(j)(k)}\left(k_{(j)(k)}\right) \right] V_{(j)} - \frac{\widetilde{b}_{\alpha(k)}}{h} \phi_{\alpha(k)} \right\} V_{(k)}$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2}\left[\widetilde{w}_{(k)(j)}\left(\phi_{(k)(j)}\right)+\widetilde{w}_{(j)(k)}\left(\phi_{(j)(k)}\right)\right]V_{(j)}-\frac{\hat{b}_{(k)}}{h}w_{(k)}\right\}V_{(k)}$$

In which $\tilde{b}_{\alpha(k)}$ and $\hat{b}_{(k)}$ are the resultant body moment and body force at material point k. $w_{(k)}$ and $\phi_{\alpha(k)}$ are the independent variables.
The resulting Euler-Lagrange equations can be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{w}_{(k)}}-\frac{\partial L}{\partial w_{(k)}}=0$$

And

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_{\alpha(k)}}-\frac{\partial L}{\partial \phi_{\alpha(k)}}=0, \qquad \alpha=x,y$$

Applying the Lagrangian equation L = T – U we can write the following equations of motion [8]:

$$\rho h\ddot{w}_{(k)}=C_s\sum_{j=1}^{\infty}\left(\frac{w_{(j)}-w_{(k)}}{\xi_{(j)(k)}}-\frac{\phi_{x(j)}-\phi_{x(k)}}{2}cos\theta\right.$$
$$\left.-\frac{\phi_{y(j)}-\phi_{y(k)}}{2}sin\theta\right)V_{(j)}+\hat{b}_{(k)}$$

$$\frac{\rho h^3}{12}\ddot{\phi}_{x(k)} = C_b \sum_{j=1}^{\infty} \left[\left(\frac{\phi_{x(j)} - \phi_{x(k)}}{\xi_{(j)(k)}} \right) cos\theta + \left(\frac{\phi_{y(j)} - \phi_{y(k)}}{\xi_{(j)(k)}} \right) sin\theta \right] cos\theta V_{(j)} + \frac{1}{2} C_s \sum_{j=1}^{\infty} \xi_{(j)(k)} \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \frac{\phi_{x(j)} - \phi_{x(k)}}{2} cos\theta - \frac{\phi_{y(j)} - \phi_{y(k)}}{2} sin\theta \right) cos\theta V_{(j)} + \tilde{b}_{x(k)}$$

And

$$\frac{\rho h^3}{12}\ddot{\phi}_{y(k)} = C_b \sum_{j=1}^{\infty} \left[\left(\frac{\phi_{x(j)} - \phi_{x(k)}}{\xi_{(j)(k)}} \right) cos\theta + \left(\frac{\phi_{y(j)} - \phi_{y(k)}}{\xi_{(j)(k)}} \right) sin\theta \right] sin\theta V_{(j)} + \frac{1}{2} C_s \sum_{j=1}^{\infty} \xi_{(j)(k)} \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} - \frac{\phi_{x(j)} - \phi_{x(k)}}{2} cos\theta - \frac{\phi_{y(j)} - \phi_{y(k)}}{2} sin\theta \right) sin\theta V_{(j)} + \tilde{b}_{x(k)}$$

It is important to note that the peridynamic interactions, obviously, exist only within the horizon of material points. In the equations the constants C_b and C_s can be expressed in terms of G and E as:

$$C_s = \frac{9E}{4\pi\delta^3}k^2, \qquad C_b = \frac{E}{\pi\delta}\left(\frac{3h^3}{4\delta^2} + \frac{27}{80}k^2\right)$$

where k^2 is the shear correction factor equal to $k^2 = \pi^2/12$, and $\nu = 1/3$. It is important to say that PD material parameters are determined for a material point whose horizon is completely embedded in the material.

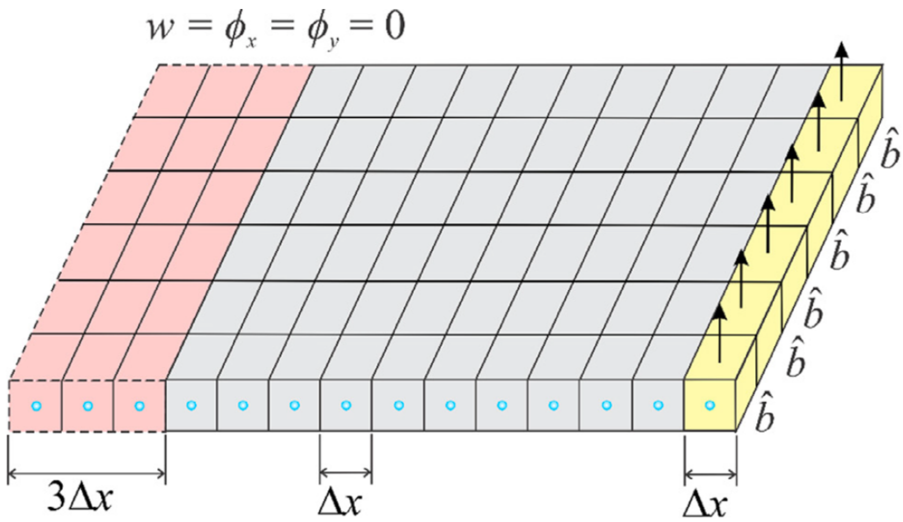
3.6 Example: Mindlin plate under pure bending and transverse force loading

Considering a static case, the derivatives of the time are equal to zero. For compare the peridynamic solution with the analytical solution has been considered:

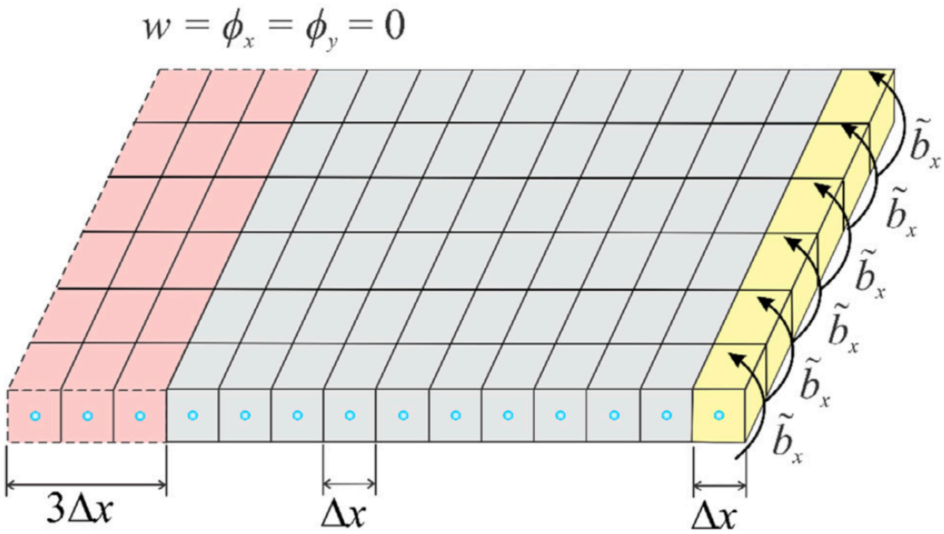
- $L = 1\ m$ (length of the plate)
- $W = 1\ m$ (width of the plate)
- $h = 0.01\ m$ (thickness of the plate)
- $E = 200\ GPa$ (Young's module)
- $\Delta x = 0.01\ m$ (distance between material points)

Furthermore, has been considered a fictitious region in the left edge for the constrained equal to $3\Delta x$. The transverse loading is $\hat{b} = 5 \times 10^8\ N/m^2$ and the bending load is$\tilde{b}_x = 3.33\ \times 10^8\ N/m$ (as shown in the next page, both applied at the right end of the plate and in a single row of material points).

Applied load case:

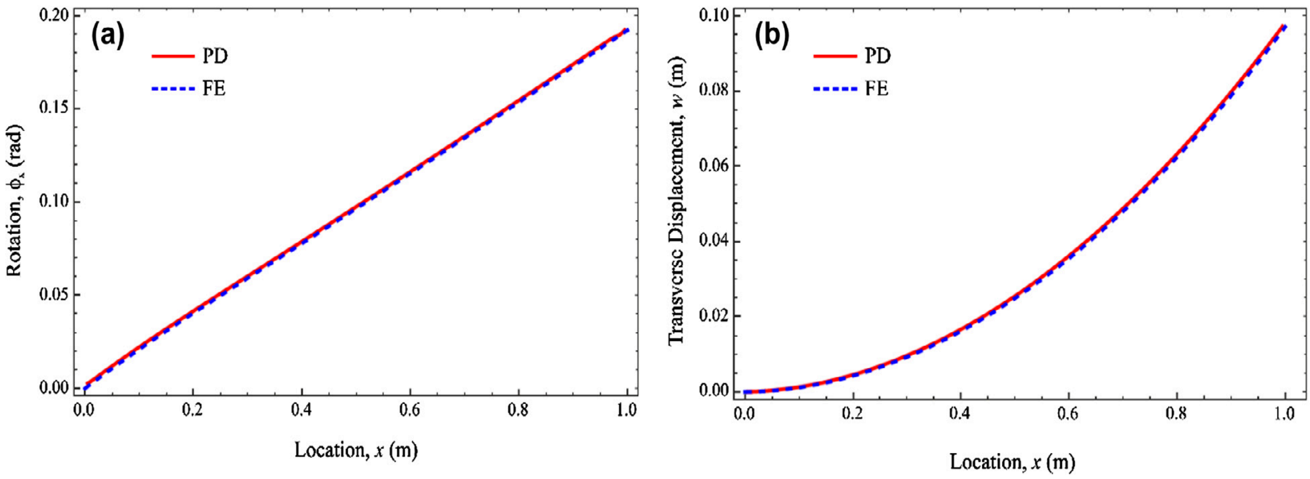


Pure bending case:

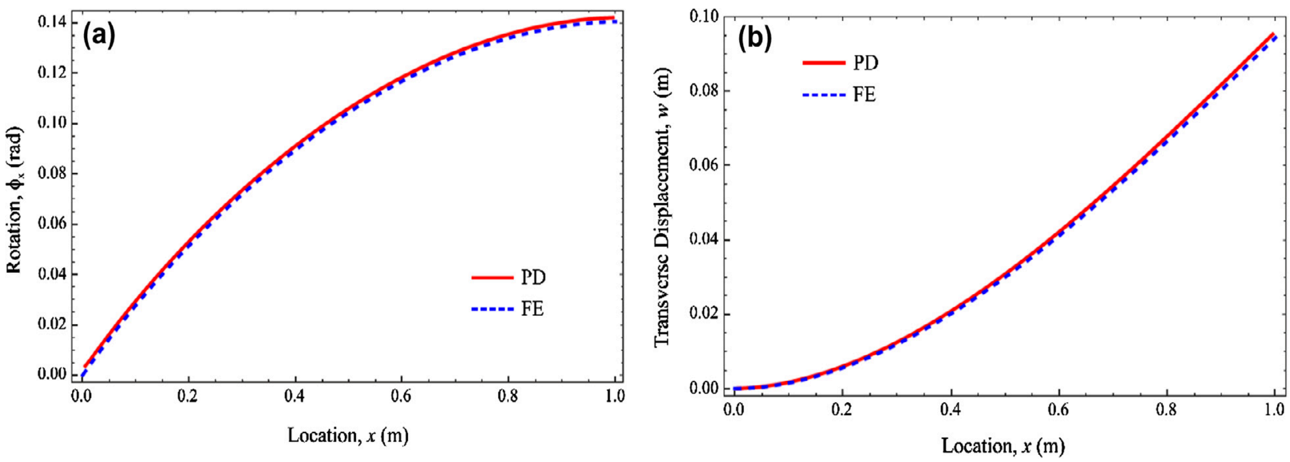


The peridynamic solution [8], now, is compared with the analytical solution (FEM).

The variation of the rotation (left) and transverse displacement (right) along a Mindlin plate under bending loading is reported in the figure below:



The variation of the rotation (left) and transverse displacement (right) along a Mindlin plate under transverse force loading is reported in the figure below:



As shown, The peridynimic and the FEM solutions agree well with each other; this means that PD equation of motion can accurately capture the deformation behaviour of a Mindlin plate.

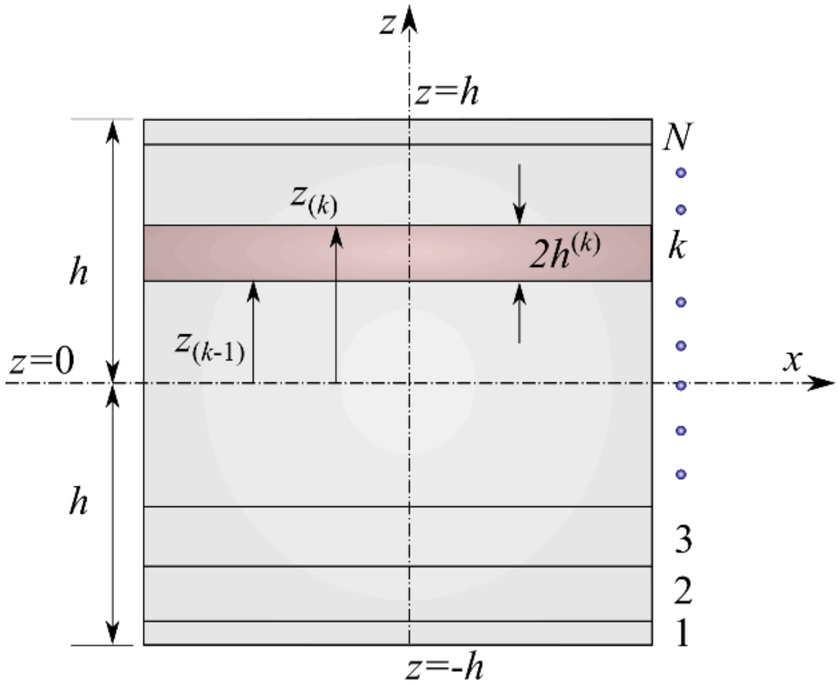
Chapter 4: Refined Zigzag Theory

The RZT formulation is very suitable for thick beams having span-to-thickness ratios up to five and does not suffer from the use of shear correction factor regardless of the material system. The RZT consists of four kinematic variables for the beams in a Cartesian coordinates system (x,z) , and the displacement components in the k^{th} layer in a laminate are expressed in terms of in-plane displacement, u, out-of-plane displacement, w, outof- plane slope, θ, and the out-of-plane zigzag amplitude , ψ, as

$$u_x^k(x,z) = u(x) + z\,\theta(x) + \theta^{(k)}(z)\,\psi(x_1)$$

$$u_z^k(x,z) = w(x)$$

where $z \in [-h, h]$ is the thickness coordinate. As shown in the figure below [11] the beam is comprised of N layers having a total thickness of $2h$.



Each layer of the beam has an arbitrary thickness of $2h^{(k)}$. The zigzag function ϕ^k varies linearly within the k^{th} layer and is defined as

$$\phi^{(k)}(\xi) = \frac{1}{2}(1-\xi)\, u_{(k-1)} + \frac{1}{2}(1-\xi)\, u_{(k)}$$

In which the local variable ξ is defined as

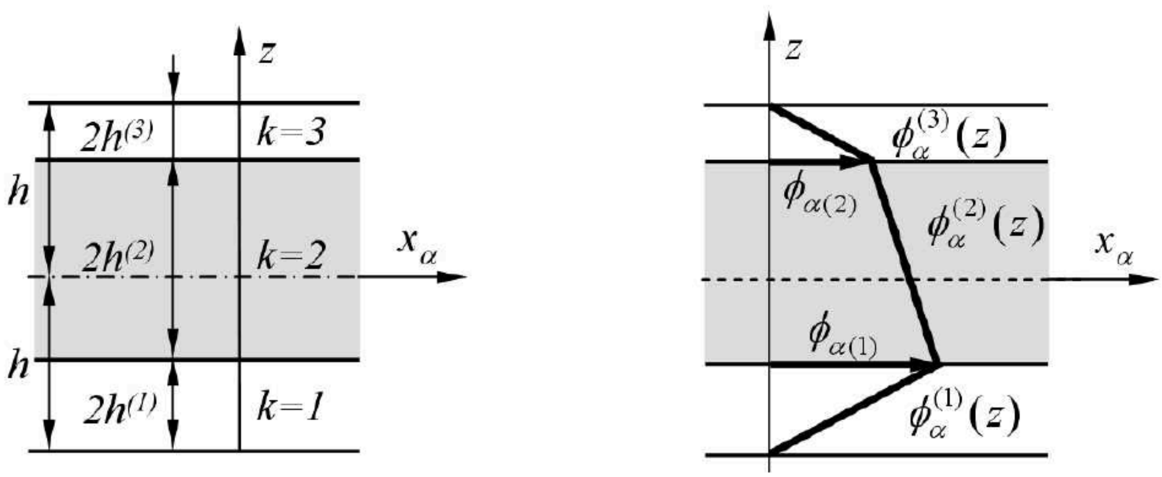
$$\xi = \frac{2\xi - \xi_{(k)} - \xi_{(k-1)}}{\xi_{(k)} - \xi_{(k-1)}} \qquad \xi_{(k-1)} \leq \xi \leq \xi_{(k)}$$

The displacement variable $u_{(k)}$ is the unknown displacement associated with the zigzag functions at the layer interfaces satisfying $u_{(0)} = u_{(N)} = 0$. The interface displacements are recursively related as

$$u_{(k)} = 2h^{(k)}\beta^{(k)} + u_{(k-1)}$$

Where the slope of the zigzag function (in the figure below [3]), $\beta^{(k)} \equiv \phi_{,z}^{(k)}$, is uniform within the k^{th} layer and can be expressed in the form of

$$\beta^{(k)} = \frac{G}{Q_{55}^{(k)}} - 1 \qquad (k = 1,2,\dots,N-1)$$



With a weighted-average transverse shear modulus [1]

$$G = \left(\frac{1}{2h} \sum_{k=1}^{N} \frac{2h^{(k)}}{\overline{Q_{55}^{(k)}}} \right)^{-1}$$

Where $\overline{Q_{55}^{(k)}}$ is the transverse shear modulus of the k^{th} layer.

The layer-wise strain components can be written in terms of displacement components as

$$\epsilon_{xx}^{(k)} = u_{,x} + z\theta_{,x} + \phi^{(k)}\psi_{,x}$$

$$\gamma_{xz}^{(k)} = w_{,x} + \theta + \phi_{,z}^{(k)}\psi$$

Where subscript comma denotes differentiation with respect to the x and z coordinates.

The generalized Hooke's law for the k^{th} orthotropic lamina, whose principal material directions are arbitrary, is specified as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{xz} \end{pmatrix}^{(k)} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix}^{(k)} \begin{pmatrix} \epsilon_{xx} \\ \gamma_{xz} \end{pmatrix}^{(k)}$$

In which $\overline{Q_{IJ}}$ represents the axial and transverse ply properties.

The governing equations of the RZT are derived based on the principal of virtual work as

$$\delta W_i + \delta V_e = 0$$

Where δW_1 and δV_E are the virtual work of internal forces and the virtual work done by an external distributed normal load, respectively.
The virtual work of internal forces can be written as

$$\delta W_i = \int_0^L \int_A \left(\sigma_{xx}^{(k)} \delta \epsilon_{xx}^{(k)} + \sigma_{xz}^{(k)} \delta \epsilon_{xz}^{(k)} \right) dAdx$$

Substituting for the stress and strain components into the last equation and integrating over the cross-section of the beam, A, the virtual work of internal forces, δW_i , becomes [1]

$$\delta W_i = \int_0^L \left(N_x \delta u_{,x} + M_x \delta \theta_{,x} + M_\phi \delta \psi_{,x} + Q_x \delta\left(w_{,x} + \theta\right) + Q_\phi \delta \psi \right) dx$$

Where the stress,moment and shear resultants are defined as

$$N_x = b \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \sigma_{xx}^{(k)} dz$$

$$\left(M_x, M_\phi\right) = b \left(\sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} z \sigma_{xx}^{(k)} dz, \qquad \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \phi^{(k)} \sigma_{xx}^{(k)} dz \right)$$

And

$$\left(Q_x, Q_\phi\right) = b(\sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \sigma_{xx}^{(k)} dz, \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \beta^{(k)} \sigma_{xx}^{(k)} dz$$

Where b is the width of the beam. The constitutive equations expressing the relation between stress resultants and derivatives of kinematic unknowns, are [1]

$$\begin{pmatrix} N_x \\ M_x \\ M_\phi \\ Q_x \\ Q_\phi \end{pmatrix} = \begin{bmatrix} A_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{12} & D_{11} & D_{12} & 0 & 0 \\ B_{13} & B_{12} & D_{22} & 0 & 0 \\ 0 & 0 & 0 & F_{11} & F_{12} \\ 0 & 0 & 0 & F_{12} & F_{22} \end{bmatrix} \begin{pmatrix} u_{,x} \\ \phi_{,x} \\ \psi_{,x} \\ w_{,x} + \theta \\ \psi \end{pmatrix}$$

Where the stiffness coefficients can be written as

$$(A_{11}, B_{12}, D_{11}) = b \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \overline{Q_{11}}^{(k)} (1, z, z^2) dz,$$

$$(B_{13}, D_{12}, D_{22}) = b \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \overline{Q_{11}}^{(k)} \phi^{(k)} \left(1, z, \phi^{(k)}\right) dz,$$

$$(F_{11}, F_{12}, F_{22}) = b \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \overline{Q_{55}}^{(k)} \left(\beta^{(k)}, -\left(\beta^{(k)}\right)^2, \left(\beta^{(k)}\right)^2\right) dz,$$

The virtual work done by an external distributed normal load can be written

$$\delta V_e = \int_0^L p(x) \delta w dx$$

Performing the cross-sectional integration and variation by parts leads to the governing equations of equilibrium and boundary conditions as [1]

Equilibrium equations:

$$N_{x,x} = 0$$

$$Q_{x,x} + p = 0$$

$$M_{x,x} - Q_x = 0$$

$$M_{\phi,x} - Q_\phi = 0$$

Boundary conditions:

$$N_x = 0 \;\; or \;\; \delta u = 0$$

$$M_x = 0 \;\; or \; \delta\theta = 0$$

$$Q_x = 0 \;\; or \;\; \delta w = 0$$

$$M_\phi = 0 \;\; or \;\; \delta\psi = 0$$

Substituting the stress, moment, and shear resultants yields the equilibrium equations in terms of the kinematic variables of the RZT:

$$A_{11}u_{,xx} + B_{12}\theta_{,xx} + B_{13}\psi_{,xx} = 0$$

$$F_{11}(w_{,xx} + \theta_{,x}) + F_{12}\psi_{,x} + p(x) = 0$$

$$B_{12}u_{,xx} + D_{11}\theta_{,xx} + D_{12}\psi_{,xx} - F_{11}\left(w_{,x} + \theta\right) - F_{12}\psi = 0$$

$$B_{13}u_{,xx} + D_{12}\theta_{,xx} + D_{22}\psi_{,xx} - F_{12}\left(w_{,x} + \theta\right) - F_{22}\psi = 0$$

Where subscript comma denotes differentiation with respect to the x coordinate.

Chapter 5: Refined Zigzag Theory by using PDDO

The PD form of the equilibrium equations can be obtained by replacing the derivatives of the kinematic variables by using the Peridynamic Differential Operator. Their PD representation can be specifically written as [1]

$$\frac{d^p}{dx^p} f(x_{(k)}) = \sum_{j=1}^{N_{(k)}} \left(f(x_{(j)}) - f(x_{(k)})\right) g_4^p(\xi_{(k)(j)}) dl_{(j)}$$

Where $\xi_{(k)(j)} = x_{(j)} - x_{(k)}, dl_{(j)} = \Delta x$ and p denotes the order of differentiation with respect to x. The field variable f represents the unknown kinematic variables of RZT, u, w, θ and ψ. Also, $g_4^p(\xi_{(k)(j)})$ is the known PD function.

Applying PDDO we can express the kinematic variables of RZT:

$$u_{,xx} = \frac{d^2u}{dx^2} = \sum_{j=1}^{N_{(k)}} \left(u(x_{(j)}) - u(x_{(k)})\right) g_4^2(\xi_{(k)(j)}) dl_{(j)}$$

$$w_{,xx} = \frac{d^2w}{dx^2} = \sum_{j=1}^{N_{(k)}} \left(w(x_{(j)}) - w(x_{(k)})\right) g_4^2(\xi_{(k)(j)}) dl_{(j)}$$

$$\theta_{,xx} = \frac{d^2\theta}{dx^2} = \sum_{j=1}^{N_{(k)}} \left(\theta(x_{(j)}) - \theta(x_{(k)})\right) g_4^2(\xi_{(k)(j)}) dl_{(j)}$$

$$\psi_{,xx} = \frac{d^2\psi}{dx^2} = \sum_{j=1}^{N_{(k)}} \left(\psi\left(x_{(j)}\right) - \psi\left(x_{(k)}\right)\right) g_4^2\left(\xi_{(k)(j)}\right) dl_{(j)}$$

$$w_{,x} = \frac{dw}{dx} = \sum_{j=1}^{N_{(k)}} \left(w\left(x_{(j)}\right) - w\left(x_{(k)}\right)\right) g_4^1\left(\xi_{(k)(j)}\right) dl_{(j)}$$

$$\theta_{,x} = \frac{d\theta}{dx} = \sum_{j=1}^{N_{(k)}} \left(\theta\left(x_{(j)}\right) - \theta\left(x_{(k)}\right)\right) g_4^1\left(\xi_{(k)(j)}\right) dl_{(j)}$$

$$\psi_{,x} = \frac{d\psi}{dx} = \sum_{j=1}^{N_{(k)}} \left(\psi\left(x_{(j)}\right) - \psi\left(x_{(k)}\right)\right) g_4^1\left(\xi_{(k)(j)}\right) dl_{(j)}$$

Substituting in the equations of Refined Zigzag Theory we can write the following equations of the RZT using PDDO:

1)
$$\begin{aligned} A_{11}&\left(\sum_{j=1}^{N_{(k)}} \left(u\left(x_{(j)}\right) - u\left(x_{(k)}\right)\right) g_4^2\left(\xi_{(k)(j)}\right) dl_{(j)}\right) \\ + B_{12}&\left(\sum_{j=1}^{N_{(k)}} \left(\theta\left(x_{(j)}\right) - \theta\left(x_{(k)}\right)\right) g_4^2\left(\xi_{(k)(j)}\right) dl_{(j)}\right) \\ + B_{13}&\left(\sum_{j=1}^{N_{(k)}} \left(\psi\left(x_{(j)}\right) - \psi\left(x_{(k)}\right)\right) g_4^2\left(\xi_{(k)(j)}\right) dl_{(j)}\right) = 0 \end{aligned}$$

2)
$$F_{11}\left(\sum_{j=1}^{N_{(k)}}\left(w\left(x_{(j)}\right)-w\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right.$$
$$\left.+\sum_{j=1}^{N_{(k)}}\left(\theta\left(x_{(j)}\right)-\theta\left(x_{(k)}\right)\right)g_4^1\left(\xi_{(k)(j)}\right)dl_{(j)}\right)$$
$$+F_{12}\left(\sum_{j=1}^{N_{(k)}}\left(\psi\left(x_{(j)}\right)-\psi\left(x_{(k)}\right)\right)g_4^1\left(\xi_{(k)(j)}\right)dl_{(j)}\right)+p(x)=0$$

3)
$$B_{12}\left(\sum_{j=1}^{N_{(k)}}\left(u\left(x_{(j)}\right)-u\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)$$
$$+D_{11}\left(\sum_{j=1}^{N_{(k)}}\left(\theta\left(x_{(j)}\right)-\theta\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)$$
$$+D_{12}\left(\sum_{j=1}^{N_{(k)}}\left(\psi\left(x_{(j)}\right)-\psi\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)$$
$$-F_{11}\left(\sum_{j=1}^{N_{(k)}}\left(w\left(x_{(j)}\right)-w\left(x_{(k)}\right)\right)g_4^1\left(\xi_{(k)(j)}\right)dl_{(j)}\right.$$
$$\left.+\sum_{j=1}^{N_{(k)}}\left(\theta\left(x_{(j)}\right)-\theta\left(x_{(k)}\right)\right)g_4^0\left(\xi_{(k)(j)}\right)dl_{(j)}\right)$$
$$-F_{12}\left(\sum_{j=1}^{N_{(k)}}\left(\psi\left(x_{(j)}\right)-\psi\left(x_{(k)}\right)\right)g_4^0\left(\xi_{(k)(j)}\right)dl_{(j)}\right)=0$$

$$
\begin{aligned}
4)\qquad & B_{13}\left(\sum_{j=1}^{N_{(k)}}\left(u\left(x_{(j)}\right)-u\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)\\
&+D_{12}\left(\sum_{j=1}^{N_{(k)}}\left(\theta\left(x_{(j)}\right)-\theta\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)\\
&+D_{22}\left(\sum_{j=1}^{N_{(k)}}\left(\psi\left(x_{(j)}\right)-\psi\left(x_{(k)}\right)\right)g_4^2\left(\xi_{(k)(j)}\right)dl_{(j)}\right)\\
&-F_{12}\left(\sum_{j=1}^{N_{(k)}}\left(w\left(x_{(j)}\right)-w\left(x_{(k)}\right)\right)g_4^1\left(\xi_{(k)(j)}\right)dl_{(j)}\right.\\
&\left.+\quad\sum_{j=1}^{N_{(k)}}\left(\theta\left(x_{(j)}\right)-\theta\left(x_{(k)}\right)\right)g_4^0\left(\xi_{(k)(j)}\right)dl_{(j)}\right)\\
&-F_{12}\left(\sum_{j=1}^{N_{(k)}}\left(\psi\left(x_{(j)}\right)-\psi\left(x_{(k)}\right)\right)g_4^0\left(\xi_{(k)(j)}\right)dl_{(j)}\right)=0
\end{aligned}
$$

The unknown quantities in the discrete form of the PD equilibrium equations are solved under specified boundary conditions. The discretized form of the equilibrium equations can be expressed in terms of the unknown kinematic variables u, w, θ and ψ as

$$F(u, b) = Lu + b = 0$$

In which the matrix L represents the coefficients arising from the PD differentiation and the PD unknowns, and the vector u contains the PD unknowns at each point. The vector b includes the known values of at each point.

Also, the boundary conditions can be expressed as constraint equations in the form:

$$Gu + d = 0$$

Where the known matrix, G contains the coefficients arising from the PD differentiation and the PD unknowns, and the vector d contains the specified known values of constraint equations.
These field and constraint equations can be combined in a variational form using Lagrange multipliers, λ, as

$$\delta u^T(Lu + b) + \delta[\lambda^T(Gu + d)] = 0$$

Where δu represents arbitrary variations of the unknown vector u. The first variation of the second term on the left-hand side yields

$$\delta u^T(Lu + b) + \delta\lambda^T(Gu + d) + \delta u^T G^T \lambda = 0$$

The resulting equation can then be written as

$$\begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix}^T \left(\begin{bmatrix} L & G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} \right) = 0$$

For arbitrary variations of δu and $\delta\lambda$, the system of algebraic equations for the solution of u and λ are obtained as

$$\begin{bmatrix} L & G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = -\begin{pmatrix} b \\ d \end{pmatrix}$$

In order to achieve the continuous variation of the transverse shear stress through the thickness of the beam, the transverse shear stress is calculated from the integration of stress equilibrium equation. The integration of the transverse shear stress can be written in the form

$$\sigma_{xz} = -\int_{-h}^{z} \sigma_{xx,x}\, dz$$

Similarly, it is possible to study the stress:the PD representation of the derivative of the in-plane stress component is expressed as

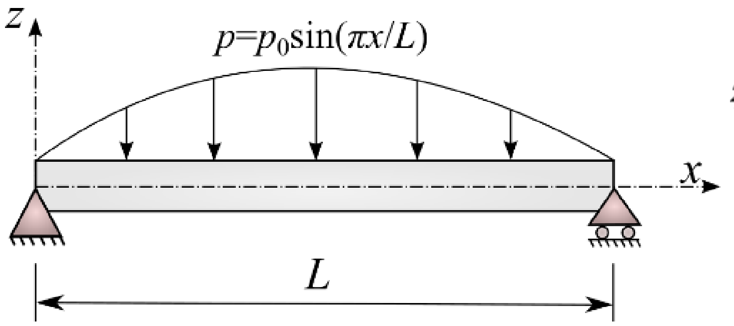
$$\frac{d}{dx}\sigma_{xx}\left(x_{(k)}, z\right) = \sum_{j=1}^{N_{(k)}} \left(\sigma_{xx}\left(x_{(j)}, z\right) - \sigma_{xx}\left(x_{(k)}, z\right)\right) g_4^1\left(\xi_{(k)(j)}\right) dl_{(j)}$$

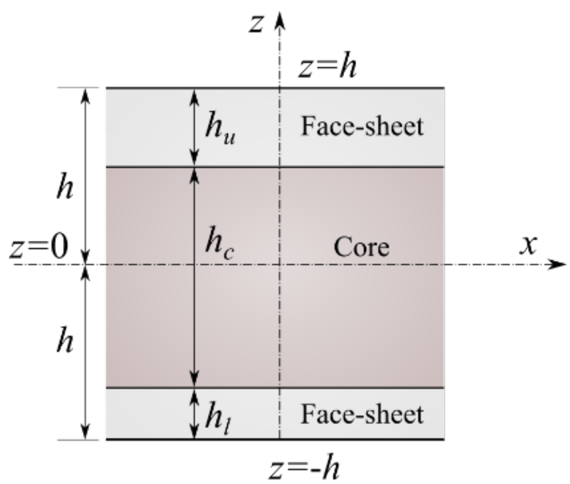
5.1 Example: Non-symmetric sandwich beam

The following example (in the figure below [1]) shows the comparison and the correspondence between the PD-RZT and Refined Zigzag Theory. It will be considered a non symmetric sandwich beam under sinusoidal transverse pressure p:

$$p = p_0 \sin\left(\frac{\pi x}{L}\right), \quad 0 \leq x \leq L$$

Where p_0 represents the amplitude of the loading.





The length is $L = 1\ m$, the width $b = 0.1\ m$. The thickness of the upper face-sheet is $h_u = 0.02$ and o the lower $h_l = 0.01$. The core has thickness $h_c = 0.07$ and finally the total thickness is $2h = 0.1$.
The horizon is $\delta = 4\Delta x$ and the discretization (Δx) is different along the beam:

$$\Delta x = 0.03\ if\ \ 0 \leq x \leq 0.3$$

$$\Delta x = 0.01\ if\ \ 0.3 \leq x \leq 0.7$$

$$\Delta x = 0.03\ if\ \ 0.7 \leq x \leq 1$$

The beam in the example is simply supported and the boundary conditions are:

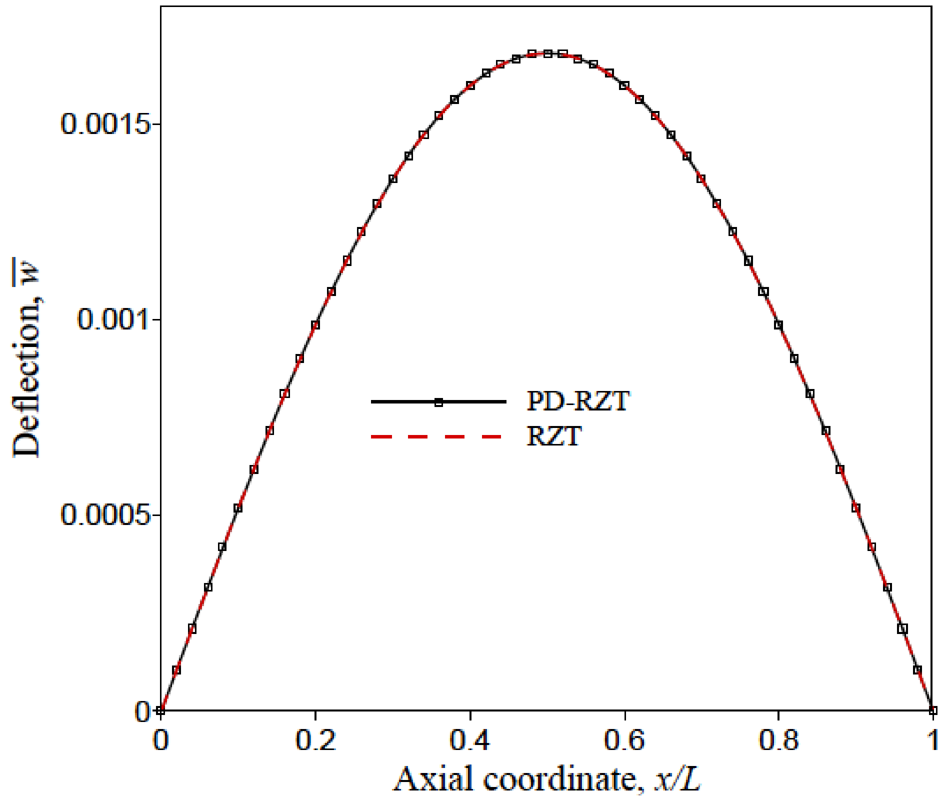
$$w = N_x = M_x = M_\phi = 0 \quad for\ x = 0, L.$$

In the results for convenience displacement and stress are normalized:

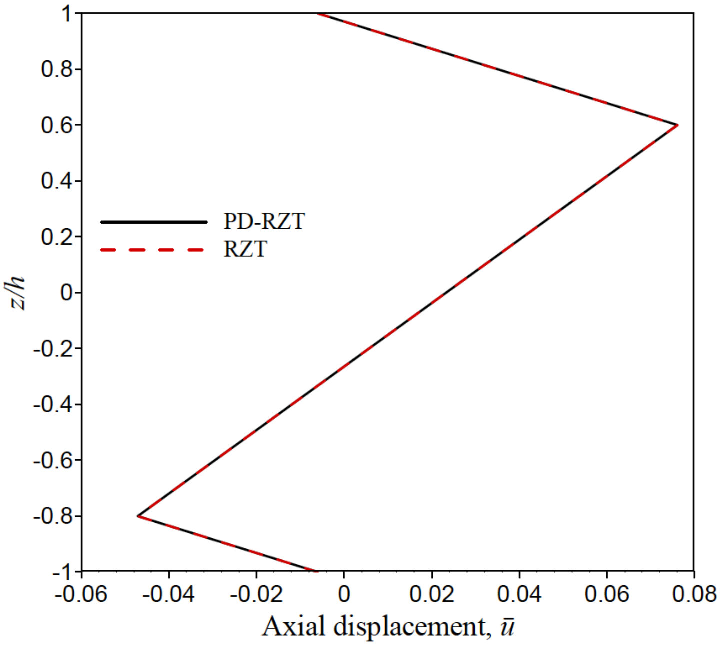
$$(\bar{u}, \bar{w}) = \frac{\pi^4 D_{11} b}{10 p_0 L^4}(u, w), \quad (\bar{\sigma}_{xx}, \bar{\sigma}_{xz}) = \frac{\pi^2 (2h)^2 b}{p_0 L^2}(\sigma_{xx}, \sigma_{xz})$$

The foam has Young module $E = 0.104\ GPa$, shear module $G = 0.04\ GPa$, poisson $\nu = 0.3$. Regarding the faces: $E_1 = 158\ GPa$, $E_2 = 10\ GPa$, $E_3 = 10\ GPa$, $\nu_{12} = 0.32$, $\nu_{23} = 0.5$, $\nu_{31} = 0.32$, $G_{12} = 6\ GPa$, $G_{23} = 3.2\ GPa$, $G_{31} = 6\ GPa$.

The PD solutions are compared with the RZT, in particular it is shown the variations of deflection along the beam and the axial displacement.
In the following figure there is a comparison between PD-RZT and RZT (analytical solution) for the normalized deflection along the beam:

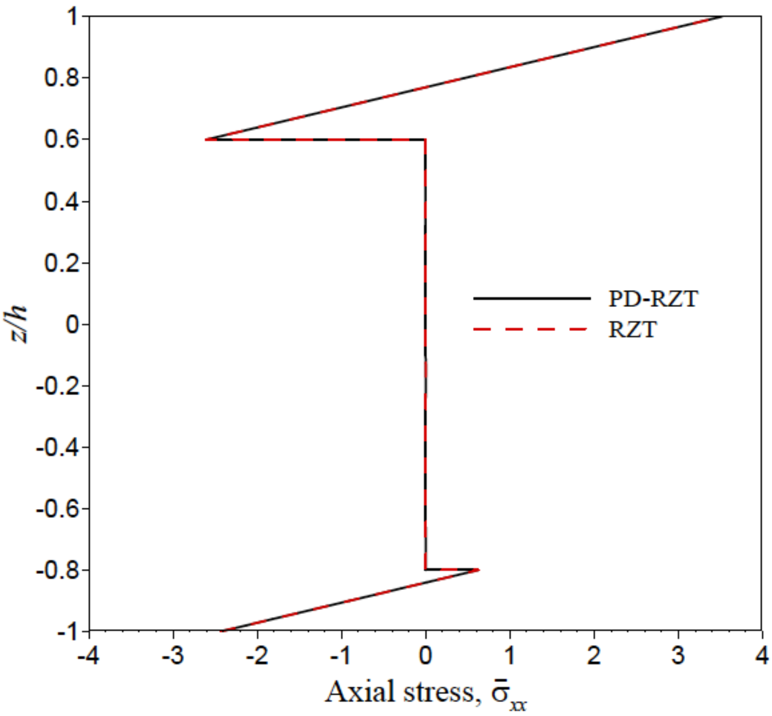


Finally it is reported a comparison between PD-RZT and RZT of axial displacement, $\bar{u}$,evalutated at $\bar{x} = 1$:

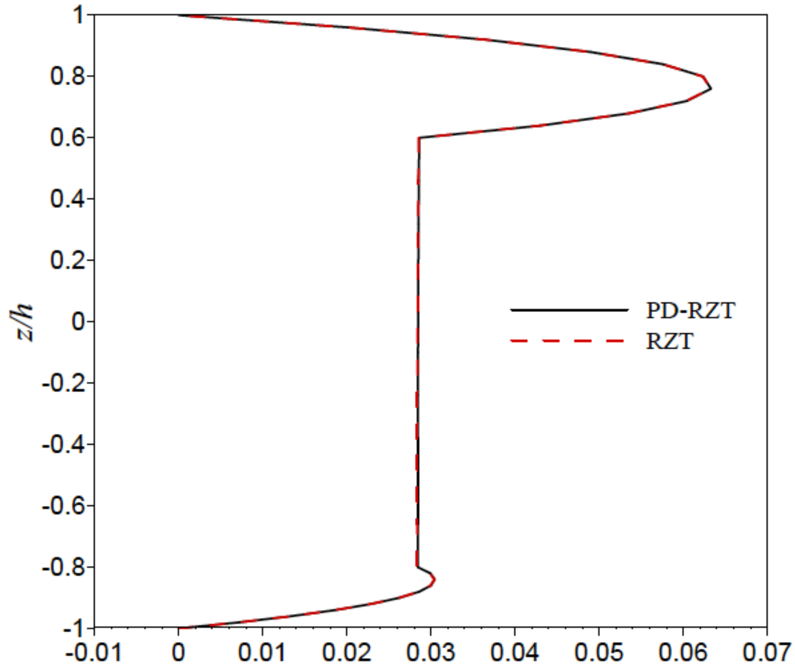


At the end of the chapter it is reported the comparison of the trend of the stresses.

The in plane stress ($\bar{\sigma}_{xx}$) is reported in the figure below:



The transverse shear stress ($\bar{\sigma}_{xz}\ in\ \bar{x} = 1, z$)) is:



Conclusion

In this thesis it highlighted that with peridynimic it is possible to represent the equations of motion of a general static or dynamic case.
In the first part I introduced the peridynamic and how it can be used to discretize a body and in the second chapter I explained the PDDO that are essential on the peridynamic to transform a differential equation in an integral equation; in order to see an applications of the peridynamic and his PDDO I have shown the equations of the kinematics and some examples.
In the second part there is an introduction of Refined Zigzag theory (with the main eequations) and in the last chapter the application of PDDO to the RZT with an example.
Furthermore it is rised that the PD model successfully captures the displacement (along the beam or the plate) in comparison to the analytical solution.
Not using differential equation, but integral equation, the problem is greatly simplified and the system of equations more easly solved. The disadvantage is the computational load that increase with the complexity of the structure and of the discretization.
Regarding the computational cost in general we note that using a uniform mesh increases the computational calculation so it is better to thicken it in the area of interest.

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