POLITECNICO DI TORINO

Master's Degree
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The Hille-Yosida theorem and some applications

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Ai miei genitori.
A Filippo, Nunzia e Riccardo.

Contents

Introduction

The present thesis is mainly focused on the Hille-Yosida theorem, which is a very powerful tool in solving evolution partial differential equations.
This work is meant to be a sort of connection between the mathematical theory of partial differential equations and its applications to physics and engineering, being the conclusion of a Master Degree in Mathematical Engineering at Politecnico di Torino. The treated topics can be well related to mechanics, thermodynamics and in general in all applied sciences which make full use such kind of problems. Some of the first definitions and theorems will be just cited and theorems will not be proved, but relevant references are present when needed.
The first chapter will be devoted to the preliminary knowledge about functional analysis, and in particular concerning the theory of linear operators, focusing on the unbounded ones and their properties which will be useful in the following pages.
The main theoretical topic of this thesis will be presented in the second chapter, in which the Hille-Yosida theorem will be treated, together with some mentions about regularity of the solutions and the case of self-adjoint operators, which are often found, for instance, in quantum mechanics. A mention of the Cauchy-Lipschitz-Picard theorem is added at the beginning, being it a very relevant and classic result about the solutions of ordinary differential equations.
In the third chapter, some of the most relevant equations from different fields of applications will be derived and then solved by means of the results achieved in the previous pages. In particular, the heat equation, the wave equation and the linearized equations of coupled sound and heat flow will be analyzed and the existence of a unique solution of this kind of problems will be proved .
The fourth and final chapter contains some results of recent studies about the so-called Hille-Yosida operators and the related Cauchy problems. This last chapter will refer mainly to a scientific article [1], and it is dedicated to the proof of the existence of different types of solutions taking advantage of the properties of the Hille-Yosida operators.

Chapter 1

Linear operators

Dealing with phisics, engineering, economy, finance or any other field requiring a certain mathematical background means to deal with *functions*, which represent basically *relations* between sets of objects. A function can represent for instance the evolution of the speed of a car in time, the oscillations of the financial value of bonds or many other things, and it comes into play, from a more general point of view, when it is necessary to describe the variation of a certain quantity with respect to another one.
When it comes to functional analysis, and even more when dealing with partial differential equations, the concept of function has to be generalized a bit. The idea of *operator* is indeed an extension of the one of function. An operator is an *application* between sets, which in turn can be normed, Banach, Hilbert and similar (an operator can easily be a "function of functions").
The theory of operators deals with these mathematical objects, trying to classify them according to their properties, which may vary depending on the sets they are defined on, the dimension of the "environment" spaces and so on. Many properties turn out to be very useful when it comes to differential equations, and provide effective tools to proove the existence and uniqueness of the solutions of problems coming from different fields of mathematics.
Although this chapter is not meant to be a thorough dissertation concerning the theory of operators, it is presented in order to give some hints about the most relevant aspects that have to be considered when trying to study evolution problems.

1.1 Linear operators

Here we are going to provide the basic definitions which are necessary to understand the following chapters. We start from the definition of linear operator (see [2]):

Definition 1.1.1. *Let X and Y be two Banach spaces. An unbounded linear operator from X to Y is a linear application $A : D(A) \subset X \to Y$ defined on a subspace $D(A) \subset X$, with values in Y. $D(A)$ is the domain of A.*
Moreover, A is said to be bounded (or continuous) if there exists a constant $c \geq 0$ such that

$$\|Au\|_Y \leq c\,\|u\|_X \quad \forall u \in D(A)$$

If the operator takes values in a scalar field, it is called functional.

Definition 1.1.2. *Let X be a normed space on $\mathbb{R}$. We define the dual space of X (denoted with X') as the vector space:*

$$X' = \{f : X \to \mathbb{R} \; : f \text{ linear and bounded}\}.$$

A simple but effective example of functional is the so-called integral function. Given, for instance, $f \in C([0,1])$ and $x_0 \in [0,1]$, it is given by:

$$F(x) = \int_{x_0}^{x} f(t)dt$$

which means that it gives, for all $x \in [0,1]$ the value of the area (with sign) under the graph of f. Other important definitions:

- Graph of A: $G(A) = \bigcup_{u \in D(A)} (u, Au) \subset X \times Y$
- Range of A: $R(A) = \bigcup_{u \in D(A)} Au \subset Y$
- Kernel of A: $N(A) = \{u \in D(A) : Au = 0\} \subset X$

Definition 1.1.3. *An operator A is said to be closed if $G(A)$ is closed in $X \times Y$.*

Definition 1.1.4. *We define $\mathcal{L}(X, Y)$ as:*

$$\mathcal{L}(X, Y) = \{A : X \to Y, A \textit{ linear and continuous}\}.$$

Definition 1.1.5. *The norm of an operator is defined by:*

$$\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}$$

To clarify the equivalence between boundedness and continuity of an operator, a classical result is the following one (see for instance [3]):

Theorem 1.1.1. *Let X and Y be normed spaces, $A : X \to Y$ a linear operator. The following are equivalent:*

- *(1) $\exists c > 0$ such that*
$$\|Ax\|_Y \leq c \|x\|_X \quad \forall x \in X$$
- *(2) $\exists c > 0$ such that*
$$\|Ax\|_Y \leq c \quad \forall x \in X, \quad \|x\|_X \leq 1$$
- *(3) A is uniformly continuous*
- *(4) A is continuous*
- *(5) A is continuous at 0*

Proof. The implications (3) $\implies$ (4),(4) $\implies$ (5) and (5) $\implies$ (1) are obvious.
(1) $\implies$ (2). Given $x \in X$, $\|x\|_X \leq 1$ we have:

$$\|Ax\|_Y \leq c \|x\|_X \leq c.$$

(2) $\implies$ (1). Take $x \in X$, $x \neq 0$. This means that $\left\| \frac{x}{\|x\|_X} \right\|_X \leq 1$, therefore:

$$(2) \implies \left\| A\left(\frac{x}{\|x\|_X}\right) \right\|_Y \leq c.$$

Therefore

$$\frac{1}{\|x\|_X} \|A(x)\|_Y \leq c$$

$$\|A(x)\|_Y \leq c \|x\|_X .$$

If $x = 0$, then $\|A(0)\|_Y = 0 \leq c \|0\|_X$. Finally we proove that (1) and (2) are equivalent.
(1) $\implies$ (3). Since A is a linear application,

$$\|Ax - Ay\|_Y = \|A(x-y)\|_Y \leq c \|x-y\|_X \quad \forall x, y \in X.$$

Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{c}$. Then when $x, y \in X$ and $\|x-y\|_X < \delta$

$$\|Ax - Ay\|_Y \leq c \|x-y\|_X < c\frac{\epsilon}{c} = \epsilon$$

therefore A is uniformly continuous.
(5) $\implies$ (1). A is continuous in 0, that means:

$$\exists \delta > 0 : \text{if } \epsilon = 1, \|x - 0\|_X < \delta \implies \|Ax - A0\|_Y \leq 1.$$

Thus, let $x \in X$, $x \neq 0$, then:

$$\left\| \frac{\delta}{2} \frac{x}{\|x\|_X} \right\|_X = \frac{\delta}{2} < \delta \implies \left\| A\left(\frac{\delta}{2} \frac{x}{\|x\|_X} \right) \right\|_Y < 1$$

$$\frac{\delta}{2} \frac{1}{\|x\|_X} \|Ax\|_Y < 1$$

$$\|Ax\|_Y < \frac{\delta}{2} \|x\|_X$$

which means that A is bounded with $c = \frac{\delta}{2}$. If $x = 0$, $A0 = 0$ and the boundedness is immediate.
□

Definition 1.1.6. *Let X,Y be normed spaces. We define the vector space*

$$B(X,Y) := \{A : X \to Y \ \ \textit{linear, bounded}\}.$$

Definition 1.1.7. *Let X, Y be normed spaces. $A \in B(X,Y)$ is said to be invertible if $\exists S \in B(Y,X)$ such that $S \circ A = I_X$, $A \circ S = I_Y$.*
$S = A^{-1}$ is called inverse of A.

There are several invertibility criteria, but we are not going to treat these topics in this dissertation, for a detailed study see [3].

1.2 Linear operators on Hilbert spaces

When dealing with Hilbert spaces, the scalar product structure allows us to define a new type of "inverse" operator, that is the adjoint.

Theorem 1.2.1. *Let H, K be Hilbert spaces and let $A \in B(H,K)$. Then $\exists! A^* \in B(K,H)$ such that*

$$(Ax, y)_K = (x, A^*y)_H \quad \forall x \in H, \forall y \in K.$$

A^* is called the adjoint operator of A.

Definition 1.2.1. *Let H be an Hilbert space, $A \in B(H)$. A is said to be self-adjoint if $A^* = A$.*

Now we are going to present a crucial theorem of Functional Analysis, very important when dealing with weak formulation of evolution problems. This theorem gives a result about existence and uniqueness of solutions of problems of the type:

$$a(u, v) = F(v)$$

where $a : H \times H \to \mathbb{R}$ is a bilinear, symmetric, continuous and coercive form, whereas $F : H \to \mathbb{R}$ is a linear form.

Theorem 1.2.2 (Lax-Milgram)**.** *Let H be a Hilbert space, $\phi : H \times H \to \mathbb{R}$ a bilinear, symmetric, continuous and coercive form. Then $\forall f \in H'$ $\exists! y \in H$ such that*

$$f(x) = \phi(x, y) \quad \forall x \in H.$$

Proof. The bilinear form $\phi : H \times H \to \mathbb{R}$ is a scalar product on H. Indeed, linearity is verified, then:

- $\phi(x, y) = \phi(y, x)$ true by hypothesis
- Thanks to coercivity, $\phi(x, x) \geq m \left\|x\right\|^2$ with $m > 0$

$$\implies \phi(x, x) \geq 0 \implies \phi(x, x) = 0 \implies 0 \geq m \left\|x\right\|^2 \implies x = 0.$$

The new scalar product induces a norm, which is in turn equivalent with the original norm $\|x\| = \sqrt{(x, x)}$. The new norm is:

$$\|x\|_\phi := \sqrt{\phi(x, x)}$$

Now, thanks to continuity and coercivity:

$$m \left\|x\right\|^2 \leq \|x\|_\phi = \phi(x, x,) \leq c \left\|x\right\|^2$$

$$\sqrt{m} \left\|x\right\| \leq \|x\|_\phi \leq \sqrt{c} \left\|x\right\|.$$

Given $f : H \to \mathbb{R}$ linear and continuous,

$$|f(x)| \leq \|f\|_{H'} \|x\| \leq \|f\|_{H'} \frac{1}{\sqrt{m}} \|x\|_\phi \quad \forall x \in H$$

which means that f is bounded aso with the new norm. This finally means that $(H, \|\cdot\|_\phi)$ is a Hilbert space and f belongs to its dual space. We can apply Riesz-Frechèt to the new space:

$$\exists! y \in H \text{ such that } f(x) = \phi(x, y) \quad \forall x \in H.$$

□

Remark 1.2.1. *We stress out the fact that these theorem is actually a generalization of Riesz-Frechèt theorem, which is applied to scalar products that are a particular case of bilinear forms. Indeed, if $\phi = (\cdot, \cdot)$, Lax-Milgram is immediately verified.*

Definition 1.2.2. *Let H be a Hilbert space on the complex field $\mathbb{C}$, $A \in B(H)$. We define the spectrum of A as:*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

On the other hand we define the resolvent of A as:

$$\rho(A) = \mathbb{C} \setminus \sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is invertible}\}.$$

1.3 Unbounded operators

In this section we are going to introduce operators which are not necessarily bounded (or continuous), thus we will refer to them as linear maps between Hilbert spaces with no additional properties involved. As an example of unbounded operator we take the following from [3].
Let $\mathcal{P}$ be the linear subspace of $C_{\mathbb{C}}[0,1]$ consisting of all polynomial functions. If $A : \mathcal{P} \to \mathcal{P}$ is the linear transformation defined by

$$T(p) = p',$$

where p' is the derivative of p, then T is not continuous.
To prove this, we need to negate the definition of boundedness.
To do this, we consider the sequence:

$$p_n(t) = t^n.$$

We notice that:

$$\|p_n\| = \sup\{|p_n(t)| : t \in [0,1]\} = 1$$

whereas

$$\|Tp_n\| = \|p_n'\| = \sup\{|p_n'(t)| : t \in [0,1]\} = \sup\{|nt^{n-1}| : t \in [0,1]\} = n.$$

Therefore there does not exist $c \geq 0$ such that $\|Tp\| \leq c\,\|p\|$ for all $p \in \mathcal{P}$.
Now we are going to introduce very important definitions taken from [2].
Let H be a Hilbert space.

Definition 1.3.1. *Let $A : D(A) \subset H \to H$ be a linear unbounded operator. A is said to be monotone if*

$$(Av, v) \geq 0 \qquad \forall v \in D(A).$$

Moreover, A is said to be maximal monotone if $R(I + A) = H$ which means

$$\forall f \in H\ \exists u \in D(A) : u + Au = f.$$

Monotone operators are also referred to as "positive", because their definition is substantially a generalization of the one of monotonically increasing functions. Indeed, taking $u, v \in H$ we have:

$$(A(u - v), u - v) = (Au - Av, u - v) \geq 0.$$

The same applies if we consider a linear $f : \mathbb{R} \to \mathbb{R}$. Here, the writing:

$$f(y - x)(y - x) = [f(y) - f(x)](y - x) \geq 0 \quad \forall x, y \in \mathbb{R}$$

means that the function is monotonically increasing.

Theorem 1.3.1. *Let A be a maximal monotone operator. Then:*

(a) $D(A)$ is dense in H

(b) A is closed

(c) for all $\lambda > 0$, $(I + \lambda A)$ is bijective on H, $(I + \lambda A)^{-1}$ is a bounded operator and

$$\left\|(I + \lambda A)^{-1}\right\|_{\mathcal{L}(H)} \leq 1.$$

Proof. (a) Let $f \in H$ such that $(f, v) = 0 \ \forall v \in D(A)$. There exists $v_0 \in D(A)$ such that $v_0 + Av_0 = f$. It holds:

$$0 = (f, v_0) = \|v_0\|^2 + (Av_0, v_0) \geq \|v_0\|^2 .$$

Thus, $v_0 = 0$ and therefore $f = 0$.

(b) We have that $\forall f \in H$ there exists a unique $u \in D(A)$ such that $u + Au = f$. Indeed, if $\overline{u}$ is another solution, then it holds:

$$(u - \overline{u}) + A(u - \overline{u}) = 0.$$

Taking the scalar product with $(u - \overline{u})$, the fact that A is monotone leads to $u - \overline{u} = 0$. On the other hand it holds:

$$\|u\| \leq \|u\|^2 + (Au, u) = (f, u) \leq \|f\| \|u\| \implies \|u\| \leq \|f\|$$

and therefore the operator $f \to u$ indicated by $(I + A)^{-1}$ is linear and bounded with

$$\left\|(I + A)^{-1}\right\|_{\mathcal{L}(H)} \leq 1.$$

To verify that A is closed, let $\{u_n\}$ be a sequence such that $u_n \in D(A) \ \forall n$, $u_n \longrightarrow u$ and $Au_n \longrightarrow f$. It holds $u_n + Au_n \longrightarrow u + f$ therefore

$$u_n = (I + A)^{-1}(u_n + Au_n) \longrightarrow (I + A)^{-1}(u + f).$$

Therefore $u = (I + A)^{-1}(u + f)$, that is $u \in D(A)$ and $u + Au = u + f$.

(c) Suppose that there exists $\lambda_0 > 0$ such that $R(I + \lambda_0 A) = H$. We want to prove that $\forall \lambda > \frac{\lambda_0}{2}$ it holds $R(I + \lambda A) = H$. Like in (b), we have that:

$$\forall f \in H \ \exists! \ u \in D(A) : u + \lambda_0 Au = f,$$

and the operator $f \to u$ indicated by $(I + A)^{-1}$ is bounded with $\left\|(I + \lambda_0 A)^{-1}\right\|_{\mathcal{L}(H)} \leq 1$. Let's try to solve

$$u + \lambda Au = f \quad \text{with } \lambda > 0, \tag{1.1}$$

which can be written as:

$$u = (I + \lambda_0 A)^{-1} \left[\frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u \right].$$

Therefore if $\left|1 - \frac{\lambda_0}{\lambda}\right| < 1$, that is $\lambda > \frac{\lambda_0}{2}$, then (1.1) admits a solution thanks to the Banach fixed point theorem (see appendix A). A is maximal monotone, therefore $I + A$ is surjective. Due to the previous passages, $I + \lambda A$ is surjective for $\lambda > \frac{1}{2}$, therefore by recurrence it holds that $I + \lambda A$ is surjective for all $\lambda > 0$.

□

Definition 1.3.2. *Let A be a maximal monotone operator. We set, for all $\lambda > 0$,*

$$J_\lambda = (I + \lambda A)^{-1}$$

and

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda).$$

J_λ is the resolvent operator of A ($\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$), whereas A_λ is the Yosida regularized of A.

Remark 1.3.1. *We notice that $\{A_\lambda\}_{\lambda>0}$ is a family of bounded operators that approximate A for $\lambda \to 0$.*

Now we will list some important properties of monotone operators and their approximants, which will be very useful in the following chapters.

Theorem 1.3.2. *Let A be a monotone operator. It holds:*

(a)
$$A_\lambda v = A(J_\lambda v) \quad \forall v \in H \quad \textit{and} \quad \forall \lambda > 0$$

(b)
$$A_\lambda v = J_\lambda(Av) \quad \forall v \in D(A) \quad \textit{and} \quad \forall \lambda > 0$$

(c)
$$\|A_\lambda v\| \leq \|Av\| \quad \forall v \in D(A) \quad \textit{and} \quad \forall \lambda > 0$$

(d)
$$\lim_{\lambda \to 0} J_\lambda v = v \quad \forall v \in H$$

(e)
$$\lim_{\lambda \to 0} A_\lambda v = Av \quad \forall v \in D(A)$$

(f)
$$(A_\lambda v, v) \geq 0 \quad \forall v \in H \quad \textit{and} \quad \forall \lambda > 0$$

(g)
$$\|A_\lambda v\| \leq \frac{1}{\lambda} \|v\| \quad \forall v \in H \quad \textit{and} \quad \forall \lambda > 0.$$

Proof. (a) It is equivalent to
$$v = J_\lambda v + \lambda A(J_\lambda v)$$
and this follows from the definition of J_λ.

(b) It holds
$$Av = \frac{1}{\lambda}[(1 + \lambda A)v - v] = \frac{1}{\lambda}(1 + \lambda A)(v - J_\lambda v)$$
and therefore
$$J_\lambda A v = \frac{1}{\lambda}(v - J_\lambda v) = A_\lambda v.$$

(c) It follows from the previous point, since $\|J_\lambda\| \leq 1$.

(d) Let $v \in D(A)$, then
$$\|(I - J_\lambda v)\| = \|v - J_\lambda v\| = \lambda \|A_\lambda v\| \leq \lambda \|Av\| \xrightarrow{\lambda \to 0} 0.$$
Thus $\lim_{\lambda \to 0} J_\lambda v = v$.
Now let $v \in H$ and $\epsilon > 0$. Since $\overline{D(A)} = H$, there exists $v_1 \in D(A)$ such that $\|v - v_1\| < \epsilon$. It holds:
$$\|J_\lambda v - v\| \leq \|J_\lambda v - J_\lambda v_1\| + \|J_\lambda v_1 - v_1\| + \|v_1 - v\|$$

$$\leq 2\|v - v_1\| + \|J_\lambda v_1 - v_1\| \leq 2\epsilon + \|J_\lambda v_1 - v_1\|$$

where the second term goes to 0 as $\lambda \to 0$. Therefore we have

$$\limsup_{\lambda\to 0} \|J_\lambda v - v\| \leq 2\epsilon \quad \forall \epsilon > 0,$$

and finally

$$\lim_{\lambda\to 0} \|J_\lambda v - v\| = 0.$$

(e) Thanks to (b) and (d) we have:

$$\|A_\lambda v - Av\| = \|J_\lambda(Av) - Av\| \xrightarrow{\lambda\to 0} 0.$$

(f) We have:

$$(A_\lambda v, v) = (A_\lambda v, v - J_\lambda v) + (A_\lambda v, J_\lambda v)$$
$$= \lambda \|A_\lambda v\|^2 + (A(J_\lambda v), J_\lambda v) \geq 0.$$

(g) It follows from the previous point, namely:

$$\lambda \|A_\lambda v\|^2 \leq (A_\lambda v, v) \leq \|A_\lambda v\| \|v\|$$

therefore

$$\|A_\lambda v\| \leq \frac{1}{\lambda} \|v\|.$$

□

Since now we are dealing with unbounded operators, some details about the self-adjoint case have to be clarified. In particular, we have to introduce an additional definition. Being H a Hilbert space, we can identify H with its dual space and consider A^* as an unbounded operator on H.

Definition 1.3.3. *Let $A : D(A) \subset H \to H$ a linear unbounded operator with $\overline{D(A)} = H$. We say that A is symmetric if*

$$(Au, v) = (u, Av) \quad \forall u, v \in D(A),$$

instead we say that A is self-adjoint if

$$A^* = A$$

and this implies $D(A^*) = D(A)$.

It is evident that if A is self-adjoint, then it is also symmetric. On the other hand, if A is symmetric, it can happen that $A \neq A^*$, as the following example (taken from [4]) shows.

Example 1. *Let $H = L^2 = L^2(0,1)$. We define operators T_1, T_2 and T_3 as follows:*

- *$D(T_1)$ consists of all absolutely continuous functions f on $[0,1]$ with derivative $f' \in L^2$,*
- *$D(T_2) = D(T_1) \cap \{f : f(0) = f(1)\}$,*
- *$D(T_3) = D(T_1) \cap \{f : f(0) = f(1) = 0\}$.*

These are dense in L^2. We define

$$T_k f = if' \quad \forall f \in D(T_k),\ k = 1{,}2{,}3$$

and claim that

$$T_1^* = T_3, \quad T_2^* = T_2, \quad T_3^* = T_1.$$

We have that $T_3 \subset T_2 \subset T_1$, therefore T_2 is a self-adjoint extension of the symmetric (but not self-adjoint) operator T_3 and that the extension T_1 of T_2 is not symmetric. Notice that

$$(T_k f, g) = \int_0^1 (if')\overline{g} = \int_0^1 f\overline{(ig')} = (f, T_m g)$$

if $f \in D(T_k)$, $g \in D(T_m)$ and $m + k = 4$, since then $f(1)\overline{g}(1) = f(0)\overline{g}(0)$. This means that $T_m \subset T_k^*$, namely:

$$T_1 \subset T_3^*, \quad T_2 \subset T_2^*, \quad T_3 \subset T_1^*.$$

Now take $g \in D(T_k^*)$ and $\phi = T_k^* g$. Let $\Phi(x) = \int_0^x \phi$. We have, for $f \in D(T_k)$,

$$\int_0^1 if'\overline{g} = (T_k f, g) = (f, \phi) = f(1)\overline{\Phi}(1) - \int_0^1 f'\Phi.$$

When $k = 1{,}2$, $D(T_k)$ contains nonzero constants, so the last equation implies $\Phi(1) = 0$. When $k = 3$, $f(1) = 0$. In each of these cases it yields:

$$ig - \Phi \in R(T_k)^\perp.$$

We have $R(T_1) = L^2$, therefore $ig = \Phi$ if $k = 1$, and since $\Phi(1) = 0$ in that case, $g \in D(T_3)$. Thus $T_1^* \subset T_3$.
If $k = 2{,}3$, $R(T_k)$ consists of all $u \in L^2$ such that $\int_0^1 u = 0$. Therefore

$$R(T_2) = R(T_3) = Y^\perp,$$

where Y is the one-dimensional subspace of L^2 that contains the constants.
Hence $ig - \Phi$ is constant, therefore g is absolutely continuous and $g' \in L^2$, that is $g \in D(T_1)$. Thus $T_3^* \subset T_2$.
If $k = 2$, then $\Phi(1) = 0$, therefore $g(0) = g(1)$, which means $g \in D(T_2)$, thus $T_2^* \subset T_2$ and this completes the proof of the claim.
The following result shows that, in case the operator A is maximal monotone, then A is symmetric if and only if A is self-adjoint.

Theorem 1.3.3. *Let A be a maximal monotone and symmetric operator. Then A is self-adjoint.*

Proof. Let $J_1 = (I + A)^{-1}$ and let's prove that J_1 is self-adjoint. Since $J_1 \in \mathcal{L}(H)$, it is sufficient to prove that

$$(J_1 u, v) = (u, J_1 v) \quad \forall u, v \in H.$$

We set $u_1 = J_1 u$, $v_1 = J_1 v$ so that

$$u_1 + Au_1 = J_1 u + AJ_1 u = J_1(I + A)u = u,$$

$$v_1 + Av_1 = J_1 v + AJ_1 v = J_1(I + A)v = v.$$

We now take the scalar products:

$$(u_1, v_1) + (Au_1, v_1) = (u, v_1)$$

$$(u_1, v_1) + (u_1, Av_1) = (u_1, v)$$

and then take the difference, which thanks to the fact that A is symmetric leads to:

$$(u_1, v) = (u, v_1)$$

meaning that J_1 is self-adjoint. Now let $u \in D(A^*)$ and $f = u + A^*u$. Given $v \in D(A)$, it holds:

$$(f, v) = (u + A^*u, v) = (u, v) + (A^*u, v) = (u, v) + (u, Av) = (u, v + Av).$$

We define $w := v + Av = (I + A)v$, thus $v = J_1 w$ and:

$$(f, J_1 w) = (u, w) \quad \forall w \in H.$$

Finally, since J_1 is self-adjoint, we get $u = J_1 f$ and therefore $u \in D(A)$. This means that $D(A^*) = D(A)$, that is A is self-adjoint. □

Chapter 2

Evolution problems

The following chapter will be devoted mainly to a crucial theorem when dealing with *evolution problems*, and it takes advantage of the theory of operators, presented in the previous chapter.
Before that, first of all it is important to recall the main result concerning the solution of systems of ordinary differential equations, which is hardly extendable to systems of partial differential equations, but still remains the reference one when dealing with Cauchy (initial value) problems.
We stress out the fact that the variables are named x and t just to have a sort of "physical" reference, in order to give a meaning to the name of "evolution problems". Obviously it could be possible to include the t variable in a sort of $n+1$ dimensional space, but this way of presenting the equations is more intuitive.
The variable x will in some way refer to the "spatial" coordinates of a certain space $\Omega \subset \mathbb{R}^n$ where n is the dimension of the space (tipically 2 or 3) and t will refer to the time variable. Thus, the solutions of these problems will have to respect some regularity requirements, typically with respect to time. Hence, for example, the writing "$u \in C([0, +\infty[, H)$" means that $u : [0, +\infty[\to H$ is continuous. We'll say that "u is continuous in t, and $\forall t \geq 0$ it takes values in H". Another example is "$u \in L^2([0, T[, H)$", that means:

$$u : [0, T[\longrightarrow H \quad , \quad t \longmapsto u(t) \quad \text{is measurable and} \quad \int_0^T \|u(t)\|_H^2 dt < +\infty.$$

2.1 The Cauchy-Lipschitz-Picard theorem

The following is a classical theorem, very useful when dealing with ordinary differential equations, but practically useless to solve partial differential ones. It is based on an extended concept of *Lipschitzianity* for applications between Banach spaces.
We recall that, given (X, d_X) and (Y, d_Y) metric spaces, a function $f : X \longrightarrow Y$ is called a *Lipschitz continuous function* if the distance (in Y) between the images of any two points in X does not exceed a certain value, quantified by the *Lipschitz constant*.
In symbols, this means that:

$$\exists K > 0 : \quad d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) \quad \forall x_1, x_2 \in X$$

From a graphical point of view it is possible to visualize very effectively this condition if we consider $f : \mathbb{R} \longrightarrow \mathbb{R}$. In this case, the definition of Lipschitzianity involves the first derivative, stating that the latter is bounded by the Lipschitz constant; therefore, for a Lipschitz continuous function, for all the values in the domain there exists a double cone, whose vertex lies on the graph of the function, so that the latter always stays outside of it.

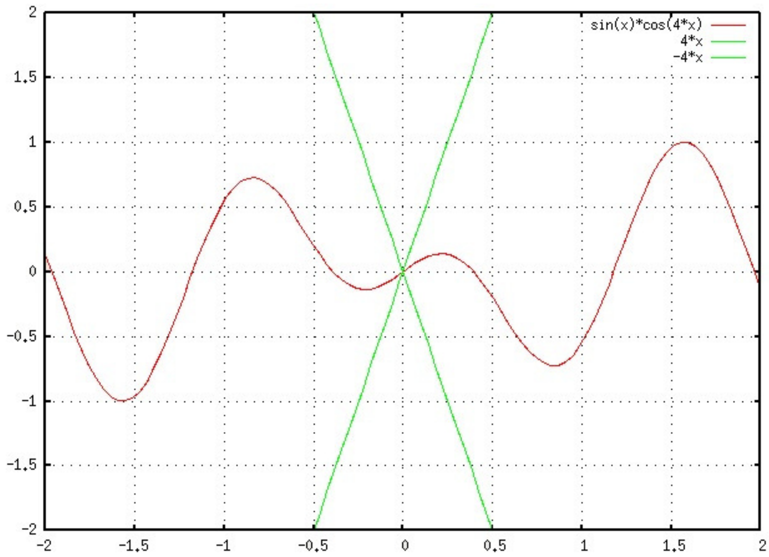

Figure 2.1: A Lipschitz continuous function ([5])

With the help of the first (and more general) definition it is possible to move to the next theorem.

Theorem 2.1.1 (Cauchy,Lipshitz,Picard)**.** *Let E be a Banach space and $F : E \longrightarrow E$ be an application such that*

$$\|Fu - Fv\|_E \leq L\,\|u - v\|_E \quad \forall u, v \in E \quad (L \geq 0).$$[1]

Then $\forall u_0 \in E\ \exists u \in C^1([0,\infty[, E)$ unique such that

$$\begin{cases} \dfrac{du}{dt} = Fu & \textit{on } [0,\infty[\\ u(0) = u_0 & (\textit{initial datum}). \end{cases} \tag{2.1}$$

Proof. The solution of problem 2.1 is such that:

$$u(t) = u(0) + \int_0^t F(u(s))ds. \tag{2.2}$$

Let $k \in \mathbb{R}$ (it will be specified later), we introduce:

$$X = \left\{ u \in C([0,\infty[; E) : \quad \sup_{t \geq 0} e^{-kt}\,\|u(t)\|_E < \infty \right\}$$

According to its definition, space X presents some important properties, which we are going to prove as in [6].

[1] *Being E a Banach space, the norm of the difference between two vectors is the distance bewtween them, induced by the norm $\|.\|_E$.*

- X is a Banach space with respect to the norm

$$\|u\|_X = \sup_{t\geq 0} e^{-kt}\,\|u(t)\|_E$$

To verify this statement, we are going to verify the definition of completeness, therefore we start from a Cauchy sequence in X. Let $\{u_n\} \subset X$ be a Cauchy sequence, which means that:

$$\forall \epsilon > 0 \quad \exists N_\epsilon : \forall m, n > N_\epsilon \qquad \|u_n - u_m\|_X < \epsilon.$$

By definition, fix $t \geq 0$

$$\exists N = N_{\epsilon,t,k} : \forall m, n > N_{\epsilon,t,k} \qquad \|u_n - u_m\|_X < \epsilon e^{-tk}$$

which is:

$$\sup_{s\geq 0} e^{-sk}\,\|u_n(s) - u_m(s)\|_E < \epsilon e^{-tk}.$$

By eliminating the sup symbol and taking any value of $t \geq 0$, we can simplify the exponential term and get:

$$\|u_n(t) - u_m(t)\|_E < \epsilon, \quad \forall t \geq 0$$

meaning that $\{u_n(t)\}$ is a Cauchy sequence in E, which is a Banach space. In turn, this means that $u_n(t)$ converges to a certain element in E, and we shall call it:

$$u(t) := \lim_{n\to\infty} u_n(t).$$

Now we have to prove that $u \in X$. First, we verify that $u \in C([0, \infty[; E)$. To do this, we observe that:

$$\exists N_1 : \quad \forall n > N_1 \qquad \|u(t) - u_n(t)\|_E < \frac{\epsilon}{3}$$

and, since $\{u_n(t_0)\} \longrightarrow u(t_0)$ as well,

$$\exists N_2 : \quad \forall n > N_2 \qquad \|u(t_0) - u_n(t_0)\|_E < \frac{\epsilon}{3}$$

and finally, since u_n is continuous,

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \qquad \|u_n(t) - u_n(t_0)\|_E < \frac{\epsilon}{3} \quad \text{for } |t - t_0| < \delta.$$

By taking $n \geq max\{N1, N2\}$ and $|t - t_0| < \delta$:

$$\|u(t) - u(t_0)\|_E \leq \|u(t) - u_n(t)\|_E + \|u_n(t) - u_n(t_0)\|_E + \|u_n(t_0) - u(t_0)\|_E \leq \epsilon.$$

Taking $\epsilon > 0$ large enough, we have $\|u(t) - u_n(t)\|_E < \epsilon$ and since $\{u_n\} \in X$ it holds $\sup_{t\geq 0} e^{-tk}\,\|u_n(t)\|_E \leq \infty$. The latter means that:

$$\exists M > 0 : \quad e^{-tk}\,\|u_n(t)\|_E < M, \quad \forall t \geq 0$$

and therefore:

$$\|u_n(t)\|_E < Me^{tk} \quad \forall t \geq 0.$$

Now:

$$\|u(t)\|_E \leq \|u(t) - u_n(t)\|_E + \|u_n(t)\|_E \leq \epsilon + Me^{tk} < Me^{tk} \quad \forall \epsilon > 0.$$

Thus, due to the arbitrary choice of t, it yelds:

$$\exists M > 0 : \qquad \sup_{t\geq 0} e^{-tk} \|u(t)\|_E < M$$

and this means that $u \in X$. Now there's left to show that $\|u_n(t) - u(t)\|_X \longrightarrow 0$.
We restart from the Cauchy sequence in X:

$$\forall \epsilon > 0 \quad \exists N_\epsilon > 0 : \quad \forall m, n > N_\epsilon \qquad \|u_n - u_m\|_X < \epsilon$$

therefore:

$$\|u_n(t) - u_m(t)\|_E < \epsilon e^{tk}, \quad \forall t \geq 0.$$

Taking the limit for $m \longrightarrow 0$ and sup on the left hand side, we get finally:

$$\|u_n - u\|_X < \epsilon \qquad \forall \epsilon > 0$$

which means:

$$\|u_n - u\|_X \longrightarrow 0$$

and X is Banach with its defined norm.

- $\forall u \in X$ the function
$$(\phi u)(t) = u_0 + \int_0^t F(u(s))ds$$
belongs to X.

Again, to prove this statement we have to verify that both $\phi u \in C([0, +\infty[, E)$ and $\sup_{t\geq 0} e^{-tk} \|(\phi u)(t)\|_E < \infty$. We start from the latter, using the triangle inequality in the definition of ϕu:

$$\|(\phi u)(t)\|_E \leq \|u_0\|_E + \int_0^t \|F(u(s)) - F(u(0))\|_E \, ds + \int_0^t \|F(u_0)\|_E \, ds$$

and thanks to the Lipschitzianity of F:

$$\|(\phi u)(t)\|_E \leq \|u_0\|_E + L \int_0^t \|u(s) - u_0\|_E \, ds + t \|F(u_0)\|_E$$

$$\leq \|u_0\|_E + Lt \|u_0\|_E + L \int_0^t \|u(s)\|_E \, ds + t \|F(u_0)\|_E \tag{2.3}$$

and this last inequality will be the reference one. We need other passages in order to obtain some inequalities, namely:

$$L \int_0^t \|u(s)\|_E \, ds \leq L \int_0^t e^{sk} \|u\|_X \, ds \leq L \|u\|_X \frac{e^{tk} - 1}{k}$$

therefore:

$$e^{-tk} L \int_0^t \|u(s)\|_E \, ds \leq L \|u\|_X \frac{1 - e^{-tk}}{k} \leq \frac{L}{k} \|u\|_X .$$

leading to:

$$L \int_0^t \|u(s)\|_E \, ds \leq \frac{L}{k} \|u\|_X . \tag{2.4}$$

The second inequality is:

$$e^{-tk}\|u_0\|_E \leq \|u_0\|_E \,. \tag{2.5}$$

Moreover, since $e^{-tk} < 1/tk$, it holds:

$$e^{-tk}Lt\|u_0\| \leq \frac{L}{k}\|u_0\|_E \tag{2.6}$$

and

$$e^{-tk}t\|F(u_0)\|_E \leq \frac{1}{k}\|u_0\|_E \,. \tag{2.7}$$

Finally, we multiply equation (2.3) by e^{-tk} and substitute (2.4), (2.5), (2.6), (2.7) in it, obtaining:

$$e^{-tk}\|(\phi u)(t)\|_E \leq \|u_0\|_E + \frac{L}{k}\|u_0\|_E + \frac{L}{k}\|u\|_X + \frac{1}{k}\|F(u_0)\|_E \qquad \forall t \geq 0$$

which finally means exactly:

$$\sup_{t\geq 0} e^{-tk}\|(\phi u)(t)\|_E < \infty.$$

To prove that $\phi u \in C([0,+\infty[, E)$, we observe that $t \longmapsto \int_0^t F(u(s))ds$ is continuous and therefore:

$$\lim_{t\to t_0}\|(\phi u)(t) - (\phi u)(t_0)\|_E = \lim_{t\to t_0}\left\|\int_0^t F(u(s))ds - \int_0^{t_0} F(u(s))ds\right\|_E = 0 \qquad \forall t_0 \geq 0$$

so we finally proved that $\phi u \in X$.

- ϕ is a Lipschitz continuous function in X:

$$\|\phi u - \phi v\|_X \leq \frac{L}{k}\|u - v\|_X \,, \qquad \forall u, v \in X.$$

We have just to write down the X norm. Let $u, v \in X$, then:

$$\|\phi u - \phi v\|_X = \sup_{t\geq 0} e^{-tk}\left\|\int_0^t (F(u(s)) - F(v(s)))ds\right\|_E \leq \sup_{t\geq 0} e^{-tk}L\int_0^t \|u(s) - v(s)\|_E\, ds$$

$$\leq \sup_{t\geq 0} e^{-tk}L\int_0^t e^{sk}\|u - v\|_X\, ds \leq \sup_{t\geq 0} e^{-tk}L\|u - v\|_X \frac{e^{tk} - 1}{k}$$

and finally, as expected:

$$\|\phi u - \phi v\|_X \leq \frac{L}{k}\|u - v\|_X \,.$$

We are ready to conclude the proof of the theorem. From the last property of space X it is easy to notice that if $k < L$, ϕ admits a fixed point (see appendix A), which is solution of (2.2). Thus, the existence of a solution is proved.

About the uniqueness of the solution, let u and $\overline{u}$ be two solutions of (2.1). We define

$$h(t) = \|u(t) - \overline{u}(t)\|_E$$

and it holds thanks to (2.2):

$$h(t) = \int_0^t \|F(u(s)) - F(\overline{u}(s))\|_E\, ds \leq L\int_0^t h(s)ds \qquad \forall t \geq 0$$

therefore, using Gronwall lemma, we obtain $h \equiv 0$. □

We observe that this theorem has been much studied, and it is possible to demonstrate that the lipschitzianity of F is in general not necessary for the existence of the solution of a Cauchy problem, whereas it comes strongly into play when dealing with uniqueness. Various examples of this phenomenon could be brought here, we give just a hint of what could happen in case that condition is not respected.
For instance, consider the Cauchy problem:

$$\begin{cases} \frac{dy}{dt}(t) = \frac{3}{2}\sqrt[3]{y(t)} & \text{in } \mathbb{R} \\ y(1) = 0. \end{cases}$$

The problem can be easily solved by separation of variables when $t \neq 0$, while in that point the field $F(y) = \frac{3}{2}\sqrt[3]{y}$ does not respect the lipschitzianity condition anymore. The solutions of the problem are (if $k \geq 1$):

$$y_k(t) = \begin{cases} 0 & \text{if } t < k \\ \pm\sqrt{(t-k)^3} & \text{if } t \geq k. \end{cases}$$

This means that, once fixed the parameter k, the solution is unique on the interval $[-\infty, k]$, but once the graph reaches point $t = k$, the curve branches out in infinite other solutions, in a way similar to the bristles of a brush (indeed, this phenomenon is referred to as the "Peano's brush", or "Peano's mustache").

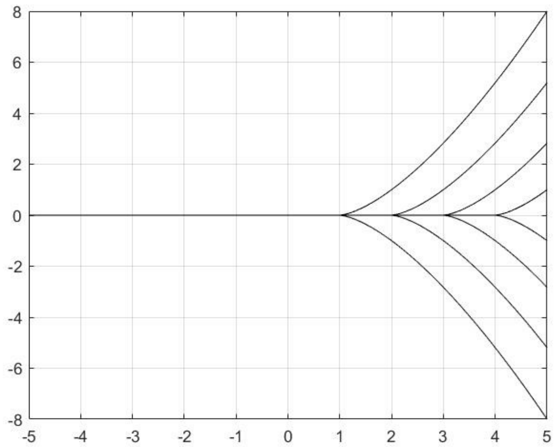

Figure 2.2: Solutions of problem 2.1 with $k \in \mathbb{N}$, $k \geq 1$ ("Peano's mustache").

2.2 The Hille-Yosida theorem

The next theorem, despite its quite long and technical demonstration, turns out to be a very useful tool and allows to assess the existence and uniqueness of the solution of certain types of evolution problems.

Theorem 2.2.1 (Hille-Yosida)**.** *Let A be a maximal monotone operator in a Hilbert space H. Then, for every $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty[; H) \cap C([0, +\infty[; D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \text{in } [0, +\infty[\\ u(0) = u_0. \end{cases} \tag{2.8}$$

Moreover it holds

$$\|u(t)\| \leq \|u_0\| \qquad \text{and} \qquad \left\| \frac{du}{dt} \right\| (t) = \|Au(t)\| \leq \|Au_0\| \qquad \forall t \geq 0.$$

Proof. It will be presented in six steps. First of all we will discuss about the uniqueness of the solution, then the remaining five steps will be dealing with the existence and the two inequalities.
Step 1.
Let u and $\overline{u}$ be two solutions of (2.8). Then, by taking the scalar product (in H) of the difference between the two equations and the difference between the solutions, it yields:

$$\left(\frac{d}{dt}(u - \overline{u}), u - \overline{u} \right) = -(A(u - \overline{u}), u - \overline{u}) \leq 0$$

where the last inequality holds because A is maximal monotone.
It is useful to recall that, if $f \in C^1([0, +\infty[, H)$, then:

$$|f|^2 \in C^1([0, +\infty[, \mathbb{R}) \qquad \text{and} \qquad \frac{d}{dt}|f|^2 = 2\left(\frac{df}{dt}, f \right).$$

Therefore:

$$\frac{1}{2}\frac{d}{dt}|u(t) - \overline{u}(t)|^2 = -(A(u - \overline{u}), u - \overline{u}) \leq 0.$$

Thus, the function $t \to |u(t) - \overline{u}(t)|$ is decreasing on $[0, +\infty[$.
Since $|u(0) - \overline{u}(0)| = 0$ because the initial datum is unique, it follows:

$$|u(t) - \overline{u}(t)| = 0 \qquad \forall t \geq 0.$$

The latter, together with the fact that the same function is decreasing, means that $u(t) = \overline{u}(t) \quad \forall t \geq 0$.
The following part of this demonstration will use the results concerning maximal monotone operators, namely the ones about the sequence of bounded operators A_λ (Yosida approximants) that approximate the unbounded operator A when $\lambda \to 0$. We observe that, thanks to Cauchy-Lipshitz-Picard Theorem 2.1, we know that there exists a unique solution of the problem:

$$\begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \qquad \text{in } [0, +\infty[\\ u_\lambda(0) = u_0 \in D(A). \end{cases} \tag{2.9}$$

Moreover, it will be useful to recall this:

Lemma 2.2.2. *Let $\varphi \in C^1([0, +\infty[; H)$ a function satisfying*

$$\frac{d\varphi}{dt} + A_\lambda \varphi = 0 \qquad \text{in} \qquad [0, +\infty[. \tag{2.10}$$

Then the functions $t \longmapsto \|\varphi(t)\|$ and $t \longmapsto \left\| \frac{d\varphi}{dt}(t) \right\| = \|A_\lambda \varphi(t)\|$ are decreasing on $[0, +\infty[$.

Proof. We take the scalar product of (2.10) with φ and get:

$$\left(\frac{d\varphi}{dt}, \varphi\right) + (A_\lambda \varphi, \varphi) = 0.$$

Thanks to theorem 1.3.2, we know that $(A_\lambda \varphi, \varphi) \geq 0$, therefore

$$\frac{1}{2}\frac{d}{dt}|\varphi|^2 \leq 0$$

namely $||\varphi(t)||$ is a decreasing function.
Finally, being A_λ a bounded operator, (2.10) gives that $\varphi \in C^\infty$ and

$$\frac{d}{dt}\left(\frac{d\varphi}{dt}\right) + A_\lambda\left(\frac{d\varphi}{dt}\right) = 0.$$

and with the same argument as before, we get that $||\frac{d\varphi}{dt}(t)||$ is a decreasing function as well. □

Step 2.
Due to the previous lemma, starting from (2.9), it holds:

$$\left\|\frac{du_\lambda}{dt}(t)\right\| = \|A_\lambda u_\lambda(t)\| \leq \|Au_0\| \quad \forall t \geq 0, \quad \forall \lambda > 0. \tag{2.11}$$

Step 3.
We are going to prove that $\forall t \geq 0$, u_λ converges (uniformly in t, for $\lambda \to 0$) to a limit, we shall call it $u(t)$, on every limited interval $[0, T]$. Let λ,μ be real positive numbers. The following relation holds:

$$\frac{du_\lambda}{dt} - \frac{du_\mu}{dt} + A_\lambda u_\lambda - A_\mu u_\mu = 0.$$

We now take the scalar product of the latter with $u_\lambda - u_\mu$, and get:

$$\left(\frac{du_\lambda}{dt} - \frac{du_\mu}{dt}, u_\lambda - u_\mu\right) + (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) = 0$$

that is:

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda - u_\mu\|^2 + (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) = 0. \tag{2.12}$$

Thanks to the properties of the Yosida Regularized (A_λ) and of the Resolvent (J_λ) as defined in the previous chapter, it is possible to derive the following inequality:

$$\begin{cases} (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) = (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu) \\ = (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) + (A(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu) \\ \geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) \end{cases}$$

namely:

$$(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) \geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu). \tag{2.13}$$

From (2.12) and (2.13) it follows:

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda - u_\mu\|^2 + (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) \leq 0$$

and expanding, due to the Cauchy-Schwarz inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|u_\lambda - u_\mu\right\|^2 + \lambda\left\|A_\lambda u_\lambda\right\|^2 - \lambda\left\|A_\mu u_\mu\right\|\left\|A_\lambda u_\lambda\right\| - \mu\left\|A_\lambda u_\lambda\right\|\left\|A_\mu u_\mu\right\| - \mu\left\|A_\mu u_\mu\right\|^2 \leq 0.$$

Finally we use (2.11):

$$\frac{1}{2}\frac{d}{dt}\left\|u_\lambda - u_\mu\right\|^2 \leq 2(\lambda+\mu)\left\|Au_0\right\|^2$$

and integrating:

$$\left\|u_\lambda(t) - u_\mu(t)\right\| \leq \sqrt{(\lambda+\mu)t}\left\|Au_0\right\|.$$

This last result shows that $\forall t \geq 0$, the sequence u_λ is a Cauchy sequence. Being H a Hilbert space by definition, it is complete and the sequence is therefore converging towards a limit, we shall call it $u(t)$. Taking the limit for $\mu \longrightarrow 0$, we get:

$$\left\|u_\lambda(t) - u(t)\right\| \leq 2\sqrt{\lambda t}\left\|Au_0\right\| \tag{2.14}$$

meaning that the convergence is uniform [2] in t on every limited interval $[0,T]$, and $u \in C([0,+\infty[,H)$.
Step 4.
With this step we're going to obtain the same convergence results of step 3, but related to the first derivative with respect to time. If $u_0 \in D(A^2)$[3], then it is possible to prove that $\frac{du_\lambda}{dt}$ converges for $\lambda \longrightarrow 0$, $\forall t \geq 0$ uniformly on every bounded interval $[0,T]$. Indeed, let $v_\lambda = \frac{du_\lambda}{dt}$ so that v_λ verifies (2.8). Following the same procedure of step 3 we get:

$$\frac{1}{2}\frac{d}{dt}\left\|v_\lambda - v_\mu\right\|^2 \leq (\left\|A_\lambda v_\lambda\right\| + \left\|A_\mu v_\mu\right\|)(\lambda\left\|A_\lambda v_\lambda\right\| + \mu\left\|A_\mu v_\mu\right\|). \tag{2.15}$$

Due to the lemma, $\left\|v_\lambda(t)\right\|$ is decreasing, which means:

$$\left\|A_\lambda v_\lambda\right\| \leq \left\|A_\lambda v_\lambda(0)\right\| = \left\|A_\lambda \frac{du_\lambda}{dt}(0)\right\| = \left\|A_\lambda A_\lambda u_\lambda(0)\right\| = \left\|A_\lambda A_\lambda u_0\right\|, \tag{2.16}$$

and analogously:

$$\left\|A_\mu v_\mu\right\| \leq \left\|A_\mu v_\mu(0)\right\| = \left\|A_\mu A_\mu u_0\right\| \tag{2.17}$$

Now we use some properties that were presented in the previous chapter, namely theorem 1.3.2, therefore since $Au_0 \in D(A)$ we can write:

$$A_\lambda A_\lambda u_0 = J_\lambda A J_\lambda A u_0 = J_\lambda J_\lambda A A u_0 = J_\lambda^2 A^2 u_0,$$

therefore, since $\left\|J_\lambda\right\| \leq 1$,

$$\left\|A_\lambda A_\lambda u_0\right\| \leq \left\|A^2 u_0\right\| \qquad , \qquad \left\|A_\mu A_\mu u_0\right\| \leq \left\|A^2 u_0\right\|. \tag{2.18}$$

Taking the two expressions in (2.18) an substituting in (2.15), we get:

$$\frac{1}{2}\frac{d}{dt}\left\|v_\lambda - v_\mu\right\|^2 \leq 2(\lambda+\mu)\left\|A^2 u_0\right\|^2.$$

[2]With "uniform convergence" we mean the convergence in the "uniform" (or "sup") norm $\|f\|_\infty = \sup\{|f(x)| : x \in D(f)\}$

[3]Here the symbol A^2 means $A \circ A$, therefore $u_0 \in D(A^2)$ means that both u_0 and Au_0 belong to $D(A)$. The definition is well-posed, since the range of A sill lies inside H.

By integration, it is immediate to state that $v_\lambda(t)$ is converging for $\lambda \longrightarrow 0$, $\forall t \geq 0$ uniformly in t on every bounded interval, and this is the convergence of the first derivative of $u(t)$ that we expected.
Step 5.
We proved that a solution of (2.8) exists with the more restrictive hypotheses of step 4, that is $u_0 \in D(A^2)$. The convergence results on both $u_\lambda(t)$ and $\frac{du_\lambda}{dt}$ guarantee that $u \in C^1([0, +\infty[, H)$ and $\frac{du_\lambda}{dt} \longrightarrow \frac{du}{dt}$ for $\lambda \longrightarrow 0$, uniformly on $[0, T]$. Let's write (2.9) in the form:

$$\frac{du_\lambda}{dt}(t) + A(J_\lambda u_\lambda(t)) = 0. \tag{2.19}$$

We observe that:

$$\|J_\lambda u_\lambda(t) - u(t)\| \leq \|J_\lambda u_\lambda(t) - J_\lambda u(t)\| + \|J_\lambda u(t) - u(t)\|$$

$$\leq \|J_\lambda\| \, \|u_\lambda(t) - u(t)\| + \|J_\lambda u(t) - u(t)\| \longrightarrow 0$$

the latter going to 0 because of the properties of the operator J_λ as $\lambda \longrightarrow 0$, and this means that:

$$J_\lambda u_\lambda(t) \longrightarrow u(t).$$

Therefore, since A is a closed operator, from (2.19) we deduce that $u(t) \in D(A)$ and it holds:

$$\frac{du}{dt}(t) + Au(t) = 0.$$

We know that $u \in C^1([0, +\infty[; H)$, therefore $t \longmapsto Au(t)$ is continuous on $[0, +\infty[$ (in H) and consequently $u \in C([0, +\infty[, D(A))$. We finally got a solution of (2.8) satisfying $\|u(t)\| \leq \|u_0\| \quad \forall t \geq 0$ and

$$\frac{du}{dt}(t) \leq \|Au_0\|.$$

Before going on with the last step, it is necessary to prove that $D(A^2)$ is dense in $D(A)$.

Lemma 2.2.3. *$D(A^2)$ is dense in $D(A)$.*

Proof. We remark that $D(A)$ is a normed space with the graph norm $\|v\| + \|Av\|$, therefore it will be necessary to obtain convergence of sequences of both v and Av in $D(A)$.
Let $\overline{u_0} = J_\lambda u_0$, so that $\overline{u_0} \in D(A)$ and $\overline{u_0} + \lambda A\overline{u_0} = u_0$. This in turn means that $A\overline{u_0} \in D(A)$, that is $\overline{u_0} \in D(A^2)$.
Thanks to the properties of operators J_λ and A_λ, it holds:

$$A\overline{u_0} = AJ_\lambda u_0 = A_\lambda u_0 = J_\lambda A u_0$$

and consequently

$$\lim_{\lambda \to 0} \|\overline{u_0} - u_0\| = \lim_{\lambda \to 0} \|J_\lambda u_0 - u_0\| = 0$$

$$\lim_{\lambda \to 0} \|A\overline{u_0} - u_0\| = \lim_{\lambda \to 0} \|J_\lambda A u_0 - Au_0\| = 0$$

which proves the thesis. □

Step 6.
Let $u_0 \in D(A)$. Thanks to lemma 2.2.3 there exists a sequence $(u_{0n}) \in D(A^2)$ such that $u_{0n} \longrightarrow u$ and $Au_0 \longrightarrow Au_0$. From step 5 we know about the existence of a solution for the problem:

$$\begin{cases} \dfrac{du_n}{dt} + Au_n = 0 \quad \text{in } [0,+\infty[\\ u_n(0) = u_{0n}. \end{cases} \tag{2.20}$$

Moreover it holds:

$$\|u_n(t) - u_m(t)\| \leq \|u_{0n} - u_{0m}\| \longrightarrow 0 \qquad \text{for} \quad m,n \longrightarrow \infty$$

$$\left\| \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t) \right\| \leq \|Au_{0n} - Au_{0m}\| \longrightarrow 0 \qquad \text{for} \quad m,n \longrightarrow \infty$$

with $u \in C^1([0,+\infty[, H)$. Taking the limit in (2.20), thanks to the fact that A is closed, it follows that $u \in C([0,+\infty[, D(A))$ and that u satisfies (2.8). □

The great importance of this last result is that it allows to define an entire class of problems according to the related operator A. In fact, let $t \geq 0$ and $S_A(t) : D(A) \longrightarrow D(A)$ such that it takes the initial datum u_0 and associates to it $u(t)$, i.e. the solution of (2.8) already obtained. Since $\|S_A(t)u_0\| \leq \|u_0\|$, the application $S_A(t)$ can be extended by continuity and density to a linear and continuous operator from H in itself. The following properties can be easily prooved:

- $\forall t \geq 0, \quad S_A(t) : H \longrightarrow H$ is a linear and continuous operator and $\|S_A(t)\|_{\mathfrak{L}(H)} \leq 1$
- $$\begin{cases} S_A(t_1 + t_2) = S_A(t_1) \circ S_A(t_2) \quad \forall t_1 \geq 0, \ \forall t_2 \geq 0 \\ S_A(0) = I \end{cases}$$
- $\lim_{t\to 0} \|S_A(t)u_0 - u_0\| = 0 \qquad \forall u_0 \in H.$

The family of operators $S(t)$ in $\mathfrak{L}(H)$, defined for all values of the parameter $t \geq 0$ and satisfying the tre already mentioned properties is by definition a *continuous contractions semigroup*. Conversely, it can be prooved that given a continuous contractions semigroup, there exists one and only one maximal monotone operator A such that $S(t) = S_A(t) \quad \forall t \geq 0$. Therefore, there is a one-to-one correspondence between maximal monotone operators and continuous contractions semigroups. This turns out to be the starting point of the theory of semigroups, since each semigroup corresponds to a maximal monotone operator, and in turn to an evolution problem of the type (2.8) and its solution.
In addition it is obvious that, once solved problem (2.8), it is possible to solve problems like the following:

$$\begin{cases} \dfrac{du}{dt} + Au + \lambda u = 0 \quad \text{in } [0,+\infty[, \ \lambda \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

just by substituting:

$$v(t) = e^{\lambda t}u(t).$$

Then $v(t)$ satisfies:

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad \text{in } [0,+\infty[\\ v(0) = u_0. \end{cases}$$

2.3 Hille-Yosida: regularity of the solutions

It is possible, by adding more hypotheses to the theorem, to obtain solutions which are more regular in the variable t. Referring to the way the proof of Hille-Yosida theorem was presented, we define by recurrence the space:

$$D(A^k) = \{v \in D(A^{k-1}); \quad Av \in D(A^{k-1})\}, \qquad k \text{ integer} \geq 2.$$

In other words, $u \in D(A^k)$ means that $u, Au, A^2u, \ldots, A^{k-1}u \in D(A)$.

Lemma 2.3.1. *$D(A^k)$ is a Hilbert space with respect to the scalar product:*

$$(u, v)_{D(A^k)} = \sum_{j=0}^{k}(A^j u, A^j v),$$

and the corresponding norm is:

$$\|u\|_{D(A^k)} = \left(\sum_{j=0}^{k}(A^j u, A^j u)\right)^{1/2} = \left(\sum_{j=0}^{k}\left\|A^j u\right\|^2\right)^{1/2}$$

Proof. Since $(u, v)_{D(A^k)}$ is a finite sum of scalar products, it is a scalar product itself. Thus, it will be sufficient to proove the completeness of the space. Let $\{u_n\} \in D(A^k)$ be a Cauchy sequence. That is, $\forall \epsilon > 0 \quad \exists N_\epsilon > 0 : \quad \forall m, n > N_\epsilon$,

$$\|u_n - u_m\|_{D(A^k)} < \epsilon$$

meaning that:

$$\left(\sum_{j=o}^{k}\left\|A^j(u_n - u_m)\right\|^2\right)^{1/2} < \epsilon$$

$$\|u_n - u_m\|^2 + \|Au_n - Au_m\|^2 + \cdots + \left\|A^k u_n - A^k u_m\right\|^2 < \epsilon^2 \qquad \forall m, n > N_\epsilon$$

$$\|u_n - u_m\|^2 < \epsilon^2$$

$$\|u_n - u_m\| < \epsilon.$$

We prooved that $\{u_n\}$ is also a Cauchy sequence in H, which is a Hilbert space and in particular complete. Therefore, u_n converges to a limit, say u_0, in H, as well as Au_n A^2u_n,...,$A^k u_n$ that converge respectively to some limits, say u_1, u_2,...,u_k. Since A is maximal monotone, it is also a closed operator. The fact that $\{u_n\} \subset D(A^k)$ implies that $\{u_n\} \subset D(A^j)$ for $j = 0, \ldots, k$. Now $\{u_n\} \subset D(A)$, $u_n \to u_0 \in H$, $Au_n \to u_1 \in H$. Since A is closed, this means that $u_0 \in D(A)$ and $u_1 = Au_0$. Again, $\{u_n\} \subset D(A^2)$ and we saw that $Au_n \to Au_0$. Also, $A(Au_n) = A^2u_n \to u_2$ and since A is closed, it follows that $Au_0 \in D(A)$ and $u_1 = Au_0$.
Repeating this argument in an inductive way it is evident that $u_j = A^j u_0$ and $A^j u_0 \in D(A)$ for $j = 0, \ldots, k$. This in turn implies that $u_0 \in D(A^k)$ and the Cauchy sequence $\{u_n\}$ converges to u_0 in $D(A^k)$, hence $D(A^k)$ is complete with the norm induced by the scalar product above. □

The space already defined allows to classify the solutions of the evolution problem according to the exponent k of the space $D(A^k)$ to which they belong. In other words, the higher the exponent, the more regular the solution will be in time.

Theorem 2.3.2. *Let $u_0 \in D(A^k)$, with $k \geq 2$. Then the solution of problem 2.8 satisfies:*

$$u \in C^{k-j}([0,+\infty[, D(A^j)) \qquad for \quad j = 0,1,\dots,k.$$

Proof. Let's begin with $k = 2$. Let $H_1 = D(A)$ be the Hilbert space equipped with the already defined scalar product $(u,v)_{D(A)}$. The operator $A_1 : D(A_1) \subset H_1 \to H_1$ defined as

$$\begin{cases} D(A_1) = D(A^2) \\ A_1 u = Au \qquad \text{for } u \in D(A_1) \end{cases} \tag{2.21}$$

which is basically A, but restricted to $D(A^2) \subset D(A)$, is maximal monotone in H_1, being A maximal monotone.
Hence it is possible to apply Theorem 2.2.1 to A_1 in the space H_1, meaning that it exists a function

$$u \in C^1([0,\infty[; H_1) \cap C([0,\infty[; D(A_1))$$

that means

$$u \in C^1([0,\infty[; D(A)) \cap C([0,\infty[; D(A^2))$$

such that

$$\begin{cases} \dfrac{du}{dt} + A_1 u = 0 \quad \text{on } [0,+\infty[\\ u(0) = u_0. \end{cases}$$

There is only left to prove that $u \in C^2([0,+\infty[; H)$, being $H = D(A^0)$. We know that A is a linear operator from H_1 to H, and $u \in C^1([0,\infty[; H_1)$, therefore thanks to linearity we get $Au \in C^1([0,+\infty[; H)$ and

$$\frac{d}{dt}(Au) = A\left(\frac{du}{dt}\right).$$

Due to equation 2.8,

$$\frac{du}{dt} = -Au \qquad \text{with} \quad Au \in C^1([0,+\infty[; H)$$

and therefore $\frac{du}{dt} \in C^1([0,+\infty[; H)$, that is $u \in C^2([0,+\infty[; H)$ and

$$\frac{d}{dt}\left(\frac{du}{dt}\right) + A\left(\frac{du}{dt}\right) = 0 \qquad \text{on} \quad [0,+\infty[. \tag{2.22}$$

Now let $k \geq 3$. We will use an induction argument: let's say the theorem is valid until order $k-1$ and let $u_0 \in D(A^k)$. We have already prooved that $u \in C^2([0,+\infty[; H) \cap C^1([0,+\infty[, D(A))$ and that (2.22) is satisfied. Let $v = \frac{du}{dt}$, then it holds:

$$v \in C^1([0,+\infty[; H) \cap C([0,+\infty[, D(A)),$$

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad \text{on } [0,+\infty[\\ v(0) = -Au_0. \end{cases}$$

Now, since $v_0 \in D(A^{k-1})$ we know, thanks to the recurrence hypothesis, that

$$v \in C^{k-1-j}([0,+\infty[, D(A^j)) \qquad \text{for} \quad j = 0,1,\dots,k-1,$$

which means that if considering u, the order has to be increased by 1 because $v = \frac{du}{dt}$:

$$u \in C^{k-j}([0, +\infty[, D(A^j)) \qquad \text{for} \quad j = 0,1, \dots, k-1.$$

The only order left to check is $j = k$ itself, so we apply (2.3) with $j = k - 1$ and get:

$$v = \frac{du}{dt} \in C([0, +\infty[; D(A^{k-1}))$$

and since by (2.8) we have that $\frac{du}{dt} = -Au$ we have that

$$Au \in C([0, +\infty[; D(A^{k-1}))$$

which means that

$$u \in C([0, +\infty[; D(A^k))$$

and the theorem is proved. □

2.4 The self-adjoint case

An additional condition imposed to operator A allows to obtain an existence and uniqueness result for problem 2.8, but with less restrictive hypotheses on the initial datum. In particular, this is the case when A is self-adjoint. We recall that, being A maximal monotone, it is sufficient for it to be symmetric in order to be self-adjoint too (see chapter 1). This specific case is very common when dealing with partial differential equations derived from engineering models [7].

Theorem 2.4.1. *Let A be a maximal monotone and self-adjoint operator. Then, for all $u_0 \in H$ it exists a unique function*

$$u \in C([0, +\infty[; H) \cap C^1([0, +\infty[; H) \cap C([0, +\infty[; D(A))$$

such that

$$\begin{cases} \dfrac{du}{dt} + Au = 0 & \text{in }]0, +\infty[\\ u(0) = u_0 \end{cases}$$

Morover, it holds:

$$\|u(t)\| \leq \|u_0\| \qquad \text{and} \qquad \left\| \frac{du}{dt}(t) \right\| \leq \|Au(t)\| \leq \frac{1}{t} \|u_0\| \qquad \forall t > 0 \tag{2.23}$$

As anticipated, we can notice that now u_0 has not necessarily to belong to $D(A)$, and the conclusion is weaker than the one in 2.2.1, since here $\frac{du}{dt}(t)$ might even "explode" when $t \longrightarrow 0$.

Proof. As concerning the uniqueness of the solution, we proceed just as we did for theorem 2.2.1. Actually there is no difference with the previously described procedure, since the coincidence between two solutions of the problem is given by the fact that A is monotone.
To prove the existence, we will follow two main steps. The first will use a restricted hypothesis on u_0, then the second will use the results obtained in the previous pages.
Step 1.
Let $u_0 \in D(A^2)$ and let u be the solution of (2.8), given by Theorem 2.2.1. We already know that J_1, as defined in the previous chapter, is self-adjoint. Moreover, being the scalar product

homogeneus with respect to scalars, J_λ is symmetric as well and therefore self-adjoint, because it is a bounded operator. We claim that A_λ is self-adjoint, indeed:

$$(A_\lambda u, v) = \left(\frac{1}{\lambda}(I - J_\lambda)u, v\right) = \frac{1}{\lambda}(u, v) - \frac{1}{\lambda}(J_\lambda u, v) = \left(u, \frac{1}{\lambda}(v - J_\lambda v)\right) = (u, A_\lambda v).$$

We now use the approximation used to prove theorem 2.2.1, namely:

$$\begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \quad \text{in } [0. + \infty[\\ u_\lambda(0) = u_0 \end{cases} \tag{2.24}$$

We multiply by u_λ and integrate on $[0, T]$, getting:

$$\left(\frac{du_\lambda}{dt}, u_\lambda\right) + (A_\lambda u_\lambda, u_\lambda) = 0$$

$$\frac{1}{2}\left\|u_\lambda(T)\right\|^2 + \int_0^T (A_\lambda u_\lambda, u_\lambda)dt = \frac{1}{2}\left\|u_0\right\|^2. \tag{2.25}$$

Starting again from (2.24), we multiply it by $t\frac{du_\lambda}{dt}(t)$ and integrate on $[0, T]$, thus:

$$t\left(\frac{du_\lambda}{dt}, \frac{du_\lambda}{dt}\right) + (A_\lambda u_\lambda, t\frac{du_\lambda}{dt}) = 0$$

$$\int_0^T \left\|\frac{du_\lambda}{dt}(t)\right\|^2 tdt + \int_0^T \left(A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t)\right)tdt = 0. \tag{2.26}$$

Morevover, since $A_\lambda^* = A_\lambda$, it holds:

$$\frac{d}{dt}(A_\lambda u_\lambda, u_\lambda) = \left(A_\lambda \frac{du_\lambda}{dt}, u_\lambda\right) + \left(A_\lambda u_\lambda, \frac{du_\lambda}{dt}\right) = 2\left(A_\lambda u_\lambda, \frac{du_\lambda}{dt}\right).$$

Therefore we take the second term of (2.26):

$$\int_0^T (A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t))tdt = \frac{1}{2}\int_0^T \frac{d}{dt}(A_\lambda u_\lambda, u_\lambda)tdt$$

and integrate by parts:

$$\frac{1}{2}\int_0^T \frac{d}{dt}(A_\lambda u_\lambda, u_\lambda)tdt = \frac{1}{2}(A_\lambda u_\lambda(T), u_\lambda(T))T - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt. \tag{2.27}$$

Thanks to lemma 2.2.2 we know that $t \to \frac{du_\lambda}{dt}(t)$ is decreasing, therefore:

$$\int_0^T \left\|\frac{du_\lambda}{dt}(t)\right\|^2 tdt \geq \left\|\frac{du_\lambda}{dt}(T)\right\|^2 \frac{T^2}{2} \tag{2.28}$$

Now we take (2.28) and (2.27) and substitute in (2.26), getting:

$$\left\|\frac{du_\lambda}{dt}(T)\right\|^2 \frac{T^2}{2} + \frac{1}{2}(A_\lambda u_\lambda(T), u_\lambda(T))T - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt \leq 0$$

thus:
$$\int_0^T (A_\lambda u_\lambda, u_\lambda)dt \geq \left\| \frac{du_\lambda}{dt}(T) \right\|^2 T^2 + (A_\lambda u_\lambda(T), u_\lambda(T))T$$
and substituting this last into (2.25):
$$\frac{1}{2}\|u_\lambda(T)\|^2 + \left\| \frac{du_\lambda}{dt}(T) \right\|^2 T^2 + (A_\lambda u_\lambda(T), u_\lambda(T))T \leq \frac{1}{2}\|u_0\|^2$$
we simplify the non negative terms, which yields to:
$$\left\| \frac{du_\lambda}{dt}(T) \right\|^2 \leq \frac{1}{T^2}\|u_0\|^2$$
which means that:
$$\left\| \frac{du_\lambda}{dt}(T) \right\| \leq \frac{1}{T}\|u_0\| \qquad \forall T > 0. \tag{2.29}$$
Now, in step 5 of the proof of theorem 2.2.1 we got that:
$$\frac{du_\lambda}{dt} \longrightarrow \frac{du}{dt} \qquad \text{as} \qquad \lambda \longrightarrow 0$$
therefore (2.23) is prooved.
Step 2.
Now we extend the result to the hypotheses of the theorem. Therefore, let $u_0 \in H$ and $\{u_o\}$ in $D(A)$ be a sequence, such that $u_{0n} \longrightarrow u_0$. Let u_n be the solution of the problem:
$$\begin{cases} \dfrac{du_n}{dt} + Au_n = 0 \quad \text{in } [0, +\infty[\\ u_n(0) = u_{0n} \end{cases}$$
Thanks to theorem 2.2.1 and to the result of the previous step, it holds:
$$\|u_n(t) - u_m(t)\| \leq \|u_{0n} - u_{0m}\| \qquad \forall m, n \qquad \forall t \geq 0$$
and
$$\left\| \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t) \right\| \leq \frac{1}{t}\|u_{0n} - u_{0m}\| \qquad \forall m, n \qquad t > 0.$$
Thus, both $u_n(t)$ and $\frac{du_n}{dt}(t)$ uniformly converge to the limits $u(t)$ and $\frac{du}{dt}(t)$, the former on $[0, +\infty[$ and the latter on $[\delta, +\infty[$ with $\delta > 0$. Therefore:
$$u \in C([0, +\infty[; H) \cap C^1(]0, +\infty[; H).$$
Being A maximal monotone, it is also closed. This means that $u(t) \in D(A)$ and verifies:
$$\frac{du}{dt} + Au = 0 \qquad in \qquad]0, +\infty[$$
which prooves the theorem. □

Chapter 3

Some applications

In this chapter we are going to use all the existence and uniqueness results of the previous chapters in order to solve some of the most relevant and famous evolution problems.
As already anticipated, a large amount of phenomena in nature can be modelled with partial differential equations, and the first part of this chapter will be devoted to the deduction, starting from some basic assumptions, of equations that come from physics or engineering. Then, each problem will be reformulated in order to be able to apply in a correct way the Hille-Yosida theorem. This will be enough to state whether there is a solution for a problem or not and, if it exists, whether it is the only one.
In this chapter we will extensively make use of integration by parts and Green formulas in $\mathbb{R}^n$ (see appendix B for a short summary).

3.1 Partial differential equations arising from physics

3.1.1 Diffusion equation: heat conduction

When dealing with thermodynamics, the main quantities which come into play are basically scalar fields like temperature, pressure and density of the considered substance. It is known from physics that *heat* is a form of *energy* which can be exchanged among gases, fluids and rigid bodies. The mechanisms of this exchange are generally classified into three categories: conduction, convection and radiation. We are now interested in deriving a mathematical model able to describe the conduction of heat in a rigid body, and we are going to follow the procedure described in [8].
Partial differential equations that govern fluid mechanics, thermodynamics, mechanics of the continua and other fields of physics are generally descending from balance laws. This is true also for the *heat equation*, which basically comes out from the balance of energy of a system. Therefore, we're going to separately define each quantity that plays a role, and then put them togethyer to derive the model.
Specifically, we will use the conservation law that states that the rate of variation of internal energy in an arbitrary unit volume V is equal to the net heat flux through the boundaries, plus the contribution of an external heat source (if present). To this aim, first of all let r be the rate of heat per unit mass that comes from outside the rigid body, and let's say the body has a

constant density ρ. If $e = e(\vec{x}, t)$[1] is the internal energy per unit mass, the total energy in V is:

$$\int_V \rho e d\vec{x}$$

and, assuming the differentiation under the integral sign to be allowed, its rate of variation is:

$$\frac{d}{dt}\int_V \rho e d\vec{x} = \int_V \rho \frac{\partial e}{\partial t} d\vec{x}.$$

Now let's introduce the heat flux vector $\vec{q}$. It contains information about the direction of the flux and its intensity (in terms of speed) per unit surface it passes through. Therefore, if $d\sigma$ is the unit element of the boundary ∂V with (external) unit normal vector $\hat{n}$, the entering heat flux through ∂V is given by:

$$-\int_{\partial V} \vec{q}\hat{n} d\sigma = -\int_V \mathrm{div}\vec{q} d\vec{x}$$

and the equality is given by the Gauss theorem (or divergence theorem, see appendix B). The contribution of the external source is instead:

$$\int_V \rho r d\vec{x}.$$

Thus, we are now able to write down the full energy balance law, which is:

$$\int_V \rho \frac{\partial e}{\partial t} = -\int_V \mathrm{div}\vec{q} d\vec{x} + \int_V \rho r d\vec{x}.$$

Since this equation has to be satisfied for any arbitrary volume V, the integrand function has to be null, and this yields:

$$\rho \frac{\partial e}{\partial t} = -\mathrm{div}\vec{q} + \rho r \tag{3.1}$$

which is the fundamental law of heat conduction. It is necessary, however, to define both $\vec{q}$ and e in a more explicit way, so as to derive a single scalar equation in the unknown T, which is the temperature of the rigid body. This goal is usually reached with the help of constitutive equations, in this case meaning the Fourier's law and the proportionality of the internal energy of the system with respect to its absolute temperature. Fourier's law states that:

$$\vec{q} = -k\nabla T$$

where $k > 0$ is the thermal conductivity of the body, and the minus sign represents the fact that the heat flows in the opposite direction of the temperature gradients.
Instead, the second constitutive law is:

$$e = c_v T$$

where c_v is the specific heat at constant volume of the material. Both c_v and k in most of the applications can be considered to be constant, since their variations are usually negligible. Finally, with these additions, (3.1) becomes:

$$\frac{\partial T}{\partial t} = \frac{k}{c_v \rho}\Delta T + \frac{r}{c_v} \tag{3.2}$$

[1]Here $\vec{x}$ refers to a generic point in space, therefore it is a vector of $\mathbb{R}^n$, with $n = 1,2$ or 3.

which is the diffusion equations with the coefficients $D = k/(c_v \rho)$ and $f = r/c_v$. We observe that, if we define $A = -\Delta$, the equation in this form is the non homogeneous version of equation (2.8), the latter being valid once no external heat sources are considered.[2]

3.1.2 Wave equation: oscillations of a string

Another common example of evolution problem is given by the wave equation, generally coming out when dealing with mechanics or electromagnetism. The usual deduction of the equation (found in [8]), in the mono-dimensional case, concerns the study of the transversal vibrations of a string, the latter satisfying the following requirements:

- Oscillations are small, in a way that these are much smaller than the string length
- Vibration is considered to be vertical
- The displacement of a single point can be considered to be depending on time and on the position along the string
- The string is perfectly flexible, meaning that it does not resist bending
- Friction is negligible.

Then, to derive the model it is necessary again to refer to balance laws. In particular, now we will consider the conservation[3] of mass and the balance of linear momentum. We define $\rho_0 = \rho_0(x)$ and $\rho = \rho(x,t)$ as the linear mass density respectively at rest and at time t. Then we consider the string element Δs corresponding to the interval $[x, x + \Delta x]$ (figure 3.1). The conservation of

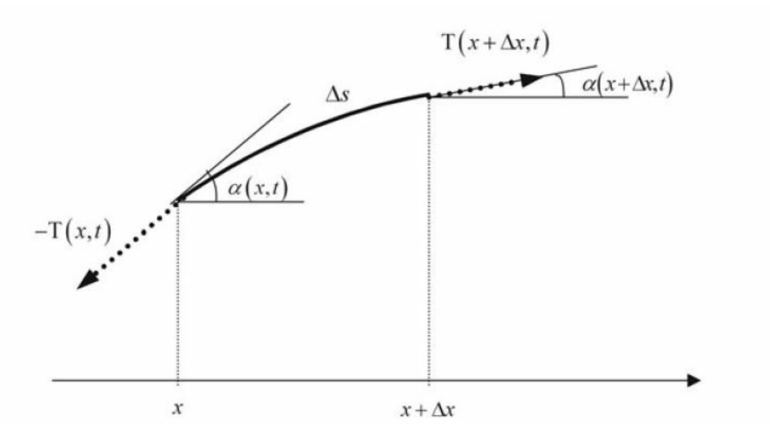

Figure 3.1: String segment with tensions at the extrema([8]).

mass states that:

$$\rho_0(x)\Delta x = \rho(x,t)\Delta s \tag{3.3}$$

[2]This means that the system is isolated only from a thermal point of view. In case also no external mass sources were present, then the system is said to be *isolated* in its very physical meaning.

[3]It is "conservation" rather than "balance" of mass, because the latter does not present source terms in its balance law.

whereas to derive the balance of momentum it is necessary to balance the forces. For the horizontal component, since the motion is assumed to be vertical, it holds:

$$\tau(x + \Delta x, t)\cos\alpha(x + \Delta x, t) - \tau(x, t)\cos\alpha(x, t) = 0.$$

We now divide by Δx and take the limit:

$$\lim_{\Delta x \to 0}\left[\frac{\tau(x + \Delta x, t)\cos\alpha(x + \Delta x, t) - \tau(x, t)\cos\alpha(x, t)}{\Delta x}\right] = 0$$

which is:

$$\frac{\partial}{\partial x}[\tau(x, t)\cos\alpha(x, t)] = 0$$

meaning that:

$$\tau(x, t)\cos\alpha(x, t) = \tau_0(t) \geq 0. \tag{3.4}$$

To make the full balance, we need to take into account all the forces which play a role in the system. To do this, we consider $f(x, t)$ as the resultant, per unit mass, of all external forces, being them either concentrated loads or distributed loads. Therefore, on a single string element acts a force equal to:

$$\rho(x, t)f(x, t)\Delta s = \rho_0(x)f(x, t)\Delta x$$

As concerning the vertical component of the tension, it holds:

$$\tau_v(x, t) = \tau(x, t)\sin\alpha(x, t) = \tau_0\tan\alpha(x, t) = \tau_0(x, t)u_x(x, t)$$

being u_x the tangent, point by point, of angle $\alpha(x, t)$. Therefore, the vertical component of the tension is:

$$\tau_v(x + \Delta x, t) - \tau_v(x, t) = \tau_0(t)[u_x(x + \Delta x, t) - u_x(x, t)].$$

Now it is sufficient to write down Newton's second law, being u_{tt} the acceleration of the points of the string. Therefore, by considering all the forces:

$$\rho_0(x)\Delta x u_{tt} = \tau_0(t)[u_x(x + \Delta x, t) - u_x(x, t)] + \rho_0(x)f(x, t)\Delta x$$

which is:

$$u_{tt} = \frac{\tau_0(t)}{\rho_0(x)}\left[\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x}\right] + f(x, t)$$

and for $\Delta x \longrightarrow 0$ we get:

$$u_{tt} - c^2 u_{xx} = f \tag{3.5}$$

which is the mono-dimensional wave equation, where:

$$c(x, t) = \sqrt{\frac{\tau_0(t)}{\rho_0(x)}}$$

is the wave propagation speed. In particular, the latter is constant when both $\tau_0(t)$ and $\rho_0(x)$ are constant, meaning that the string is perfectly elastic and homogeneous.
Again, if no external forces are present ($f = 0$) we are back to the homogeneous evolution problem, which can be approached with theorem 2.2.1 once defined the operator $A = -c^2\frac{\partial^2}{\partial x^2}$ [4].

[4] In $\mathbb{R}^n$ the equation assumes the same form, except for the operator, which becomes $A = -c^2\Delta$

3.1.3 Wave equation: free axial vibrations of rectilinear rods

When it comes to applied mechanics and vibration mechanics, a lot of systems are modeled in a discrete way, i.e. they are treated as they were constituted by a finite number of points or rigid bodies. In the majority of cases this approach is more than sufficient to describe the behaviour of mechanisms or structures, even if it is evidently a simplification.
A different approach consists in modeling the structures as *distributed parameters systems*, and this leads to partial differential equations. Here we present an interesting example of equation describing the axial vibrations of a rod[5], as it is deduced in [7].
Let's consider a constant cross-section rod clamped to an end, made by linear, elastic and isotropic material and let's suppose each section behaves as a rigid body. We are going to describe the axial displacement of each cross-section as a function u of the axial coordinate x and of time t.
According to figure 3.2, section S_1 starting from position x will move towards position $x + u$, while section S_2 will move from $x + dx$ to $x + dx + u + \frac{\partial u}{\partial x}dx$.

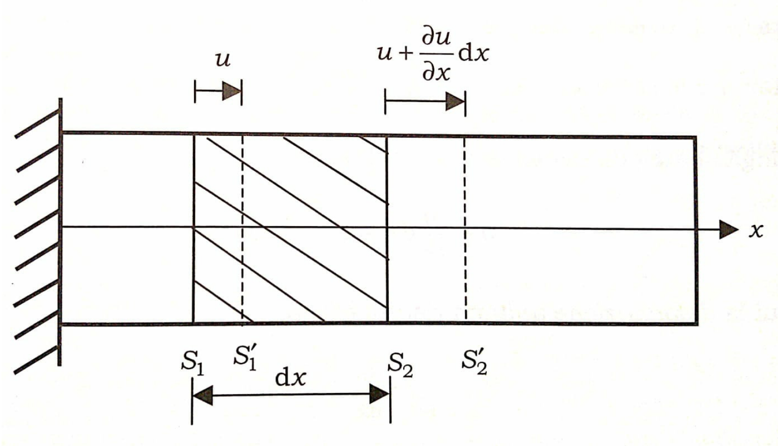

Figure 3.2: The clamped rod subjected to axial vibrations ([7])

Now let μ be the mass per unit length (linear mass density) of the rod. We balance the forces [6] acting on a rod element, namely the two axial reactions due to the motion plus the inertia, the latter being proportional to the linear mass density and to the second derivative of the displacement with respect to time (Newton's law, figure 3.3).
Thus, we get with the dynamic balance:

$$N + \frac{\partial N}{\partial x}dx - N - \mu dx\frac{\partial^2 u}{\partial t^2} = 0$$

which is:

$$\frac{\partial N}{\partial x} = \mu\frac{\partial^2 u}{\partial t^2}.$$

[5]Typically, in mechanics, "rods", "shafts", and "beams" are classified according to the way they are loaded. So, respectively, they undergo axial, torsional and flexural vibrations.

[6]This approach to derive the equations of motion of a system is called "Newtonian approach", and it is opposite to the Lagrangian, or "energetic" one, which has its origins in the calculus of variations ([9]).

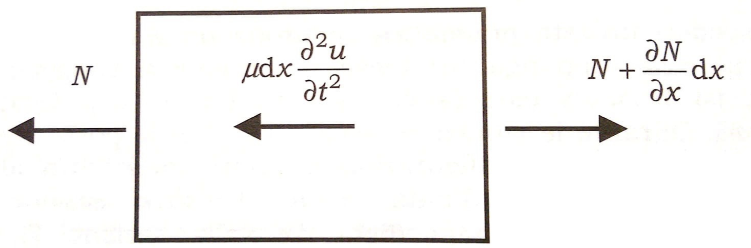

Figure 3.3: External forces on a single rod element due to axial vibrations. ([7])

Now, we want an equation in the unknown u, therefore we need, as in the previously described heat equation, constitutive equations expressing N in terms of u or its derivatives. In particular, being:

$$\epsilon = \frac{\partial u}{\partial x}$$

we hark back to Hooke's law, which states that[7]:

$$\sigma = E\epsilon$$

where E is the Young modulus of the considered material, which describes its behaviour in the linear range, and this in turn leads to:

$$N = AE\frac{\partial u}{\partial x}.$$

Taking the first derivative, supposing both A and E are constant:

$$\frac{\partial N}{\partial x} = AE\frac{\partial^2 u}{\partial x^2}$$

which gives, if substituted in the balance equation:

$$AE\frac{\partial^2 u}{\partial x^2} = \mu\frac{\partial^2 u}{\partial t^2}. \quad (3.6)$$

The deduced one is already a wave equation, but defining $\rho = \mu/a$ and $c = \sqrt{E/\rho}$ we get:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (3.7)$$

where c is the speed at which the waves are propagated along the rod.
We observe that the heat equation, as it was deduced, represents a diffusion phenomenon, thus only the first derivative of the unknown quantity with respect to time appears. Instead, when dealing with mechanics, due to Newton's law the derivatives of the quantity with respect to time are present up to the second order and the wave equation is obtained. This is just an example of how the mathematical description of a phenomenon is strongly affected by its physical origin.

[7]Here $\sigma = N/A$ is the axial force per unit area, also called "normal stress".

These two examples of partial differential equations (heat and wave equation) cover a large variety of physical models. It is worthy to outline that, in case stationary[8] processes are considered, the derivatives with respect to time can be neglected from the equations. Thus, starting from (3.5), we get:

$$\Delta u = f$$

which is the *Poisson equation*, that is the stationary part of the problem. This equation is obtained both from the heat equation and from the wave one, and in case, respectively, no external heat source or external load are considered, we obtain the *Laplace equation*:

$$\Delta u = 0.$$

These equations, with suitable boundary and initial conditions, have been deeply studied and there are many results concerning existence and uniqueness of the solutions under certain assumptions.
Besides the physical interpretation of the problems, partial differential equations are usually classified into *elliptic*, *parabolic* and *hyperbolic* ones. The heat equation is the classical example of parabolic problem, whereas the wave equation is a hyperbolic one. To examine in depth the mathematical aspects of this classification, see [10].

3.2 Heat equation

In this section, we are going to apply mainly the Hille-Yosida theorem in order to get results about existence, uniqueness and regularity of the solutions of the heat equation. First of all, it is necessary to define the domain, which is a bounded subset $\Omega \subset \mathbb{R}^n$, being Γ its boundary. The idea is to impose equation (3.2) on this domain defining the unknown function u as its temperature point by point, and imposing boundary conditions which can be:

- *Dirichlet boundary conditions*: the value of the unknown u is imposed on Γ, meaning that we suppose to know the exact value of the temperature of the body on its boundaries. This could be the case in which the body is in contact with a fixed temperature heat source, or thermostat.

- *Neumann boundary conditions*: here, instead of the value of u, it is imposed the value of the normal derivative of u on Γ, which represents the heat flux exiting from Ω.

- *Robin boundary conditions*: this is the mixed case, i.e. we impose a value of the temperature at the boundary plus a certain amount of heat flux.

All these b.c. (boundary conditions) can be *homogeneous* or not, depending on whether the assumed value for u (or $\frac{\partial u}{\partial n}$) is null or not.
Then, since we are interested in an evolution problem, we need to define an interval for the variable t, which is $]0, +\infty[$. Therefore, the global domain will be a cilynder[9] Q, defined as:

$$Q = \Omega \times]0, +\infty[$$

[8]For instance, in fluid dynamics a stationary process could be a flux in a pipe, where the velocity profiles do not change in time.

[9]We stress out the fact that if $\Omega \subset \mathbb{R}^2$, Q is effectively a 3-dimensional cylinder, whereas if $n > 2$, it is a $n+1$-dimensional one, but the meaning of the equation remains unchanged.

and with lateral boundary:

$$\Sigma = \Gamma \times]0, +\infty[.$$

Finally initial conditions are imposed, meaning that we are supposed to know the initial value of the temperature of the body.
We are going to use also the following:

Lemma 3.2.1. *For all $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$ of the equation*

$$-\Delta u + u = f \qquad in \quad \Omega$$

with any of the previously presented boundary conditions.

Proof. To proof this lemma it is sufficient to apply Lax-Milgram Theorem 1.2.2 defining suitable Hilbert spaces and bilinear forms. In particular, for the problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases} \tag{3.8}$$

we use $H = H_0^1(\Omega)$. Given $\varphi \in H_0^1(\Omega)$, we multiply (3.8) by φ and integrate on Ω, obtaining:

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in H_0^1(\Omega).$$

The latter always has a unique solution, given by Lax-Milgram theorem with the bilinear form:

$$a(u, \varphi) = \int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi$$

and the linear form:

$$\Phi(\varphi) = \int_\Omega f\varphi.$$

Instead, when considering the problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

it is sufficient to consider the Hilbert space $H^1(\Omega)$ and the same forms as before. □

Another result concerning regularity of problem (3.8) is the following:

Theorem 3.2.2 (Regularity for Dirichlet problem)**.** *Let Ω be an open set, of class C^2, with bounded Γ, or $\Omega = \mathbb{R}_+^N$ [10]. Let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ satisfying:*

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in H_0^1(\Omega).$$

Then $u \in H^2(\Omega)$ and $\exists C = C(\Omega)$ such that $\|u\|_{H^2} \le C \|f\|_{L^2}$. Moreover, if Ω is C^{m+2} and if $f \in H^m(\Omega)$, then

$$u \in H^{m+2}(\Omega) \quad with \quad \|u\|_{H^{m+2}(\Omega)} \le c \|f\|_{H^m(\Omega)}$$

in particular, if $m > N/2$, then $u \in C^2(\overline{\Omega})$. Finally, if Ω is C^∞ and if $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$

[10] *Here with $\mathbb{R}_+^n$ we mean $\mathbb{R}^{n-1} \times [0, +\infty[$, where the last dimension could reasonably represent time.*

The theorem states that a certain regularity of the solution is achieved when Dirichlet boundary conditions are considered. Instead of proving it, we just give a short and simple example in the monodimensional case of problem (3.8).
Let $\Omega = [0,1]$, $f \in L^2(0,1)$ and $u \in H_0^1(0,1)$.Thus, the weak formulation of (3.8) is:

$$u \in H_0^1(0,1) \ : \ \int_0^1 u'v' + \int_0^1 uv = \int_0^1 fv \quad \forall v \in H_0^1(0,1).$$

Since we are considering the monodimensional case, the solution u is also continuous on $(0,1)$ and the boundary conditions are satisfied in a classical way. It is possible to write the same problem in the following form:

$$u \in H_0^1(0,1) \ : \ \int_0^1 u'v' = \int_0^1 (f-u)v \quad \forall v \in H_0^1(0,1)$$

Being $u \in H_0^1(0,1)$, its first derivative is still in $L^2(0,1)$, meaning that it can be derived (in general as a distribution), therefore the last equation says that:

$$-(u'', v)_{L^2(0,1)} = (f-u, v)_{L^2(0,1)} \quad \forall v \in \mathcal{D}(0,1)$$ [11]

and this leads to:

$$u'' = u - f \quad \text{in } \mathcal{D}'(0,1)$$

where the right hand side belongs to $L^2(0,1)$. This in turn implies that the second derivative of u is in $L^2(0,1)$, therefore $u \in H^2(0,1) \subset C^1([0,1])$. Moreover, if $f \in C([0,1])$, then $u \in C^2([0,1])$ and it is also a classical solution. For a complete proof of the theorem, including the continuous dependence of the solution from data, see [2].
Now it comes to the resolution of the heat equation by means of the tools we presented.

Theorem 3.2.3. *Let $u_0 \in L^2(\Omega)$. Then there exists a unique function $u(x,t) : \overline{\Omega} \times [0,+\infty[\longrightarrow \mathbb{R}$ satisfying:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{3.9}$$

and

$$u \in C([0,+\infty[, L^2(\Omega)) \cap C(]0,+\infty[; H^2(\Omega) \cap H_0^1(\Omega)) \tag{3.10}$$

$$u \in C^1(]0,+\infty[; L^2(\Omega)). \tag{3.11}$$

Moreover

$$u \in C^\infty(\overline{\Omega} \times [\epsilon, +\infty[) \qquad \forall \epsilon > 0.$$

Finally $u \in L^2((0,+\infty), H_0^1(\Omega))$ and it holds:

$$\frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} + \int_0^T \|\nabla u(t)\|^2_{L^2(\Omega)} \, dt = \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} \quad \forall T > 0. \tag{3.12}$$

[11] We recall that $\mathcal{D}(\Omega) = \{\phi \in C^\infty(\Omega) : \text{supp}\phi = K \subset \Omega \text{ compact}\}$, whereas $\mathcal{D}'(\Omega)$ is its dual space, i.e. the set of all linear and bounded functionals on $\mathcal{D}(\Omega)$.

Proof. In order to apply Hille-Yosida Theorem, we consider the Hilbert space $L^2(\Omega)$, and define the unbounded operator $A : D(A) \subset H \longrightarrow H$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = -\Delta u$. Now, it is sufficient to verify the hypotheses of the theorem.

- A is monotone:
$$(Au, u)_{L^2(\Omega)} = \int_\Omega (-\Delta u)u = \int_\Omega |\nabla u|^2 \geq 0$$
- A is maximal monotone. Indeed, thanks to lemma 3.2.1,
$$\forall f \in L^2(\Omega), \exists!\ u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ solution of}$$
$$u - \Delta u = f$$
or, in other words, $R(A + I) = H = L^2(\Omega)$.
- A is self-adjoint. Since it is maximal monotone, it is sufficient to verify its simmetry, that is:
$$(Au, v)_{L^2(\Omega)} = \int_\Omega (-\Delta u)v = \int_\Omega \nabla u \nabla v$$
and
$$(u, Av)_{L^2(\Omega)} = \int_\Omega u(-\Delta v) = \int_\Omega \nabla u \nabla v$$

From theorem 3.2.2 it comes that $D(A^l) \subset H^{2l}(\Omega)$ with continuous embedding, specifically:
$$D(A^l) = \{u \in H^{2l}(\Omega) : u = \Delta u = \ldots = \Delta^{l-1}u = 0 \text{ on } \Gamma\}.$$
Thanks to theorem 2.4.1 we know that
$$u \in C^k(]0, +\infty[, D(A^l)) \quad \forall k, \forall l$$
therefore
$$u \in C^k(]0, +\infty[, H^{2l}(\Omega)) \quad \forall k, \forall l$$
and finally this means that:
$$u \in C^k(]0, +\infty[, C^k(\overline{\Omega})) \quad \forall k, \forall l$$
where the last equation comes from the Sobolev embedding theorems (see [2]).
To prove 3.12, we consider the function
$$\varphi(t) = \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)}$$
which, thanks to (3.11) is C^1 on $]0, +\infty[$ and:
$$\varphi'(t) = \left(u(t), \frac{du}{dt}(t)\right)_{L^2(\Omega)} = (u, \Delta u)_{L^2(\Omega)} = -\int_\Omega |\nabla u|^2 = -\|\nabla u(t)\|^2_{L^2(\Omega)}.$$
Letting $0 < \epsilon < T < \infty$ it holds:
$$\varphi(T) - \varphi(\epsilon) = \int_\epsilon^T \varphi'(t)dt = -\int_\epsilon^T \|\nabla u(t)\|^2_{L^2(\Omega)} dt.$$
The last expression becomes (3.12) as $\epsilon \longrightarrow 0$, so that $\varphi(\epsilon) \longrightarrow \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$. □

We remark that the heat equation has a strong regularizing effect, i.e.:

$$u \in C^\infty(\overline{\Omega} \times [\epsilon, +\infty[) \qquad \forall \epsilon > 0,$$

even if u_0 is only an L^2 function.
Unlike in the previous chapter, as a consequence of additional hypotheses on the initial datum u_0, the solution becomes more regular only in a neighbourhood of $t = 0$, whereas in $t = 0$ it can still be discontinuous. To show these properties, we consider three different cases, each one respecting the hypotheses of the Hille-Yosida theorem.

Theorem 3.2.4. *Let $u_0 \in H_0^1(\Omega)$, then the solution of (3.9) satisfies*

$$u \in C([0, +\infty[; H_0^1(\Omega)) \cap L^2((0, +\infty); H^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2((0, +\infty); L^2(\Omega)).$$

Moreover it holds

$$\int_0^T \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \|\nabla u(T)\|_{L^2(\Omega)}^2 = \frac{1}{2} \|\nabla u_0\|_{L^2(\Omega)}^2 .$$

Proof. In this case the Hilbert space we consider is $H_1 = H_0^1(\Omega)$ with the scalar product:

$$(u, v)_{H_1} = \int_\Omega \nabla u \nabla v + \int_\Omega uv.$$

The unbouded operator is $A_1 : D(A_1) \subset H_1 \longrightarrow H_1$ defined as

$$\begin{cases} D(A_1) = \{u \in H^3(\Omega) \cap H_0^1(\Omega) : \; \Delta u \in H_0^1(\Omega)\} \\ A_1 u = -\Delta u. \end{cases}$$

Let's verify the properties of operator A_1:

- A_1 is monotone:

$$(A_1 u, u)_{H_1} = \int_\Omega \nabla(-\Delta u) \nabla u + \int_\Omega (-\Delta u) u$$
$$= \int_\Omega \Delta u \Delta u + \int_\Omega |\nabla u|^2 = \int_\Omega |\Delta u|^2 + \int_\Omega |\nabla u|^2 \geq 0$$

- A_1 is maximal monotone: thanks to lemma 3.2.1, $\forall f \in H^1(\Omega)$ it exists a unique $u \in H^3(\Omega) \cap H_0^1(\Omega)$ solution of $u - \Delta u = f$. If $f \in H_0^1(\Omega)$, then

$$\Delta u \in H_0^1(\Omega)$$

  therefore

$$u \in D(A_1).$$

- A_1 is symmetric and therefore self-adjoint:

$$(u, A_1 v)_{H_1} = \int_\Omega \nabla u \nabla(-\Delta v) + \int_\Omega u(-\Delta v) = \int_\Omega \Delta u \Delta v + \int_\Omega \nabla u \nabla v$$
$$(A_1 u, v)_{H_1} = \int_\Omega \nabla(-\Delta u) \nabla v + \int (-\Delta u) v = \int_\Omega \Delta v \Delta u + \int_\Omega \nabla u \nabla v.$$

Then, thanks to theorem 2.4.1, when $u_0 \in H_0^1(\Omega)$ we have that $u \in C([0,+\infty[, H_0^1(\Omega))$. Again, to prove Theorem 3.2.4 we consider the function

$$\varphi(t) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)}$$

which is C^∞ on $]0,\infty[$ and satisfies:

$$\varphi'(t) = \left(\nabla u(t), \nabla \frac{du}{dt}(t)\right)_{L^2(\Omega)} = \left(-\Delta u(t), \frac{du}{dt}(t)\right)_{L^2(\Omega)} = -\left\|\frac{du}{dt}(t)\right\|^2_{L^2(\Omega)}$$

the last passage being justified by the heat equation itself. When $0 < \epsilon < T < \infty$, it holds:

$$\varphi(T) - \varphi(\epsilon) = -\int_\epsilon^T \left\|\frac{du}{dt}(t)\right\|^2_{L^2(\Omega)} dt$$

that is:

$$\varphi(T) - \varphi(\epsilon) + \int_\epsilon^T \left\|\frac{du}{dt}(t)\right\|^2_{L^2(\Omega)} dt = 0.$$

As before, it is sufficient to send $\epsilon \longrightarrow 0$. □

Theorem 3.2.5. *Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then it holds*

$$u \in C([0,+\infty[, H^2(\Omega) \cap L^2(\Omega)) \cap L^2((0,+\infty), H^3(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2((0,+\infty), H^1(\Omega)).$$

Proof. We now consider the Hilbert space $H_2 = H^2(\Omega) \cap H_0^1(\Omega)$ with the scalar product

$$(u,v)_{H_2} = (\Delta u, \Delta v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)}.$$

The unbouded operator is $A_2 : D(A_2) \subset H_2 \longrightarrow H_2$ defined as:

$$\begin{cases} D(A_2) = \{u \in H^4(\Omega) : u \in H_0^1(\Omega) \text{ and } \Delta u \in H_0^1(\Omega)\} \\ A_2 u = -\Delta u \end{cases}$$

which is maximal monotone and self adjoint in H_2.
We can apply theorem 2.4.1 to A_2 in H_2.
Now we consider the function:

$$\varphi(t) = \frac{1}{2}\|\Delta u(t)\|^2_{L^2(\Omega)}$$

which is C^∞ on $]0,\infty[$ and it holds:

$$\varphi'(t) = \left(\Delta u(t), \Delta \frac{du}{dt}(t)\right)_{L^2(\Omega)} = (\Delta u(t), \Delta^2 u(t))_{L^2(\Omega)} = -\|\nabla \Delta u(t)\|^2_{L^2(\Omega)}.$$

Given $0 < \epsilon < T < \infty$, we integrate and get:

$$\frac{1}{2}\|\Delta u(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\Delta u(\epsilon)\|^2_{L^2(\Omega)} = -\int_\epsilon^T \|\nabla \Delta u(t)\|^2_{L^2(\Omega)} dt$$

that is:

$$\frac{1}{2}\|\Delta u(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\Delta u(\epsilon)\|^2_{L^2(\Omega)} + \int_\epsilon^T \|\nabla \Delta u(t)\|^2_{L^2(\Omega)} dt = 0.$$

Considering the limit for $\epsilon \longrightarrow 0$, it turns out that $u \in L^2(0,\infty, H^3(\Omega))$ and, thanks to the heat equation itself, $\frac{du}{dt} \in L^2((0,+\infty), H^1(\Omega))$. □

3.3 Wave equation

This section is devoted to the solution of the wave equation with $\Omega \subset \mathbb{R}^n$, which is a generalization of the ones previously deduced starting from physical considerations. Again, the Hille-Yosida theorem turns out to be very effective when dealing with this kind of problems.
Let $\Omega \subset \mathbb{R}^n$ be an open set and let Γ be its boundary. Like in the previous section, we define:

$$Q = \Omega \times]0, \infty[$$

and

$$\Sigma = \Gamma \times]0, \infty[.$$

Now we are ready to state and proof the following:

Theorem 3.3.1. *Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then there exists a unique solution of the problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } Q \\ u = 0 & \text{on } \Gamma \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{3.13}$$

with

$$u \in C([0, \infty[; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty[; H_0^1(\Omega)) \cap C^2([0, \infty[; L^2(\Omega)).$$

Moreover it holds $\forall t \geq 0$

$$\left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 = \|v_0\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 . \tag{3.14}$$

We observe that (3.14) represents the conservation of mechanical energy if, for example, the unknown $u(x, t)$ represents the position of a massive particle.

Proof. To proof this theorem we need first of all to bring back the problem to a first order one, in order to be able to properly apply the Hille-Yosida theory. To do this, we rewrite (3.13) in the following way:

$$\begin{cases} v = \dfrac{\partial u}{\partial t} & \text{in } Q \\ \dfrac{\partial v}{\partial t} - \Delta u = 0 & \text{in } Q \end{cases} \tag{3.15}$$

and defining $U = \begin{pmatrix} u \\ v \end{pmatrix}$ the problem becomes:

$$\frac{dU}{dt} + AU = 0$$

with

$$AU = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} U = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ Deltau \end{pmatrix} . \tag{3.16}$$

Now we consider the Hilbert space $H = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the scalar product

$$(U_1, U_2)_H = \int_\Omega \nabla u_1 \nabla u_2 + \int_\Omega u_1 u_2 + \int_\Omega v_1 v_2$$

where we considered

$$U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \quad , \quad U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

with $u_1, u_2 \in H_0^1(\Omega)$ and $v_1, v_2 \in L^2(\Omega)$ so that $U_1, U_2 \in H$. The unbounded operator is defined like in (3.16), and its domain is:

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \cap H_0^1(\Omega)$$

Now we verify that the operator $A + I$ is maximal monotone.

- $A + I$ is monotone. Indeed, given $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$:

  $$(AU, U)_H + (U, U)_H = -\int_\Omega \nabla v \nabla u - \int_\Omega uv + \int_\Omega (-\Delta u)v + \int_\Omega u^2 + \int_\Omega |\nabla u|^2 + \int_\Omega v^2$$

  The first term cancels out with the third one, due to the second Green's identity, thus:

  $$-\int_\Omega uv + \int_\Omega u^2 + \int_\Omega v^2 + \int_\Omega |\nabla u|^2 \geq 0.$$

  The inequality is immediately verified by evaluating the discriminant of the quadratic equation:

  $$-ab + a^2 + b^2 + c \geq 0 \quad , \quad \text{with} \quad c \geq 0.$$

- $A + I$ is maximal monotone. It is sufficient to prove that $A + 2I$ is surjective. Given $F = \begin{pmatrix} f \\ g \end{pmatrix} \in H$, the system that has to be considered is:

  $$\begin{cases} -v + 2u = f & \text{in } \Omega \\ -\Delta u + 2v = g & \text{in } \Omega \end{cases}$$

  with $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$, i.e. $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$. We multiply by 2 the first equation, then sum with the second one and get:

  $$-\Delta u + 4u = 2f + g.$$

  The latter, due to lemma 3.2.1, has a unique solution u. Then v is obtained by difference: $v = 2u - f$.

Now we can apply Hille-Yosida theorem, and therefore we can state that the problem:

$$\begin{cases} \dfrac{dU}{dt} + AU = 0 & \text{in } [0, +\infty[\\ U(0) = U_0 \end{cases} \tag{3.17}$$

has the unique solution

$$U \in C^1([0, +\infty[, H) \cap C([0, +\infty[, D(A)).$$

To prove the conservation of energy, we multiply the wave equation by $\frac{\partial u}{\partial t}$ and integrate on Ω. We get:

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}dx = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left\|\frac{\partial u}{\partial t}(x,t)\right\|_H^2 dx$$

and

$$\int_\Omega (-\Delta u)\frac{\partial u}{\partial t}dx = \int_\Omega \nabla u \frac{\partial}{\partial t}(\nabla u)dx = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla u|^2 dx.$$

Finally, integrating on $[0,t]$, we get (3.14). □

3.4 Coupled sound and heat flow

In this section we are going to make use of the Hille-Yosida Theorem when dealing with equations deriving from fluid-dynamics. In particular, when considering infinitesimal motions of a compressible fluid, energy can be transfered not only thanks to the motion of the fluid itself, but also due to thermal conduction. The system we are going to consider comes out from the linearized equations for conservation of mass, momentum and energy of the fluid, i.e. the linearized Navier-Stokes equations.

Let $\Omega \subset \mathbb{R}^n$ be a fixed domain, with $\Gamma = \partial\Omega$ bounded and smooth. The linearized equations, as formulated in [11] are:

$$\begin{cases} \dfrac{\partial w}{\partial t} = c\, div\vec{u} \\ \dfrac{\partial u}{\partial t} = c\nabla w - c\nabla e \\ \dfrac{\partial e}{\partial t} = \sigma\Delta e - (\gamma-1)c\nabla\cdot\vec{u} \end{cases}$$

where c is the isothermal sound speed, $\gamma > 1$ is the ratio of specific heats and $\sigma > 0$ is the thermal conductivity. We assume $e = w = 0$ on Γ, $\forall t \geq 0$. By taking the divergence of the second equation, like in [6], we get a system composed by two equations only. If we consider a more general problem and we add proper initial conditions we get:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = c^2\Delta w - c^2\Delta e + m^2 w & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial e}{\partial t} = \sigma\Delta e - (\gamma-1)\dfrac{\partial w}{\partial t} & \text{in } \Omega\times(0,\infty), \\ e = w = 0 & \text{on } \Gamma\times[0,\infty), \\ w(x,0) = w_0(x), \dfrac{\partial w}{\partial t}(x,0) = v_0(x), e(x,0) = e_0(x) & \text{on } \Omega \end{cases} \tag{3.18}$$

with $\sigma > 0$, $\gamma > 1$, $c > 0$ and $m \in \mathbb{R}$.

Now we can apply the Hille-Yosida Theorem, like in [12]. First, we will state the following

Theorem 3.4.1 (Existence and uniqueness). *Suppose $w_0, e_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then there exists a unique solution (w,e) of (3.18) satisfying:*

$$w \in C([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)),$$

$$e \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)).$$

Furthermore, for some $\alpha > 0$ the following estimates hold:

$$\begin{aligned} &\|w(t)\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|v(t)\|_{L^2(\Omega)}^2 + \frac{1}{\gamma-1}\|e(t)\|_{H^1(\Omega)}^2 \\ &\leq e^{2\alpha t}\left(\|w_0\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|v_0\|_{L^2(\Omega)}^2 + \frac{1}{\gamma-1}\|e_0\|_{H^1(\Omega)}^2\right) \quad \forall t \geq 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\|v(t)\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\left\|c^2\Delta w(t) + m^2 w(t) - c^2\Delta e(t)\right\|_{L^2(\Omega)}^2 \\ &\qquad + \frac{1}{\gamma-1}\|(\gamma-1)v(t) - \sigma\Delta e(t)\|_{H^1(\Omega)}^2 \\ &\leq e^{2\alpha t}\Bigg(\|v_0\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\left\|c^2\Delta w_0 + m^2 w_0 - c^2\Delta e_0\right\|_{L^2(\Omega)}^2 \\ &\qquad + \frac{1}{\gamma-1}\|(\gamma-1)v_0 + \sigma\Delta e_0\|_{H^1(\Omega)}^2\Bigg) \quad \forall t \geq 0. \end{aligned} \tag{3.20}$$

Proof. To prove the theorem, we have to write the system as a system of first order equations, so as to able to identify a suitable operator A, that is:

$$\begin{cases} \dfrac{\partial w}{\partial t} = v \\ \dfrac{\partial v}{\partial t} = c^2\Delta w + m^2 w - c^2\Delta e \\ \dfrac{\partial e}{\partial t} = \sigma\Delta e - (\gamma-1)v. \end{cases} \tag{3.21}$$

We now define a new variable:

$$U = \begin{pmatrix} w \\ v \\ e \end{pmatrix}$$

thanks to which we can write the system above as:

$$\begin{cases} \dfrac{dU}{dt} + AU = 0 \quad \text{on } (0,+\infty) \\ U(0) = U_0 \end{cases} \tag{3.22}$$

where the operator $A : D(A) \subset H \longrightarrow H$ is defined as follows:

$$A = \begin{pmatrix} 0 & -I & 0 \\ -(c^2\Delta + m^2 I) & 0 & c^2\Delta \\ 0 & (\gamma-1)I & -\sigma\Delta \end{pmatrix}$$

with

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$$

and

$$H = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega).$$

The space H is a Hilbert space with scalar product: if $U_i = \begin{pmatrix} w_i \\ v_i \\ e_i \end{pmatrix}$ then we get

$$\begin{aligned}(U_1, U_2) = \int_\Omega \nabla w_1 \cdot \nabla w_2 + \int_\Omega w_1 w_2 + \frac{1}{c^2}\int_\Omega v_1 v_2 + \frac{1}{\gamma-1}\int_\Omega \nabla e_1 \cdot \nabla e_2 + \frac{1}{\gamma-1}\int_\Omega e_1 e_2 \\ = (w_1, w_2)_{H^1(\Omega)} + \frac{1}{c^2}(v_1, v_2)_{L^2(\Omega)} + \frac{1}{\gamma-1}(e_1, e_2)_{H^1(\Omega)}\end{aligned}$$

and induced norm:

$$\|U\|_H^2 = \|w\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|v\|_{L^2(\Omega)}^2 + \frac{1}{\gamma-1}\|e\|_{H^1(\Omega)}^2 .$$

Now that the space is correctly defined, it is sufficient to verify the hypotheses of the Hille-Yosida Theorem in order to show the existence and uniqueness of problem (3.22), i.e. of (3.18).

- $A + \alpha I$ is a monotone operator for any positive constant

$$\alpha > \max\left\{|m|, \left(\frac{1}{2} + \frac{m^2}{2c^2}\right), c^2\left(\frac{1}{2} + \frac{m^2}{2c^2}\right), \frac{\gamma-1}{2}\right\}.$$

Indeed, we have:

$$\begin{aligned}(AU, U) &= \int_\Omega \nabla(-v)\cdot\nabla w + \int_\Omega -vw + \frac{1}{c^2}\int_\Omega (-c^2\Delta w - m^2 w + c^2\Delta e)v \\ &+ \frac{1}{\gamma-1}\int_\Omega \nabla((\gamma-1)v - \sigma\Delta e)\cdot\nabla e + \frac{1}{\gamma-1}\int_\Omega ((\gamma-1)v - \sigma\Delta e)e \\ &= \int_\Omega v\Delta w - \int_\Omega vw + \frac{1}{c^2}\int_\Omega (-c^2\Delta w - m^2 w + c^2\Delta e)v \\ &- \frac{1}{\gamma-1}\int_\Omega ((\gamma-1)v - \sigma\Delta e)\Delta e + \frac{1}{\gamma-1}\int_\Omega ((\gamma-1)v - \sigma\Delta e)e \\ &= -\left(1 + \frac{m^2}{c^2}\right)\int_\Omega vw + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 + \int_\Omega ve + \frac{\sigma}{\gamma-1}\int_\Omega e\Delta e \\ &= -\left(1 + \frac{m^2}{c^2}\right)\int_\Omega vw + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 + \int_\Omega ve + \frac{\sigma}{\gamma-1}\int_\Omega |\nabla e|^2 \\ &\geq -\left(1 + \frac{m^2}{c^2}\right)\int_\Omega vw + \int_\Omega ve.\end{aligned}$$

and since

$$vw \leq |v||w| \leq \frac{v^2 + w^2}{2} \quad \text{and} \quad ve \geq -|v||e| \geq -\frac{v^2 + e^2}{2}$$

we get:

$$\begin{aligned}(AU, U) &\geq -\left(\frac{1}{2} + \frac{m^2}{2c^2}\right)\int_\Omega (v^2 + w^2) - \frac{1}{2}\int_\Omega (v^2 + e^2) \\ &= -\left(1 + \frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2} + \frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2.\end{aligned}$$

Since now

$$\alpha > max\left\{|m|, \left(\frac{1}{2} + \frac{m^2}{2c^2}\right), c^2\left(\frac{1}{2} + \frac{m^2}{2c^2}\right), \frac{\gamma-1}{2}\right\},$$

we get:

$$\begin{aligned}((A+\alpha I)U,U) &\geq -\left(1+\frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2 \\ &+\alpha\left(\int_\Omega |\nabla w|^2 + \int_\Omega w^2 + \frac{1}{c^2}\int_\Omega v^2 + \frac{1}{\gamma-1}\int_\Omega |\nabla e|^2 + \frac{1}{\gamma-1}\int_\Omega e^2\right) \\ = \left[\alpha - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\right]\int_\Omega w^2 &+ \left[\frac{\alpha}{c^2} - \left(1+\frac{m^2}{2c^2}\right)\right]\int_\Omega v^2 + \left[\frac{\alpha}{\gamma-1} - \frac{1}{2}\right]\int_\Omega e^2 \geq 0\end{aligned}$$

and therefore the first property of A is prooved.

- For a positive constant $\beta > |m|, R(A+\beta I) = H$, i.e.

$$\forall\, F \in H\ \exists\, U \in D(A) : (A+\beta I)U = F.$$

As a proof, we take a generic $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ and search for $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A)$ such that:

$$\begin{cases} -v + \beta w = f \\ -c^2\Delta w - m^2 w + c^2\Delta e + \beta v = g \\ (\gamma-1)v - \sigma\Delta e + \beta e = h. \end{cases}$$

We state $v = \beta w - f$, obtaining:

$$\begin{cases} -\Delta w + \left(\dfrac{\beta^2 - m^2}{c^2}\right) w + \Delta e = \dfrac{\beta f}{c^2} + \dfrac{g}{c^2} \\ -\Delta e + \dfrac{\beta}{\sigma} e + \dfrac{\beta(\gamma-1)}{\sigma} w = \dfrac{\gamma-1}{\sigma} f + \dfrac{h}{\sigma}. \end{cases} \tag{3.23}$$

We now want to make use of lemma 3.2.1, therefore we try to express 3.23 in the form $-\Delta\phi + k\phi = p$, where ϕ is a linear combination of w and e. Then, thanks to the lemma, we have a unique solution of the system and the property of A is proved.
We set

$$\gamma_1 = \frac{\beta^2 - m^2}{c^2}, \gamma_2 = \frac{\beta}{\sigma}, \gamma_3 = \frac{\beta(\gamma-1)}{\sigma}, f_1 = \frac{\beta}{c^2} f + \frac{g}{c^2}, f_2 = \frac{\gamma-1}{\sigma} f + \frac{h}{\sigma}$$

and (3.23) becomes:

$$\begin{cases} -\Delta w + \gamma_1 w + \Delta e = f_1 \\ -\Delta e + \gamma_2 e + \gamma_3 w = f_2. \end{cases} \tag{3.24}$$

We sum the second equation with the first multiplied by a and it yields:

$$-\Delta(aw + (1-a)e) + [(a\gamma_1 + \gamma_3)w + \gamma_2 e] = af_1 + f_2. \tag{3.25}$$

We search now for two positive constants a and k such that:

$$\phi = aw + (1-a)e \quad \text{and} \quad k\phi = (a\gamma_1 + \gamma_3)w + \gamma_2 e.$$

Actually these constants exists, and are given by:

$$a_1 = \frac{-(\gamma_2+\gamma_3-\gamma_1)+\sqrt{(\gamma_2+\gamma_3-\gamma_1)^2+4\gamma_1\gamma_3}}{2\gamma_1}$$
$$a_2 = \frac{-(\gamma_2+\gamma_3-\gamma_1)-\sqrt{(\gamma_2+\gamma_3-\gamma_1)^2+4\gamma_1\gamma_3}}{2\gamma_1}$$

$$k_1 = \frac{a_1\gamma_1+\gamma_3}{a_1}$$
$$k_2 = \frac{a_2\gamma_1+\gamma_3}{a_2}$$

(for the detailed derivation, see [6]).
Therefore, thanks to Lemma 3.2.1, equations

$$-\Delta\phi_1 + k_1\phi_1 = a_1 f_1 + f_2$$

$$-\Delta\phi_2 + k_2\phi_2 = a_2 f_1 + f_2$$

have unique solutions. In addition, thanks to (3.25), also $a_1 w+(1-a_1)e$ and $a_2 w+(1-a_2)e$ satisfy those equations, and due to uniqueness it holds:

$$\phi_1 = a_1 w + (1-a_1)e$$

$$\phi_2 = a_2 w + (1-a_2)e$$

leading to:

$$\begin{aligned} w &= \frac{(1-a_2)\phi_1-(1-a_1)\phi_2}{(1-a_2)a_1-(1-a_1)a_2} \\ e &= \frac{a_2\phi_1-a_1\phi_2}{(1-a_1)a_2-(1-a_2)a_1}. \end{aligned} \tag{3.26}$$

Thus, $w, e \in H^2(\Omega)\cap H^1_0(\Omega)$ and $v = \beta w - f$ so that $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A)$ with $(A+\beta I)U = F$.

Thanks to the previous considerations we are now able to define the problem:

$$\begin{cases} \dfrac{dV}{dt} + A_1 v = 0 & \text{on } [0,+\infty), \\ V(0) = U(0) \in D(A_1) = D(A) \end{cases} \tag{3.27}$$

where $A_1 = A + \alpha I$.
If the constant α respects the above conditions that make operator A_1 maximal monotone, then we can apply Hille-Yosida Theorem and state that (3.27) has a unique solution

$$V \in C^1([0,+\infty);H) \cap C([0,+\infty);D(A)).$$

To derive the estimates (3.19) and (3.20) , we need to recall what was proved during the proof of Theorem 2.2.1 in chapter 2, namely:

Remark 3.4.1. *If w satisfies*

$$\frac{dw}{dt} + A_\lambda w = 0$$

than $t \longmapsto |w(t)|$ is non-increasing, and moreover it holds:

$$\frac{d}{dt}\left(\frac{dw}{dt}\right) + A_\lambda\left(\frac{dw}{dt}\right) = 0$$

$$\frac{d}{dt}\left(\frac{dw}{dt} + \alpha w\right) + A_\lambda\left(\frac{dw}{dt} + \alpha w\right) = 0$$

with $t \longmapsto |\frac{dw}{dt}(t) + \alpha w(t)|$ non-increasing as well.

If we follow the proof of Theorem 2.2.1, we get:

$$\|-Au(t) + \alpha u(t)\| \leq \|-Au_0 + \alpha u_0\|$$

and thanks to the observation:

$$\|V(t)\| \leq \|U_0\| \tag{3.28}$$

and

$$\|-A_1 V(t) + \alpha V(t)\| \leq \|-A_1 U_0 + \alpha U_0\|$$
$$\|AV(t)\| \leq \|AU_0\| . \tag{3.29}$$

We now take $U(t) = e^{\alpha t} V(t)$, getting:

$$\begin{aligned}\frac{dU}{dt}(t) &= e^{\alpha t}\left[\frac{dV}{dt}(t) + \alpha V(t)\right] \\ &= e^{\alpha t}[-AV(t)] \\ &= -AU(t)\end{aligned}$$

which finally means that $U(t) = e^{\alpha t} V(t)$ is the unique solution of the problem:

$$\begin{cases}\dfrac{dU}{dt} + AU = 0 \quad \text{on } [0, +\infty), \\ U(0) = U_0.\end{cases}$$

Finally, from 3.28 and 3.29 we get:

$$\|U(t)\|^2 = e^{2\alpha t} \|V(t)\|^2 \leq e^{2\alpha t} \|U_0\|^2$$

$$\|AU(t)\|^2 = e^{2\alpha t} \|AV(t)\|^2 \leq e^{2\alpha t} \|AU_0\|^2$$

which are (3.19) and (3.20) respectively. □

Chapter 4

Recent studies

This chapter will be devoted to the presentation of a new kind of proof of existence and uniqueness of solutions for more general non-homogeneous Cauchy problems, together with new temporal regularity results. The following is a recent study, found in [1], and requires only the fundamental properties of the Yosida approximants, without involving neither Hille-Yosida Theorem nor the semigroup theory (see [13]).
The main tools these results are based on are a Banach space X and a linear operator $A : D(A) \subset X \to X$. We write $A \in HY(M, \omega)$ and say that A is a "Hille-Yosida operator" if there exist $M, \omega \in \mathbb{R}$ such that if $\lambda > \omega$ then $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and $\forall n \in \mathbb{N}$ it holds $\|(\lambda - \omega)^n (\lambda - A)^{-n}\| \leq M$ (see [14]).
With these assumptions, the following Cauchy problem has been much studied:

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{4.1}$$

(for a more detailed discussion, see the introduction of [1]).
First some new types of solutions are introduced, then the existence and uniqueness theorem is presented, together with the consequence on other types of solutions

4.1 Different types of solutions

The following definitions are reported from [1] and are meant just to give the basis for the next section. The first one was introduced in [15].

Definition 4.1.1 (Integral solution)**.** *Let $f \in L^1(0, T; X)$ and $u_0 \in X$. A function $u \in C(0, T; X)$ is called and integral solution of (4.1) if $\int_0^t u(s)ds \in D(A)$,$t \in [0, T]$ and*

$$u(t) = u_0 + A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \in [0, T]. \tag{4.2}$$

Notice that, as a consequence, we deduce $u(0) = u_0$.

By considering the limit of the solutions of problems approximating (4.1), other solutions are defined, called *Friedrichs solutions* in [16],which in [17] are connected to a functional interpretation of (4.1).

Definition 4.1.2 (L^p-solution)**.** *Let $f \in L^p(0,t;X)$ and $u_0 \in X$. A function $u \in C(0,T;X)$ is called an L^p-solution of (4.1) if there exists $\{u_k\}$ such that*

$$u_k \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A)), \quad k \in \mathbb{N}, \tag{4.3}$$

$$\lim_{k\to\infty} \|u_k' - Au_k - f\|_{L^p(0,t;X)} = 0, \tag{4.4}$$

$$\lim_{k\to\infty} \|u_k - u\|_{L^p(0,t;X)} = 0, \tag{4.5}$$

$$\lim_{k\to\infty} \|u_k(0) - u_0\| = 0. \tag{4.6}$$

Remark 4.1.1. *We stress out the fact that an L^p-solution is actually a continuous function which is not a solution of (4.1), but is the limit of the solutions of the approximated problems in L^p norm.*

Definition 4.1.3 (C-solution)**.** *Let $f \in C(0,t;X)$ and $u_0 \in X$. A function $u \in C(0,T;X)$ is called an C-solution of (4.1) if there exists $\{u_k\}$ such that*

$$u_k \in C^1(0,T;X) \cap C(0,T;D(A)), \quad k \in \mathbb{N}, \tag{4.7}$$

$$\lim_{k\to\infty} \|u_k' - Au_k - f\|_{C(0,t;X)} = 0, \tag{4.8}$$

$$\lim_{k\to\infty} \|u_k - u\|_{C(0,t;X)} = 0, \tag{4.9}$$

$$\lim_{k\to\infty} \|u_k(0) - u_0\| = 0. \tag{4.10}$$

We deduce $u(0) = u_0$.

Definition 4.1.4 (Strict solution in C)**.** *Let $f : [0,T] \to X$ and $u_0 \in X$. A function $u \in C^1(0,T;X) \cap C(0,T;D(A))$ is called a strict solution in C of (4.1) if (4.1) holds for every $t \in [0,T]$ and $u(0) = u_0$. If such a solution exists then $f \in C(0,T;X)$ and $u_0 \in D(A)$.*

Definition 4.1.5 (Strict solution in L^p)**.** *Let $f : [0,T] \to X$ and $u_0 \in X$. A function $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ is called a strict solution in L^p of (4.1) if (4.1) holds for every $t \in]0,T[$ a.e. and $u(0) = u_0$. If such a solution exists then $f \in L^p(0,T;X)$.*

It is possible to deduce, from the previous definitions, that the more restrictive is the strict C solution, followed in turn by the C-solution, the strict L^psolution and the L^p-solution, being the latter the weakest in terms of integrability conditions. In particular, it is possible to prove (see [**sinestrari**]) that a solution of each type is an integral solution, and an integral solution is L^p or C-solution if $f \in L^p(0,T;X)$ or $f \in C(0,T;X)$.
To proceed further we need also another lemma, concerning a correspondence between integral and strict solutions.

Lemma 4.1.1. *Let $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. If u is an integral solution of problem (4.1) then $v(t) := \int_0^t u(s)ds$, $t \in [0,T]$ is a strict solution in C of the problem*

$$\begin{cases} v'(t) = Av(t) + u_0 + \displaystyle\int_0^t f(s)ds, & t \in [0,T] \\ v(0) = 0. \end{cases}$$

Conversely if v is a strict solution in C of problem (4.1.1) then $u := v'$ is an integral solution of problem (4.1).

4.2 Uniqueness and temporal regularity

Referring to [**15sinestrari**], we now list some properties of the Yosida approximants of a Hille-Yosida operator $A : D(A) \subset X \to X$ of type (M, ω), and these will be useful to deduce a uniqueness theorem for the solutions of (4.1).

Theorem 4.2.1. *Given $A \in HY(M, \omega)$ the following properties hold:*

(a) $\lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1}x = x$ *if* $x \in \overline{D(A)}$.

(b) Setting for $n > \omega$, $A_n := nA(n - A)^{-1}$ *we have* $\lim_{n\to\infty} A_n x = Ax$ *if* $x \in D(A)$ *and* $Ax \in \overline{D(A)}$.

(c) $\|e^{A_n t}\| \leq Me^{\frac{n\omega t}{n-\omega}}$ *for* $t \geq 0$ *and* $n > \omega$ *and* $\|e^{A_n t}\| \leq Me^{2|\omega|t}$ *for* $t \geq 0$ *and* $n > 2\omega$.

(d) $e^{A_n t}x \in \overline{D(A)}$ *for* $n > \omega$, $t \geq 0$ *and* $x \in X$.

(e) Given $\lambda > \omega$ *there exists* $n_\lambda \in \mathbb{N}$ *such that* $n_\lambda > \omega$ *and* $(\lambda - A_n)^{-1} \in \mathcal{L}(X)$ *for* $n > n_\lambda$. *In addition* $\lim_{n\to\infty} \|(\lambda - A_n)^{-1} - (\lambda - A)^{-1}\| = 0$.

(f) There exists $T(t)x := \lim_{n\to\infty} e^{A_n t}x$ *uniformly for* $x \in K$ *a compact set of* $\overline{D(A)}$ *and* $t \in [0, T]$.

(g) Given $\lambda > \omega$, *we have* $T(t)(\lambda - A)^{-1}x = \lim_{n\to\infty} e^{A_n t}(\lambda - A_n)^{-1}x$ *uniformly for* $x \in H$ *a compact set of* X *and* $t \in [0, T]$.

The next lemma will be the intermediate step necessary to get the uniqueness result.

Lemma 4.2.2. *If* $u \in W^{1,1}(0, T; X)$ *and is such that* $u(t) \in D(A)$ *for* $t \in]0, T[$ *a.e. and*

$$u'(t) = Au(t), \quad \text{for a.e. } t \in]0, T[, \quad u(0) = 0 \tag{4.11}$$

then $u(t) = 0$.

To prove it, we are going to follow [1].

Proof. Fix $n > \omega$, $t \in]0, T]$ and define $\phi : [0, t] \to X$ as $\phi(s) = e^{A_n(t-s)}u(s)$, $0 \leq s \leq t$. We have $\phi \in W^{1,1}(0, t; X)$ and for a.e. $t \in]0, T[$, thanks to (4.11), it holds:

$$\phi'(s) = -A_n e^{A_n(t-s)}u(s) + e^{A_n(t-s)}u'(s) = e^{A_n(t-s)}(A - A_n)u(s).$$

Moreover, since:

$$A - A_n = A - nA(n - A)^{-1} = -A[-I + n(n - A)^{-1}] = -A(n - A)^{-1}A$$

we get:

$$\phi'(s) = -e^{A_n(t-s)}A(n - A)^{-1}Au(s)$$

hence

$$u(t) = \phi(t) = \int_0^t \phi'(s)ds = -\int_0^t e^{A_n(t-s)}A(n - A)^{-1}Au(s)ds.$$

We now take $\lambda \in \rho(A)$, so that $(\lambda - A)^{-1}$ is well defined, and $n > 2\omega$. Thus, we get:

$$\|(\lambda - A)^{-1}u(t)\| = \left\|\int_0^t e^{A_n(t-s)}(\lambda - A)^{-1}A(n - A)^{-1}Au(s)ds\right\|$$

$$= \left\| \int_0^t e^{A_n(t-s)} [\lambda(\lambda - A)^{-1} - I](n - A)^{-1} u'(s) ds \right\|$$
$$\leq M e^{2|\omega| t} \left\| \lambda(\lambda - A)^{-1} - I \right\| \frac{M}{n - \omega} \|u'\|_{L^1(0,T;X)} \xrightarrow{n \to \infty} 0.$$

Therefore, since $\lambda \in \rho(A)$, $u(t) = 0 \; \forall t \in [0,T]$. □

Now it comes to the uniqueness result:

Theorem 4.2.3. *A solution of each type of problem (4.1) is unique when it exists.*

Proof. We stated that a solution of each type is an integral solution, therefore it is sufficient to prove its uniqueness. If $u \in C(0,T;X)$ satisfies (4.2) with $u_0 = 0$ and $f(t) \equiv 0$ then $\phi(t) := \int_0^t u(s) ds$, $t \in [0,T]$ is a strict solution in C of (4.11) and so $\phi \equiv 0$, hence $u \equiv 0$. □

The next theorem is the main result of [**sinestrari**] and it is about existence of a strict solution in C of problem (4.1), together with its temporal regularity. The proof follows an approximation procedure which is a generalization of the Hille-Yosida Theorem, and here it will be reported as in [**sinestrari**].

Theorem 4.2.4. *Given $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;X)$ such that*

$$u_1 := Au_0 + f(0) \in \overline{D(A)} \tag{4.12}$$

there exists a unique u strict solution in C of problem (4.1) and for $t \in [0,T]$ we have

$$\|u(t)\| \leq M e^{\omega t} \left(\|u_0\| + \int_0^t e^{-\omega s} \|f(s)\| \, ds \right), \tag{4.13}$$

$$\|u'(t)\| \leq M e^{\omega t} \left(\|u_1\| + \int_0^t e^{-\omega s} \|f'(s)\| \, ds \right), \tag{4.14}$$

Proof. First of all, suppose $f \in C^2(0,T;X)$. We fix $\lambda > \omega$ and for $n > n_\lambda$ we define

$$u_{0n} := (\lambda - A_n)^{-1}(\lambda - A)u_0 = (\lambda - A_n)^{-1}(\lambda u_0 - u_1 + f(0)). \tag{4.15}$$

We know that $A_n \in \mathcal{L}(X)$, therefore the problem

$$u_n'(t) = A_n u_n(t) + f(t), \quad t \in [0,T]; \qquad u_n(0) = u_{0n} \tag{4.16}$$

has the unique solution

$$u_n(t) = e^{A_n t} u_{0n} + \int_0^t e^{A_n s} f(t-s) ds, \quad t \in [0,T]. \tag{4.17}$$

We want to prove that $\{u_n\}$ converges in $C^1(0,T;X)$ to a solution of (4.1) verifying (4.13) and (4.14).

It surely holds:

$$u_n(t) = u_{0n} + \int_0^t u_n'(s) ds, \quad t \in [0,T]$$

and in addition we have, thanks to 4.2.1:

$$\|u_{0n} - u_0\| = \left\| (\lambda - A_n)^{-1}(\lambda - A)u_0 - u_0 \right\| = \left\| [(\lambda - A_n)^{-1}(\lambda - A) - I] u_0 \right\|$$

$$\leq \left\|(\lambda - A_n)^{-1}(\lambda - A) - I\right\| \|u_0\| \leq \|(\lambda - A)\| \left\|(\lambda - A_n)^{-1} - (\lambda - A)^{-1}\right\| \|u_0\|,$$

therefore

$$\lim_{n\to\infty} u_{0n} = u_0.$$

Thus, thanks to the last two considerations, it will be sufficient to prove that $\{u_n'\}$ converges in $C(0,T;X)$.
We put (4.17) in (4.16) getting:

$$u_n'(t) = e^{A_n t} A_n u_{0n} + \int_0^t A_n e^{A_n s} f(t-s) ds + f(t). \tag{4.18}$$

We now consider the integral term and integrate by parts two times, obtaining:

$$\begin{aligned}
\int_0^t A_n e^{A_n s} f(t-s) ds &= e^{A_n t} f(0) - f(t) + \int_0^t e^{(A_n - \lambda)s} e^{\lambda s} f'(t-s) ds \\
&= e^{A_n t} f(0) - f(t) - e^{A_n t}(\lambda - A_n)^{-1} f'(0) \\
+(\lambda - A_n)^{-1} f'(t) &+ \int_0^t e^{A_n s}(\lambda - A_n)^{-1}[\lambda f'(t-s) - f''(t-s)] ds,
\end{aligned}$$

therefore:

$$\begin{aligned}
u_n'(t) = e^{A_n t}(A_n u_{0n} + f(0)) &- e^{A_n t}(\lambda - A_n)^{-1} f'(0) + (\lambda - A_n)^{-1} f'(t) \\
&+ \int_0^t e^{A_n s}(\lambda - A_n)^{-1}[\lambda f'(t-s) - f''(t-s)] ds.
\end{aligned} \tag{4.19}$$

We have:

$$\begin{aligned}
A_n u_{0n} + f(0) &= A_n(\lambda - A_n)^{-1}(\lambda - A)u_0 + f(0) \\
&= -(\lambda - A)u_0 + \lambda(\lambda - A_n)^{-1}(\lambda - A)u_0 + f(0) \\
&= -\lambda u_0 + u_1 + \lambda(\lambda - A_n)^{-1}(\lambda - A)u_0 \\
&= [\lambda(\lambda - A_n)^{-1}(\lambda - A) - \lambda]u_0 + u_1 \\
&= \lambda[(\lambda - A_n)^{-1}(\lambda - A) - I]u_0 + u_1 \\
&= \lambda(\lambda - A)[(\lambda - A_n)^{-1} - (\lambda - A)^{-1}]u_0 + u_1,
\end{aligned}$$

and therefore, thanks to 4.2.1:

$$\lim_{n\to\infty} A_n u_{0n} + f(0) = u_1.$$

From Theorem 4.2.1 (points (c),(e),(f)), we get:

$$\lim_{n\to\infty} e^{A_n t}(A_n u_{0n} + f(0)) = \lim_{n\to\infty}(A_n u_{0n} + f(0) - u_1) + \lim_{n\to\infty} e^{A_n t} u_1 = T(t)u_1,$$

$$\begin{aligned}
\lim_{n\to\infty} e^{A_n t}(\lambda - A_n)^{-1} f'(0) &= \lim_{n\to\infty} e^{A_n t}[(\lambda - A_n)^{-1} - (\lambda - A)^{-1}] f'(0) \\
&\quad + \lim_{n\to\infty} e^{A_n t}(\lambda - A)^{-1} f'(0) \\
&= T(t)(\lambda - A)^{-1} f'(0)
\end{aligned}$$

uniformly for $t \in [0,T]$. Finally we have:

$$\begin{aligned}
\lim_{n\to\infty} u_n'(t) = T(t)u_1 &- T(t)(\lambda - A)^{-1} f'(0) + (\lambda - A)^{-1} f'(t) \\
&+ \int_0^t T(s)(\lambda - A)^{-1}[\lambda f'(t-s) - f''(t-s)] ds
\end{aligned}$$

meaning that $\{u_n\}$ converges in $C^1(0,T;X)$ to a function u. Still, it has to be proved that u is a strict solution of problem (4.1).
We know that $u(t) = \lim_{n\to\infty} u_n(t)$ and $u'(t) = \lim_{n\to\infty} u_n'(t) = \lim_{n\to\infty} A_n u_n(t) + f(t)$ uniformly for $t \in [0,T]$. Therefore, given $\lambda > \omega$:

$$\lim_{n\to\infty}(\lambda - A_n)u_n(t) = \lambda u(t) - u'(t) + f(t). \tag{4.20}$$

If $n > n_\lambda$, it holds:

$$\begin{aligned} &\left\|u_n(t) + (\lambda - A)^{-1}[u'(t) - \lambda u(t) - f(t)]\right\| \\ \leq &\left\|(\lambda - A_n)^{-1}\right\| \left\|(\lambda - A_n)u_n(t) + u'(t) - \lambda u(t) - f(t)\right\| \\ &+ \left\|-(\lambda - A_n)^{-1} + (\lambda - A)^{-1}\right\| \left\|u'(t) - \lambda u(t) - f(t)\right\|. \end{aligned}$$

and this means that, by taking the limit for $n \longrightarrow \infty$:

$$u(t) = -(\lambda - A)^{-1}[u'(t) - \lambda u(t) - f(t)].$$

Applying $\lambda - A$ we deduce $u'(t) = Au(t) + f(t)$ and since $u(0) = \lim_{n\to\infty} u_n(0) = u_0$ this finally means that u is a strict solution in C of problem (4.1).
Now, to get the estimates, we take (4.17) with $n > \omega$, and thanks to 4.2.1 (c) we have:

$$\|u_n(t)\| \leq M\left(e^{\frac{\omega n}{n-\omega}}\|u_{0n}\| + \int_0^t e^{\frac{\omega n}{n-\omega}}\|f(t-s)\|\,ds\right)$$

which gives (4.13) for $n \longrightarrow \infty$. If we, instead, take (4.18) and integrate by parts:

$$u_n'(t) = e^{A_n t}\left(A_n u_{0n} + f(0) + \int_0^t e^{A_n s} f'(t-s)ds\right)$$

and again taking the limit we get (4.14).
Now suppose $f \in W^{1,1}(0,T;X)$ and let $\{g_n\} \subset C^1(0,T;X)$ be a sequence approximating f', that is:

$$\lim_{n\longrightarrow\infty}\|g_n - f'\|_{L^1(0,T;X)} = 0.$$

Given $n \in \mathbb{N}$, $t \in [0,T]$, we define:

$$f_n(t) := f(0) + \int_0^t g_n(s)ds.$$

We have that $f \in C^2(0,T;X)$ and hold both the limits:

$$\lim_{n\to\infty}\|f_n - f\|_{C(0,T;X)} = 0 \quad , \quad \lim_{n\to\infty}\|f_n' - f'\|_{L^1(0,T;X)} = 0. \tag{4.21}$$

Since by hypothesis $Au_0 + f_n(0) = u_1 \in \overline{D(A)}$, proceeding as the first part of the proof we can deduce the existence of u_n strict solution in C of the problem

$$\begin{cases} u_n'(t) = Au_n(t) + f_n(t) & t \in [0,T] \\ u_n(0) = u_0. \end{cases} \tag{4.22}$$

In addition, thanks to the temporal regularity inequalities, given $m, n \in \mathbb{N}$:

$$\|u_n'(t) - u_m(t)\| \leq Me^{|\omega|t}\|f_n' - f_m'\|_{L^1(0,T;X)}$$

hence, thanks to (4.21), $\{u_n'\}$ converges in $C(0,T;X)$.
Moreover, since $u_n(0)=u_0$, $n\in\mathbb{N}$ it is possible to deduce the existence of $u\in C^1(0,T;X)$ such that $u=\lim_{n\to\infty}u_n$ in $C^1(0,T;X)$.
Since (4.22) implies $\lim_{n\to\infty}Au_n(t)=u'(t)-f(t)$ and A is closed, this means that $u(t)\in D(A)$ and $Au(t)=u'(t)-f(t)$. Finally, given that $u(0)=\lim_{n\to\infty}u_n(0)=u_0$, u is a strict solution in C of problem (4.1).
Again, since $\{u_n\}$ converges to u in $C^1(0,T;X)$ and $\lim_{n\to\infty}A_nu_0+f_n(0)=\lim_{n\to\infty}u_n'(0)=Au_0+f(0)=u_1$, the same estimates for the solution u of (4.1) can be obtained by applying (4.13) and (4.14) to (4.22). □

4.3 L^p, C and integral solutions

Here we will report mainly three theorems concerning existence of other types of solutions, which come as a consequence of Theorem 4.2.4. Essentially the idea is to use 4.2.4 to solve approximations of the problems, and then to take the limit for $n\longrightarrow\infty$. All these results are found in [1].

Theorem 4.3.1. *Given $u_0\in\overline{D(A)}$ and $f\in L^p(0,T;X)$ there exists a unique L^p-solution u of problem (4.1) and*

$$\|u(t)\|\leq Me^{\omega t}\bigg(\|u_0\|+\int_0^t e^{-\omega s}\,\|f(s)\|\,ds\bigg),\quad t\in[0,T].\tag{4.23}$$

Proof. We define

$$u_{0n}:=n(n-A)^{-1}u_0,\quad n\in\mathbb{N}.$$

It holds $u_{0n}\in D(A)$, $Au_{0n}\in\overline{D(A)}$ and, thanks to 4.2.1 (a),

$$\lim_{n\to\infty}u_{0n}=u_0.\tag{4.24}$$

Moreover, we consider $\{f_n\}\subseteq C^1(0,T;X)$ such that

$$\lim_{n\to\infty}\|f_n-f\|_{L^p(0,T;X)}=0\tag{4.25}$$

Thanks to Theorem 4.2.4, there exists u_n strict solution in C of problem

$$\begin{cases}u_n'(t)=Au_n(t)+f_n(t),\quad t\in[0,T]\\ u_n(0)=u_{0n}.\end{cases}\tag{4.26}$$

Taking $m,n,\in\mathbb{N}$, $t\in[0,T]$ it holds the inequality:

$$\|u_n(t)-u_m(t)\|\leq Me^{|\omega|t}\bigg(\|u_{0n}-u_{0m}\|+\int_0^t e^{-\omega s}\,\|f_n(s)-f_m(s)\|\,ds\bigg)\tag{4.27}$$

therefore $\{u_n\}$ converges in $C(0,T;X)$ to a function $u\in C(0,T;X)$. Additionally it holds, due to (4.24) and (4.25):

$$\lim_{n\to\infty}\|u_n'-Au_n-f\|_{L^p(0,T;X)}=\lim_{n\to\infty}\|f_n-f\|_{L^p(0,T;X)}=0.$$

$$\lim_{n\to\infty}\|u_n(0)-u_0\|=\lim_{n\to\infty}\|u_{0n}-u_0\|=0$$

and the latter completes the set of equalities which make u a L^p-solution of problem (4.1). Estimate (4.23) is a consequence of the corresponding estimate applied to the approximated problems. □

Theorem 4.3.2. *Given $u_0 \in \overline{D(A)}$ and $f \in C(0,T;X)$ there exists a unique C-solution u of problem (4.1) and*

$$\|u(t)\| \leq Me^{\omega t}\left(\|u_0\| + \int_0^t e^{-\omega s} \|f(s)\| \, ds \right), \quad t \in [0,T]. \tag{4.28}$$

Proof. We fix $\lambda \in \rho(A)$ and, for $n > \omega$, define

$$u_{0n} := n(n-A)^{-1}[u_0 - (\lambda - A)^{-1} f(0)] + (\lambda - A)^{-1} f(0).$$

Since

$$\begin{aligned} Au_{0n} = -n[u_0 - (\lambda - A)^{-1} f(0)] \\ -n^2(n-A)^{-1}[u_0 - (\lambda - A)^{-1} f(0)] \\ -f(0) + \lambda(\lambda - A)^{-1} f(0) \end{aligned}$$

this means that $u_{0n} \in D(A)$, $Au_{0n} + f(0) \in \overline{D(A)}$ and

$$\lim_{n \to \infty} u_{0n} = u_0. \tag{4.29}$$

Moreover, if $\{g_n\} \subseteq C^1(0,T;X)$ is such that $\lim_{n\to\infty} \|g_n - f\|_{C(0,T;X)} = 0$ then setting

$$f_n(t) := g_n(t) - g_n(0) + f(0), \quad n \in \mathbb{N}, \quad t \in [0,T]$$

we have $\{f_n\} \subseteq C^1(0,T;X)$, $f_n(0) = f(0)$, $Au_{0n} + f_n(0) \in \overline{D(A)}$ and

$$\lim_{n \to \infty} \|f_n - f\|_{C(0,T;X)} = 0. \tag{4.30}$$

Thanks to Theorem 4.2.4 there exists a unique u_n strict solution in C of problem

$$\begin{cases} u_n'(t) = Au_n(t) + f_n(t), & t \in [0,T] \\ u_n(0) = u_{0n} \end{cases} \tag{4.31}$$

and given $m, n \in \mathbb{N}$ it holds:

$$\|u_n(t) - u_m(t)\| \leq Me^{|\omega| t}\left(\|u_{0n} - u_{0m}\| + \int_0^t e^{-\omega s} \|f_n(s) - f_m(s)\| \, ds \right)$$

therefore $\{u_n\}$ converges in $C^(0,T;X)$ to a function $u \in C^(0,T;X)$.
Finally, thanks to (4.29) and (4.30):

$$\lim_{n\to\infty} \|u_n' - Au_n - f\|_{C(0,T;X)} = \lim_{n\to\infty} \|f_n - f\|_{C(0,T;X)} = 0$$

$$\lim_{n\to\infty} \|u_n(0) - u_0\| = \lim_{n\to\infty} \|u_{0n} - u_0\| = 0$$

which means that u is a C-solution of problem (4.1). Again, (4.28) is a consequence of the estimate applied to each approximated problem. □

Theorem 4.3.3. *If $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$ then problem (4.1) has a unique integral solution u verifying estimate (4.23). In addition, if $f \in L^p(0,T;X)$ or $f \in C(0,T;X)$ then u is an L^p-solution or a C-solution.*

Proof. Thanks to Theorem 4.3.1, (4.1) has a unique L^p-solution if $f \in L^p(0,T;X)$, which is an integral solution as well. The same holds if $f \in C(0,T;X)$. In both cases (4.23) holds.
From another point of view, we can directly use Theorem 4.2.4 for the problem

$$\begin{cases} v'(t) = Av(t) + u_0 + \displaystyle\int_0^t f(s)ds, \quad t \in [0,T] \\ v(0) = 0 \end{cases}$$

whose strict solution in C satisfies

$$\|v'(t)\| \leq Me^{\omega t}\left(\|u_0\| + \int_0^t \|f(s)\|\, ds \right), \quad t \in [0,T].$$

Finally, thanks to 4.1.1, $u := v'$ is the integral solution of (4.1). □

Appendices

Appendix A

The fixed-point theorem

The following is a very important theorem that has been used in chapter 2. It concerns contractions, which are particular lipschitz-continuous functions on metric spaces. Before it, we have to state some definitions.

Definition A.1. *A metric space is a couple* (X, d) *such that* X *is a set and* $d : X \times X \to \mathbb{R}$ *is such that:*

- $d(x, y) \geq 0 \quad \forall x, y \in X$
- $d(x, y) = d(y, x) \quad \forall x, y \in X$
- $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$
- $d(x, y) = 0 \iff x = y.$

The latter is a more general case than a normed space, since the function d is not necessarily induced by a norm.

Definition A.2. *Let* X *be a set. A sequence* $\{x_n\} \subset X$ *is called a Cauchy sequence if*

$$\forall \epsilon > 0 \; \exists N_\epsilon : d(x_n, x_m) < \epsilon \; \forall n, m \geq N_\epsilon.$$

A sequence is said to be converging to x *(or* $\lim_{n\to\infty} x_n = x$*) if*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition A.3. *A metric space* X *is said to be complete if all its Cauchy sequences are converging.*

Theorem A.1 (fixed-point)**.** *Let* X *be a complete metric space,* $X \neq 0$ *and let* $f : X \to X$ *be a lipschitz-continuous function with constant* $c \in [0,1[$[1]*. Then there exists one and only one* $\eta \in X$ *such that* $f(\eta) = \eta$.

---

[1] *In this case* f *can be referred to as a contraction. Indeed, being* $d(f(x) - f(y)) \leq cd(x, y) \; \forall x, y \in X$*, it "shrinks" the distance between any couple of points of* X*.*

Proof. Let $x_0 \in X$ and $\{x_h\}$ the sequence defined as $x_{h+1} = f(x_h)$.
We want to prove, by induction, that

$$d(x_{h+1}, x_h) \leq c^h d(x_1, x_0) \quad \forall h \geq 0. \tag{A.1}$$

If $h = 0$, (A.1) is verified. Using the lipschitz-continuity of f and assuming (A.1) to be true for a certain $h > 0$ we have:

$$d(x_{h+2}, x_{h+1}) = d(f(x_{h+1}), f(x_h)) \leq cd(x_{h+1}, x_h) \leq c^{h+1} d(x_1, x_0),$$

therefore (A.1) is true fot $h + 1$, and by induction for all $h > 0$.
We want now to prove that $\forall h \geq 0$ and $\forall j \geq 1$ it holds:

$$d(x_{h+j}, x_h) \leq c^h \left(\sum_{i=0}^{j-1} c^i \right) d(x_1, x_0). \tag{A.2}$$

If $j = 1$, (A.2) is equivalent to (A.1) and therefore it is true. We assume it to be true for a certain $j \geq 1$, and take:

$$d(x_{h+j+1}, x_h) \leq d(x_{h+j+1}, x_{h+j}) + d(x_{h+j}, x_h)$$

which thanks to (A.1) leads to:

$$d(x_{h+j+1}, x_{h+j}) + d(x_{h+j}, x_h) \leq c^{h+j} d(x_1, x_0) + c^h \left(\sum_{i=0}^{j-1} c^i \right) d(x_1, x_0) = c^h \left(\sum_{i=0}^{j} \right) d(x_1, x_0).$$

Thus, (A.2) is valid for $j + 1$ and therefore by induction for all $j \geq 1$.
We now observe that, since $c \in [0,1[$, we have:

$$\sum_{i=0}^{j-1} c^i \leq \frac{1}{1-c} \quad \forall j \geq 1$$

therefore, in particular it holds:

$$d(x_{h+j}, x_h) \leq \frac{c^h}{1-c} d(x_1, x_0).$$

We now want to prove that $\{x_h\}$ is a Cauchy sequence (and therefore converges, since X is complete by hypothesis). Let $\epsilon > 0$ and $\overline{h} \in \mathbb{N}$ such that

$$\frac{c^{\overline{h}}}{1-c} d(x_1, x_0) < \epsilon.$$

Let $h, k > \overline{h}$ with $k > h$, so that $k = h + j$. This leads to:

$$d(x_k, x_h) = d(x_{h+j}, x_h) \leq \frac{c^h}{1-c} d(x_1, x_0) \leq \frac{c^{\overline{h}}}{1-c} d(x_1, x_0) < \epsilon,$$

which means that $\{x_h\}$ is a Cauchy sequence. Thus, there exists $\eta > 0$ such that

$$\lim_{h \to \infty} x_h = \eta.$$

Being $\{x_{h+1}\}$ a subsequence of $\{x_h\}$, we have:

$$\lim_{h\to\infty} f(x_h) = \lim_{h\to\infty} x_{h+1} = \eta,$$

and due to the continuity of f:

$$\lim_{h\to\infty} f(x_h) = f(\eta),$$

which leads to $f(\eta) = \eta$ because the limit is unique.
Finally, assuming the existence of another element $\sigma \in X$ such that $f(\sigma) = \sigma$, this would mean that:

$$d(\eta, \sigma) = d(f(\eta), f(\sigma)) \leq cd(\eta, \sigma)$$

and therefore

$$(1 - c)d(\eta, \sigma) \leq 0$$

which means that $\eta = \sigma$, and the theorem is proved. □

Appendix B

Integration by parts in $\mathbb{R}^n$ and Green's identities

The integration formulas in n dimensions can be derived in a straightforward way starting from the divergence theorem, which we shall only state without proof.

Theorem B.1 (Divergence)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, with $\partial\Omega \in C^1$ and let $\vec{F} : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined as*

$$x \longmapsto \vec{F}(x) = (F_1(x), \dots, F_n(x))$$

be a vector field such that $F_i \in C^1(\Omega)$ $\forall i = 1, \dots, n$. Then it holds:

$$\int_{\Omega} \operatorname{div}(\vec{F}(x))dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n} dS \tag{B.1}$$

where $\vec{n} = \vec{n}(x)$ is the unitary normal vector pointing outwards of $\partial\Omega$, and

$$\operatorname{div}(\vec{F}(x)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i(x))$$

is the divergence operator applied to $\vec{F}$.

The essential meaning of this theorem is that the divergence of a vector field can be interpreted as an "infinitesimal flux", being the term on the right hand side the flux of $\vec{F}$ through the boundary $\partial\Omega$. It has many important applications, for instance consider the integral form of *Gauss's flux theorem*[1] in $\mathbb{R}^3$, which states that:

$$\int_{\partial\Omega} \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} \int_{\Omega} \rho dx$$

where $\vec{E}$ is the electrostatic vector field ($[V]$), ρ the density of electric charge per unit volume ($[\frac{C}{m^3}]$) and ϵ_0 the permittivity of free space ($[\frac{F}{m}] = [\frac{C}{Vm}]$). The divergence theorem allows to

[1]It is essentially the first of the four Maxwell's equations for electromagnetism.

express the term on the left as a volume integral instead a surface one, leading to the local formulation of the equation:

$$div\vec{E} = \frac{\rho}{\epsilon_0}$$

Starting from this, we can demonstrate the following:

Corollary B.1.1 (Integration by parts in $\mathbb{R}^n$)**.** *Under the same hypotheses of the divergence theorem, let $u, v \in C^2(\Omega)$. It Holds:*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uvn_i dS \tag{B.2}$$

where n_i is the i-th component of $\vec{n}$.

Proof. Let's consider the vector field $\vec{F}$ with all null entries but the i-th component, equal to uv. $\vec{F} \in C^1(\Omega)$, and its divergence is given by:

$$\text{div}\vec{F}(x) = \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}$$

Thanks to (B.1) we get:

$$\int_\Omega \text{div}\vec{F}(x) dx = \int_\Omega \left(\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \right) dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n} dS = \int_{\partial\Omega} uvn_i dS$$

□

We stress out the fact that the boundary term usually can be simplified due to the properties of the solutions of the considered problems.

Theorem B.2 (Green's identities)**.** *Under the hypotheses of the divergence theorem, consider $u, v, \in C^1(\overline{\Omega})$. Then the following three expressions hold:*

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} dS, \tag{B.3}$$

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega \Delta v u dx + \int_{\partial\Omega} \frac{\partial v}{\partial \vec{n}} u dS, \tag{B.4}$$

$$\int_\Omega (u\Delta v - v\Delta u) dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS. \tag{B.5}$$

Proof. To get (B.3), we apply the divergence theorem to the field $\vec{F} = \nabla u$, reminding that $\Delta u = \text{div}\nabla u$:

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \nabla u \cdot \vec{n} dS = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} dS.$$

To get (B.4), we write (B.2) substituting v with $\frac{\partial v}{\partial x_i}$:

$$\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = -\int_\Omega u \frac{\partial^2 v}{\partial x_i^2} dx + \int_{\partial\Omega} u \frac{\partial v}{\partial x_i} n_i dS$$

then sum:

$$\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = -\sum_{i=1}^n \int_\Omega u \frac{\partial^2 v}{\partial x_i^2} dx + \sum_{i=1}^n \int_{\partial\Omega} u \frac{\partial v}{\partial x_i} n_i dS$$

and thanks to the linearity of the integral we get (B.4).
To get the last expression, we write (B.4) switching u and v:

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega v\Delta u dx + \int_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}} dS$$

and subtract (B.2) from the latter, obtaining:

$$0 = \int_\Omega (\Delta u v - \Delta v u) dx + \int_{\partial\Omega} \left(\frac{\partial v}{\partial \vec{n}} u - \frac{\partial u}{\partial \vec{n}} v\right) dS.$$

□

Under the assumption that the domain Ω is bounded, the validity of the above formulas can be proved also in Sobolev spaces like $H^1(\Omega)$, making use of the embedding theorems that are not treated in this dissertation. Roughly speaking, the idea is that the terms ∇u and Δu "make sense" if, respectively, $u \in H^1(\Omega)$ or $u \in H^2(\Omega)$. In this appendix, we just give a hint about integration by parts in the mono-dimensional case, with $u, w \in H^1(I)$, with $I = (a, b) \subset \mathbb{R}$.

Theorem B.3 (Integration by parts in $H^1(I)$)**.** *Let $I = (a, b) \subset \mathbb{R}$. Then it holds:*

$$\int_I wv' = w(b)v(b) - w(a)v(a) - \int_I w'v \quad \forall w, v \in H^1(I) \tag{B.6}$$

Proof. We recall that, since $H^1(I) \subset C^0(I)$, the term $w(b)v(b) - w(a)v(a)$ is well defined. Moreover, the formula holds $\forall \phi \in \mathcal{D}(\bar{I})$, the latter begin dense in $H^1(I)$. Thus, we shall extend the result by density.
First of all, we know that given $w, v \in H^1(I)$, $\exists w_n, v_n \in \mathcal{D}(\bar{I})$ such that

$$v_n \longrightarrow v \quad \text{and} \quad w_n \longrightarrow w \quad \text{in } H^1(I)$$

It holds:

$$\int_I w_n v_n' = w_n(b)v_n(b) - w_n(a)v_n(a) - \int_I w_n' v_n.$$

Since the embedding $H^1(I) \subset C^0(I)$ is continuous, this means that the convergence is uniform, therefore:

$$w_n(b)v_n(b) \longrightarrow w(b)v(b) \quad \text{and} \quad w_n(a)v_n(a) \longrightarrow w(a)v(a) \tag{B.7}$$

Now we evaluate:

$$\begin{aligned}\left|\int_I w_n v_n' - \int_I wv'\right| &\le \left|\int_I w_n v_n' - \int_I w_n v'\right| + \left|\int_I w_n v' - \int_I wv'\right| \\ &\le \int_I |w_n||v_n' - v'| + \int_I |w_n - w||v'| \\ &\le \|w_n\|_{L^2(I)} \|v_n' - v'\|_{L^2(I)} + \|v'\|_{L^2(I)} \|w_n - w\|_{L^2(I)} \longrightarrow 0.\end{aligned}$$

Therefore it yields:

$$\int_I w_n v_n' \longrightarrow \int_I wv' \tag{B.8}$$

and analogously as before, it follows:

$$\int_I w_n' v_n \longrightarrow \int_I w'v. \tag{B.9}$$

Finally (B.6) is prooved thanks to (B.7), (B.8), (B.9). □

Appendix C

The Galerkin method

When dealing with partial differential equations, the existence of a theorem which guarantees existence and uniqueness of solutions has not to be taken for granted. Lax-Milgram or Hille-Yosida Theorems can be really helpful in several circumstances, but in others it may be necessary to search for a solution by means of approximation procedures, with a more numeric approach. This is the case of the Galerkin method, whose idea is to "project" the considered problem, which is in general set in an infinite dimensional space, on a finite dimensional subspace of the latter, leading to a system of ordinary differential equations (that can be solved in an easier way with numeric procedures). The main difference with the other approaches is that while the cited theorems just confirm or not the existence and uniqueness of the solution, here we directly search for it.
This fundamental idea allowed the arising of numerical schemes like the Finite Elements Method or FEM, widely used in structural engineering [1], and the Finite Volumes Methods or FVM, more utilized for thermo-fluid-dynamics and in general conservation laws[2]. A good reference for this kind of topics is [18], here we are just going to give a hint, for completeness, about their mathematical principle.
Let $\Omega \subset \mathbb{R}^n$ be bounded. We consider the generic initial and boundary value parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = g(x), & x \in \Omega \end{cases} \tag{C.1}$$

where f, g and u are considered to respect all the constraints in order give sense to the problem. We want to write the weak formulation of the problem. To do this, we take $v \in \mathcal{D}(\Omega)$ and take the scalar product in $L^2(\Omega)$:

$$\int_\Omega \frac{\partial u}{\partial t}(x,t), v(x)dx - \int_\Omega \Delta u(x,t)v(x)dx = \int_\Omega f(x,t)v(x)dx, \quad \forall v \in \mathcal{D}(\Omega)$$

$$\int_\Omega \frac{\partial u}{\partial t}(x,t), v(x)dx + \int_\Omega \nabla u(x,t)\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx, \quad \forall v \in \mathcal{D}(\Omega)$$

[1] Very often FEM is used to characterize mechanical structures in the frequency domain.

[2] For instance, to study and predict the velocity profile of a flux in a pipe, or to study its temperature field.

which in turn is, since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$:

$$\left(\frac{\partial u}{\partial t}(x,t), v(x)\right)_{L^2(\Omega)} + (u(x,t), v(x))_{H_0^1(\Omega)} = (f(x,t), v(x))_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Now let $\varphi \in \mathcal{D}(0,T)$. By taking the scalar product in time and integrating by parts, we get the weak formulation:

$$-\int_0^T (u(x,t), v(x))_{L^2(\Omega)} \frac{d\varphi(t)}{dt} dt + \int_0^T (u(x,t), v(x))_{H_0^1(\Omega)} \varphi(t) dt$$
$$= \int_0^T (f(x,t), v(x))_{L^2(\Omega)} \varphi(t) dt, \quad \forall v \in H_0^1 \Omega, \quad \forall \varphi \in \mathcal{D}(0,T).$$

Now we want to explicitly determine the solution of the problem, or at least estimate it. We consider $\{w_k\}$ as the Hilbert basis of $L^2(\Omega)$ constituted by eigenfunctions of operator $-\Delta$ with Dirichlet boundary conditions, i.e.:

$$(w_k, w_j)_{L^2(\Omega)} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$(w_k, w_j)_{H_0^1(\Omega)} = \begin{cases} \lambda_j & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

where λ_j is the eigenvalue of $-\Delta$ corresponding to the j-th eigenfunction. We set:

$$u_m(x,t) := \sum_{k=1}^m c_k(t) w_k(x) \quad , \quad f_m(x,t) := \sum_{k=1}^m \bar{f}_k(t) w_k(x)$$

with $\bar{f}_k(t) = (f(x,t), w_k)_{L^2(\Omega)}$, $m \in \mathbb{N}$.
The following step is to consider the approximated problem, i.e. the projection of (C.1) on $span\{w_1, \dots, w_m\}$:

$$\left(\frac{\partial u_m}{\partial t}(x,t), v(x)\right)_{L^2(\Omega)} + (u_m(x,t), v(x))_{H_0^1(\Omega)} = (f_m(x,t), v(x))_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

which is:

$$\sum_{k=1}^m \frac{\partial c_k}{\partial t}(t)(w_k(x), v(x))_{L^2(\Omega)} + \sum_{k=1}^m c_k(t)(w_k(x), v(x))_{H_0^1(\Omega)} = \sum_{k=1}^m \bar{f}_k(t)(w_k(x), v(x))_{L^2(\Omega)}.$$

Now, by taking $v = w_j$ we get:

$$\frac{\partial c_j}{\partial t}(t) + \lambda_j c_j(t) = \bar{f}_j(t) \quad \forall j = 1, \dots, m. \tag{C.2}$$

Therefore, (C.2) is a system of ordinary differential equations of the first order with the m unknowns $c_1, \dots, c_m$, which can be solved with a numerical procedure; instead, hyperbolic problems lead to systems of the second order. The idea is that u_m is an approximation of the solution, and

$$\lim_{m \to \infty} u_m = u$$

solves (C.1).

References

[1] Eugenio Sinestrari. «Hille-Yosida operators and Cauchy problems». In: *Semigroup Forum* 82 (2011), pp. 10–34.

[2] H. Brezis. *Analisi funzionale: teoria e applicazioni*. Serie di matematica e fisica. Liguori, 1986.

[3] B. Rynne and M.A. Youngson. *Linear Functional Analysis*. Springer Undergraduate Mathematics Series. Springer London, 2007.

[4] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1991.

[5] Wikipedia contributors. *Lipschitz continuity — Wikipedia, The Free Encyclopedia*. [Online; accessed 28-August-2019]. 2019. URL: https://en.wikipedia.org/w/index.php?title=Lipschitz_continuity&oldid=906024986.

[6] Apratim De. «Hille-Yosida theorem and some Applications». Central European University, 2017.

[7] A. Fasana and S. Marchesiello. *Meccanica delle vibrazioni*. CLUT, 2006.

[8] S. Salsa. *Equazioni a Derivate Parziali: Metodi, Modelli E Applicazioni*. Unitext: La Matematica Per il 3+2. Springer, 2004.

[9] C. Lanczos. *The Variational Principles of Mechanics*. Dover Books On Physics. Dover Publications, 1986.

[10] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Methods of Mathematical Physics by R. Courant and D. Hilbert v. 1 and 2. Wiley, 1953.

[11] A. Carasso. «Coupled sound and heat flow and the method of least squares». In: *Mathematics of Computation* 29 (1975), pp. 447–463.

[12] A. Matsubara and T. Yokota. «Applications of the Hille-Yosida theorem to the linearized equations of coupled sound and heat flow». In: *AIMS Mathematics* 1 (2016), pp. 165–177.

[13] J.A. Goldstein. *Semigroups of Operators and Applications*. Oxford University Press, 1985.

[14] E Sinestrari. «On the Hille-Yosida Operators». In: *LECTURE NOTES IN PURE AND APPLIED MATHEMATICS* (1994), pp. 537–537.

[15] G. Da Prato and E. Sinestrari. «Differential operators with non dense domain». In: *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* Ser. 4, 14 (1987), pp. 285–344.

[16] G. Da Prato. «Sommes d'opérateurs linéaires et équations différentielles opérationnelles». In: *J. Math. Pures Apple.(9)* 54 (1975), pp. 305–387.

[17] E. Sinestrari. «Product semigroups and extrapolation spaces». In: *Semigroup Forum*. Vol. 71. 1. Springer. 2005, pp. 102–118.

[18] A. Quarteroni. *Modellistica Numerica per Problemi Differenziali*. UNITEXT. Springer Milan, 2016.

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