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On network games with coordinating and anti-coordinating agents

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Summary

Network games are useful in modeling strategic interactions over interconnected systems. When the individual decisions are binary, agents might prefer one action over the other according to the number of neighbors that are choosing it. For instance, in the spread of innovations and beliefs, the utility of an agent choosing a certain action increases with the number of neighbors taking the same action. On the other hand, in games that model dispersion of crowds or division of work, the utility of an agent decreases with the number of neighbors playing the same action. Special cases of the two situations above are modeled as network coordination games and network anti-coordination games, respectively. Thanks to their wide use and the simplicity of their definition, such games have been largely studied in the literature and many results have been proved.

This thesis studies heterogeneous network games where both coordinating and anti-coordinating agents are present. Specifically, the aim of the thesis is to find analytical conditions for the existence of Nash equilibria in the general case where coordinating and anti-coordinating agents have heterogeneous thresholds and interact in the same network.

We first show that the network coordination game, as well as the network anti-coordination game, maintain their well-known potential property even if the players have different incentives in choosing an action over the other. As far as we know, this result is not known in the literature. Furthermore, we present a formulation of the heterogeneous network coordination game in terms of a network coordination game with stubborn players and we show that these two formulations are equivalent. The same method can be applied for the anti-coordination case. We also exhibit a complete characterization of the Nash equilibria of the heterogeneous network anti-coordination game over the complete graph. The analysis is done in terms of the cumulative distribution function of thresholds and generalizes the ideas of the linear-threshold model introduced by Granovetter in 1978 to the case of anti-coordinating

agents.

The main contribution of the thesis is to provide a sufficient condition on the network structure for the existence of Nash equilibria in a mixed coordination anti-coordination game. Specifically, the condition is based on the idea of cohesiveness introduced by Morris in 1997: a subset of the node set is q-cohesive if for any agent in the subset the sum of the weights of the edges pointing to other nodes in the subset is at least a fraction q of the total degree. In the work, we proved that if the subset of the coordinating agents is sufficiently cohesive then the existence of at least one Nash equilibrium is guaranteed over any possible network.

Contents

Introduction

Social interactions influence many decisions of our lives. For instance, when an individual has to choose which product to buy or who to vote for, he might decide according to the opinions of his friends and acquaintances. Network game theory gives essential tools to model and study the interactions of agents who are connected through a network and act according to the behavior of those around them [3]. In the theory, games are defined on graphs: each player adopts a strategy and his outcome depends on both his choice and the actions taken by his neighbors.

This work focuses on two opposite situations that are prominent and widely used in the applications. More specifically, when an innovation is launched on the market, people have higher incentives in adopting the new technology if a large number of friends adopts it too. On the other hand, if a resource is shared, agents might be interested in taking other solutions than those adopted by the people around them. These two relevant examples are referred as games of *strategic complements* and *strategic substitutes* respectively [2],[7]. The main assumption is that the payoffs of the players when choosing an action versus another can either increase or decrease according to the set of the neighbors taking the same action. In particular, in a game of strategic complements, the motivation of a player to adopt a strategy (or more generally a "higher" strategy) increases with the number of neighbors adopting the (higher) strategy. Conversely, in games of strategic substitutes players have opposite incentives.

In the thesis, we concentrate on games having binary action set, which are called *binary network games*. Specifically, *coordinating agents* prefer an action if enough neighbors are playing it, while *anti-coordinating agents* do not want too many friends to take the same action as them. Furthermore, every player is equipped with a personal threshold that captures his incentive in taking one action over the other with no external influences. We remark that the two cases are canonical examples of games with strategic complements and strategic substitutes, respectively.

Network coordination and anti-coordination games have been largely studied in the literature. The focus is usually on whether players will reach equilibrium states where they are all satisfied. Nash equilibria are indeed action configurations where every agent has no incentives in changing *unilaterally* his strategy. Furthermore, one can be interested in predicting which states are they likely to attain and how fast they will get there, depending on the definition of the dynamics, the network structure and the initial condition.

A big challenge in the analysis of strategic interactions over interconnected systems is the inherent complexity of social networks and in general it is not easy to draw conclusions without focusing on specific graph structures. Nevertheless, games of strategic complements and strategic substitutes satisfy some important monotonicity properties that allowed to achieve many results, especially in the case where all the players have the same behavior and the only form of heterogeneity is given by the network structure.

Indeed, it is well-known that, in the homogeneous symmetric case where all the players have the same threshold and the graph is undirected, both the network coordination game and the network anti-coordination game satisfy the potential property introduced by Monderer and Shapley in 1996 [9]. A strategic game is potential if it is possible to express in one single global function the incentive of all the players to change their actions. This is a very strong property: it not only does guarantee the existence of at least one Nash equilibrium, but it is sufficient to prove that the best response dynamics converges to the set of Nash equilibria with probability one in finite time regardless of the topology and the initial condition. Even though a subset of the Nash equilibria can be found by maximizing the potential property, a complete characterization of the Nash equilibria is often computationally unfeasible and hard to find, especially for the anti-coordination case.

Games with coordinating agents, in particular, well-behave in many situations. For instance, a consensus configuration is always a Nash equilibria regardless of the network structure. Morris [11] provided a full characterization of the Nash equilibria in the network coordination game. This characterization is based on the idea of *cohesiveness* and permits to find sufficient and necessary conditions for contagion. In simple words, if the interactions among players are binary, a subset of the vertex set is q-cohesive if for any player in the subset it holds that at least a fraction q of his neighbors belongs to the subset. As we shall see, this concept is fundamental in this thesis. Many other results have been accomplished in terms of the convergence of network dynamics and how the behavior of the system depends on the network structure in the symmetric case [4],[8],[10].

The analysis of the games with heterogeneous thresholds is more delicate. This problem was firstly addressed by Granovetter in [6] by proposing a linear-threshold model to study a fully-connected population of coordinating agents. In the model, the dynamics is completely described in terms of the threshold cumulative distribution function. Similar results can be generalized for configuration models [13].

We recall that the existence of Nash equilibria is not an issue for games with coordinating agents since the consensus is always an equilibrium state. On the other hand, the existence of Nash equilibria is not in principle guaranteed for the anti-coordination case and in general the characteristics of the equilibrium configurations are not trivial. Ming Cao et al [12] proved in 2016 that anti-coordinating agents, as well as coordinating agents, tend to reach satisfactory situations also with heterogeneous thresholds. In fact, they showed that a Nash equilibrium is almost surely reached through an asynchronous or partially synchronous best-response dynamics on every network topology and with any distribution of thresholds. They conclude the paper by pointing out that, while complexity of the network structure and population heterogeneity are not sufficient to cause nonconvergence issues, a possible source of irregular behaviors is given by mixture of coordinating and anti-coordinating agents.

The aim of the thesis is to investigate the existence of Nash equilibria in the general case of heterogeneous coordinating and anti-coordinating agents interacting over the same network.

In the first chapter, we provide the background definitions that underlie the case studies of the thesis. We start by recalling the basic elements of graph theory and game theory. Then, we explain how the two theories are combined in network games. Furthermore, we give the definition of potential games and we highlight some of the properties that they share.

In the second chapter, we introduce network coordination games with heterogeneous thresholds and we make some relevant observations. In particular, we show that the network coordination game preserves the potential property also when the pairwise interactions are not symmetric. Furthermore, we provide a formulation of the game in terms of a network coordination games with stubborn players.

In the third chapter, we define the network game with anti-coordinating agents and we make similar considerations as the ones of the previous chapter. Specifically, we prove that the anti-coordination game with heterogeneous thresholds is a potential game and we provide an analogous way to rewrite the game in terms of an anti-coordination game with stubborn players. Thanks to the potential property, the existence of at least one Nash equilibrium is guaranteed over any possible network. In this chapter, we also provide a complete characterization of the Nash equilibria of the game when the population is fully-connected. In this particular topology, the game can be studied in terms of the cumulative distribution function of the thresholds, in a similar way to the linear-threshold model.

In the fourth chapter, we finally give the definition of the mixed coordination anti-coordination game where heterogeneous coordinating and anti-coordinating agents interact over the same network. We begin the analysis by observing that it is enough to introduce one edge between a coordinating and anti-coordinating agent to lose the potential property. Thereafter, we recall the definition of alpha-cohesiveness and we apply it to our case. For instance let us assume that coordinating players have homogeneous thresholds q. We observe that if the set of the coordinating agent is q-cohesive, then they are in equilibrium in a consensus configuration regardless of the actions of the players outside the subset. According to this observation, we prove that the existence of at least one Nash equilibrium is guaranteed when the set of coordinating agent is sufficiently cohesive. We conclude the chapter by investigating the existence of Nash equilibria in some simple graph structures where the previous condition is not satisfied.

In the last chapter, we draw the conclusions and we discuss some future research directions.

Chapter 1

Network games

The aim of this chapter is to provide the background definitions that are needed to understand the following sections.

In particular, we present the basic elements of *graph theory*, which is fundamental for networks analysis, and *game theory*, which is used to model strategic interactions among rational agents. As we shall see in the last section, the two theories can be combined in *network game theory*, which aims to model situations where adaptive players interact over interconnected systems.

Furthermore, we introduce the concept of potential games, which is crucial for the aim of the thesis. In fact, potential games share important properties and one of the main purposes of the thesis is to investigate how they change when the potential game is perturbed and, in particular, when two different potential games are mixed together.

1.1 Graphs

Graph theory, a discrete mathematics sub-branch, is a strong tool for the study of complex systems such as networks. In this section, we present the basic elements of the theory and we give some relevant examples.

The interconnection among elements in a network is represented through *graphs*, mathematical structures that model pairwise relations between units.

Definition 1. A *(directed weighted) graph* is a triple

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$$

where $\mathcal{V}$ is the countable set of *nodes* (or *vertices*), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of *edges* or *links* and $W \in \mathbb{R}_+^{\mathcal{V}\times\mathcal{V}}$ is the *weight matrix*, such that $W_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$, namely if (i,j) is an edge.

In other words, the main aspects of the network are captured in three features:

1. The set of vertices $\mathcal{V}$ is composed of the units that constitute the network. Nodes in a graph may represent people or objects, economic individuals or biological entities, geographical points or elements in a mains.

2. The set of links $\mathcal{E}$ tells us if there exists a direct connection between two elements of the network. Specifically, a link is an ordered pair (i,j), $i,j \in \mathcal{V}$ and its presence represents the existence of an ordered relation between node i and node j. For example, if nodes are rational individuals, it may mean that node i influences node j or conversely that node i gets influenced by node j. On the other hand, if we are dealing with a transport network, it can indicate a direct way from i to j. A link (i,i) from a node to itself is called *self-loop*.

3. It may occur that distinct edges have non-identical meaning in the network. The *weight matrix* W associates a positive scalar W_{ij} to each edge (i,j) and it is used to diversify the links. This quantity may measure how strong is the influence of one node over another or, in the case of network flows, the conductance or the capacity of the edge.

The *size* of the graph is given by the number of vertices in $\mathcal{V}$ and it is denoted by $n = |\mathcal{V}|$. The set $\mathcal{V}$ is typically, but not always, finite. If $n < \infty$, we can relabel the nodes of the graph and identify $\mathcal{V}$ with the set $\{1,2,\ldots,n\}$. In this way, we obtain $W_{ij} \in \mathbb{R}^{n\times n}$.

Observation 1. More generally, we identify two graphs when they are isomorphic, that is when we can obtain one graph from the other through a relabeling of the nodes. Formally, two graphs $\mathcal{G}^{(i)} = (\mathcal{V}^{(i)}, \mathcal{E}^{(i)}, W^{(i)})$ for $i = 1,2$ are called *isomorphic* if there exists a bijection $f : \mathcal{V}^{(1)} \to \mathcal{V}^{(2)}$ such that:

1. $(i,j) \in \mathcal{E}^{(1)}$ if and only if $(f(i), f(j)) \in \mathcal{E}^{(2)}$

2. $W_{ij}^{(1)} = W_{f(i)f(j)}^{(2)}$ for all $i,j \in \mathcal{V}^{(1)}$.

1.1.1 Types of graphs

In the theory, different types of graphs are defined according to the weight matrix W. Specifically, a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is called

- *unweighted* if $W \in \{0,1\}^{V \times \mathcal{V}}$, namely if $W_{ij} = 1$ for all $(i,j) \in \mathcal{E}$ and $W_{ij} = 0$ for every $(i,j) \notin \mathcal{E}$. Note that the weight matrix can be entirely obtained from the set of the edges. Therefore, an unweighted graph is often denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In this case, the weight matrix is called *adjancency matrix*.

- *undirected* if $W = W'$ is symmetric, that is if $W_{ij} = W_{ji}$ for all $i, j \in \mathcal{V}$. This is the same as asking that if $(i,j) \in \mathcal{E}$ then $(j,i) \in \mathcal{E}$ too and, more specifically, the two links must have the same weight. Therefore, the two directed links (i,j) and (j,i) can be represented by one undirected link $\{i,j\}$. This notation is often used to emphasize the fact that the order does not matter.

- *simple* if it is undirected, unweighted and contains no self-loops, that is $W_{ii} = 0$ for all $i \in \mathcal{V}$.

Unless it is explicitly said or evident from the context, when we mention a graph we intend the more general definition of a directed weighted graph. Note that this definition is still not the most general. For instance, it does not consider *multigraphs*, which are graphs where it is allowed to have more than one link between two vertices.

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. A graph $\mathcal{H} = (\mathcal{U}, \mathcal{F}, Z)$ such that $\mathcal{U} \subseteq \mathcal{V}$, $\mathcal{F} \subseteq \mathcal{E}$ and $Z_{ij} \leq W_{ij}$ for $i, j \in \mathcal{U}$ is called *subgraph* of $\mathcal{G}$.There are two types of subgraphs that are particularly relevant in the theory:

- Given $\mathcal{U} \subseteq \mathcal{V}$, the *subgraph* of $\mathcal{G}$ *induced* by $\mathcal{U}$ is given by $\mathcal{G}[\mathcal{U}] = (\mathcal{U}, \mathcal{F}, W_{|\mathcal{U} \times \mathcal{U}})$ where
$$\mathcal{F} = \{(i,j) \in \mathcal{E} : i, j \in \mathcal{U}\}$$
In words, given a subset of the nodes, the induced subgraph is obtain by removing all the links that involve nodes which do not belong to the given subset, namely nodes in $\mathcal{V} \setminus \mathcal{U}$.

- Given $\mathcal{F} \subseteq \mathcal{E}$, the *spanning subgraph* is a graph $\mathcal{H} = (\mathcal{V}, \mathcal{F}, Z)$ having
$$Z_{ij} = \begin{cases} W_{ij} & \text{if } (i,j) \in \mathcal{F} \\ 0 & \text{if } (i,j) \notin \mathcal{F} \end{cases} \qquad i, j \in \mathcal{V}$$

In this case, we start from a subset of the edges: we keep the same set of nodes and we obtain the subgraph by removing the links that are not in the given subset.

1.1.2 Neighborhood and degree

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, we introduce the following notation that will be useful in the next sections. The *out-neighborhood* of node $i \in \mathcal{V}$ is defined as

$$N_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$$

and its elements are called *out-neighbors*. Similarly, the set of the *in-neighbors* of node $i \in \mathcal{V}$, namely its *in-neighborhood*, is given by

$$N_i^- = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$$

Furthermore, we define the *out-degree* and the *in-degree* of node $i \in \mathcal{V}$, respectively, as

$$w_i = \sum_{j \in \mathcal{V}} W_{ij} \qquad \text{and} \qquad w_i^- = \sum_{j \in \mathcal{V}} W_{ji}$$

If the graph is undirected, $N_i = N_i^-$ and $w_i = w_i^-$, namely the out-neighborhood and in-neighborhood coincide, as well as the out-degree and the in-degree. In this case, we naturally refer to N_i as the *neighborhood* of node $i \in \mathcal{V}$, and to w_i as its *degree*. In the matter of directed graphs, the term degree is sometimes used in place of out-degree.

We can obtain a compact notation for the out-degree and in-degree of the nodes by defining

$$w := \mathbb{1}W \qquad \text{and} \qquad w^- := W'\mathbb{1}$$

We refer to w and w^-, respectively, as the *out-degree* and the *in-degree vectors* of the graph. Following the same idea, the *total degree* of the graph is given by $\mathbb{1}'W\mathbb{1}$, while

$$\overline{w} = \frac{1}{n}\mathbb{1}'W\mathbb{1}, \qquad n = |\mathcal{V}|$$

is the *average degree*. If $w = w'$, the graph is called *balanced* since all its nodes are so ($w_i = w_i^-$ for all $i \in \mathcal{V}$). Furthermore, a graph is *regular* if $w = w' = \overline{w}\mathbb{1}$.

Notice that, if the graph is simple, the total degree is an even number (*hand-shaking lemma*). This comes straightforward from the fact that the total degree corresponds to the number of edges of the graph, which is even, namely

$$\mathbb{1}'W\mathbb{1} = \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} W_{ij} = \sum_{(i,j)\in\mathcal{E}} 1 = |\mathcal{E}|$$

In a simple graph, the number of edges is even since we are counting each undirected edge $\{i,j\}$ twice: one as (i,j) and one as (j,i).

1.1.3 Reachability

Reachability is a fundamental concept in graph theory. In fact, determining if a graph or a subgraph is connected or disconnected is important to investigate the strength of the connections among elements in the network.

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. We recall that a *walk* from a vertex i to a vertex j is a sequence of nodes $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$, where l is the *length* of the walk, such that $\gamma_0 = i$, $\gamma_l = j$ and $(\gamma_k, \gamma_{k+1}) \in \mathcal{E}$ for $k = 0, \dots, l-1$. In other words, in a walk every two consecutive nodes are linked by an edge.

Definition 2. A node j is *reachable* from a node i if there exists a walk from node i to node j.

A walk such that $\gamma_0 = \gamma_l$ is called a *circuit*. A *path* is a walk with no repeated nodes, except eventually $\gamma_0 = \gamma_l$, namely where each transition node appears at most once ($\gamma_i \neq \gamma_j$ for any $i,j \in \{0,\dots,l-1\}$). If $\gamma_0 = \gamma_l$ and $l \geq 3$, the path is called a *cycle*. We remark that self-loops $(i,i) \in \mathcal{E}$ and walks such as (i,j,i), with $(i,j),(j,i) \in \mathcal{E}$, are circuits but not cycles ($l < 3$).

Definition 3. A graph $\mathcal{G}$ is called *circuit-free* if it does not contain any circuit, *acyclic* if it has no cycles.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, the *distance* between a node i and a node j is defined as the length of the shortest path from i to j if j is reachable from i, while it is set to infinity otherwise. In formulas,

$$\mathrm{dist}(i,j) = \begin{cases} \min_{\substack{\gamma=(\gamma_0,\dots,\gamma_l) \\ \text{s.t. } \gamma \text{ path}}} l & \text{if } j \text{ is reachable from } i \\ +\infty & \text{otherwise} \end{cases}$$

The *diameter* of a graph $\mathcal{G}$ is the maximum distance between two nodes of the graph, namely

$$\mathrm{diam}(\mathcal{G}) = \max_{i,j\in\mathcal{V}} \mathrm{dist}(i,j)$$

Definition 4. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is *strongly connected* if, for any $i, j \in \mathcal{V}$, node j is reachable from node i, which is the same as asking $\text{diam}(\mathcal{G}) < \infty$.

A *connected component* (or shortly *component*) of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is an induced subgraph $\mathcal{G}' = \mathcal{G}[\mathcal{V}']$ such that

1. $\mathcal{G}'$ is strongly connected
2. $i \in \mathcal{V}'$, $(i, j) \in \mathcal{E}$ implies $j \in \mathcal{V}'$

In other words, a component is a maximal connected subgraph. With the term *maximal*, we mean that we cannot find a bigger subset of nodes such that the induced subgraph is connected and contains the component.

Note that, if we identify the component with its vertex set, connected components form a partition of the nodes set of the graph. The component of a node $i \in \mathcal{V}$ is the component that contains i. The component with the maximum number of nodes is called *largest connected component* of the graph.

1.1.4 Examples of graphs

In this unit, we present three important families of graphs and we give some relevant examples. As we shall see in the next chapters, those examples are very useful in the applications. Indeed, they represent a fundamental first step in the analysis of the behavior of the more complicated real networks. Their simplicity usually permits rigorous and general results. Furthermore, they represent extreme cases and they can be used to underline the main characteristics of the network and study their effects.

Regular graphs

Recall that in a regular graph all nodes have the same degree. A regular graph where nodes have degree equal to k is called a *k-regular* graph or regular graph of degree k.

Let $n \in \mathbb{N}$. Some examples of regular graphs are:

- The *ring graph*, which is a 2-regular simple graph, namely a simple graph where each node has degree equal to 2. Such a graph may appear as a circle where each vertex communicates with the preceding node and next one. The ring graph with n nodes is denote by $\mathcal{C}_n$.

- The *complete graph*, which is a regular graph of degree $k = n - 1$, namely each node has degree $n - 1$. In other words, the complete graph is a simple graph where each vertex is linked to all the other vertices of the graphs. As a consequence, the graph has $\binom{n}{2}$ undirected edges, which is the maximum number of links in a simple graph. The complete graph with n nodes is denoted with $\mathcal{K}_n$ and it is used to model *fully connected* networks.

Consider two complete graphs with n_1 and n_2 nodes respectively. The *barbell graph* is the simple graph obtained by adding one link between a node in the first graph and a node in the second graph. In the applications, it may represent the extreme case of two fully connected groups which communicate just over one link, namely two very closed communities that are not completely isolated.

Bipartite graphs

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is called *bipartite* if there exists a partition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ such that $W_{ij} = 0$ for all $i, j \in \mathcal{V}_1$ and $W_{ij} = 0$ for all $i, j \in \mathcal{V}_2$. Namely, a graph is bipartite if we can divide the nodes set in 2 subsets such that the subgraphs induced by the subsets have only isolated nodes, that is there are no edges between nodes in the same subset.
The following proposition gives a characterization of bipartite graphs.

Proposition 1. A graph is bipartite if and only if it contains no cycles of odd length.

Let us introduce an important subfamily of bipartite graphs.

Definition 5. A *tree* is a connected acyclic simple graph.

Graphs in this family have many interesting properties. For instance, note that all trees are bipartite. Indeed, trees are special cases of Proposition 1 since they contain no cycles at all.

Moreover, it is possible to prove that a connected simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n nodes and m undirected edges is a tree if and only if $m = n - 1$.

Consider $n \in \mathbb{N}$. Two examples of trees with n nodes are:

- The *line graph*, which is denoted by $\mathcal{L}_n$. A line graph is a ring graph with one removed link, namely $\mathcal{L}_n = (\{1, \dots, n\}, \mathcal{E})$ such that $\mathcal{E} = \{(i, i+1), (i+1, i), i = 1, \dots, n-1\}$.

- The *star graph*, where all nodes but one have degree one. As a consequence, the remaining node has degree $n-1$. A star graph with n nodes is denoted by $\mathcal{S}_n$.

A *leaf* is a node with degree one. In a line graph, there are 2 leaves, while a star graph has $n-1$ leaves. In general, the number of leaves of a tree is least 2 and at most $n-1$.

Random graphs

Previously, we gave examples of deterministic graphs. In this last part, we explore some techniques for generating random graphs, which are graphs drawn from given probability distributions. In this way, we can study a wider spectrum of networks, which still preserve some common characteristics that vary according to the chosen probability distribution.

A well-known family of random graphs is such of Erdös-Renyi random graphs [5],[1]. This class of graphs is vastly used in the applications for the simplicity of their definition,.

Definition 6. The *Erdös-Renyi random graph* $\mathcal{G}(n,p)$, with $n \in \mathbb{N}$ and $p \in [0,1]$, is a random graph with vertex set $\mathcal{V} = \{1,2,\ldots,n\}$ where each edge $\{i,j\} \in \mathcal{E}$ is present independently with probability p.

In other words, we start with an empty graph with vertex set $\mathcal{V} = \{1,\ldots,n\}$ and perform $\binom{n}{2}$ Bernoulli experiments inserting edges independently with probability p.

Observation 2. The properties of those random graphs have been largely studied and many results in terms of degree distribution and connectivity have been achieved, although they are not strictly necessary for the scope of the thesis. We just show a straightforward result on the expected number of edges, which is twice the number of undirected edges that is by definition a binomial random variable, namely

$$\mathbb{E}[|\mathcal{E}|] = 2\binom{n}{2}p = 2\frac{n(n-1)p}{2} = n(n-1)p$$

As a consequence, the expected average degree is $\mathbb{E}[\overline{w}] = \mathbb{E}[|\mathcal{E}|/n] = (n-1)p$. This fact should be considered when choosing the parameters p and n.

Let us point out that Erdös-Renyi random graphs can be also defined by setting the number of undirected edges of the graph, instead of giving the probability of an edge to appear.

Definition 7. The *uniform random graph* $\mathcal{G}(n,m)$, with $n \in \mathbb{N} = \{1,2,\dots\}$ and $0 \leq m \leq \binom{n}{2}$, is a random graph with n vertices and m edges where the m edges are drawn uniformly random from the set of all possible edges. Equivalently, we can define $\mathcal{G}_{n,m}$ as the family of all labeled graphs with vertex set $V = \{1,2,\dots,n\}$ and exactly m edges and assign to each graph $\mathcal{G} \in \mathcal{G}_{n,m}$ the probability

$$\mathbb{P}(\mathcal{G}) = \binom{\binom{n}{2}}{m}^{-1}.$$

In this case, we start with an empty graph with vertex set $\mathcal{V} = \{1,\dots,n\}$ and insert m edges in such a way that all possible $\binom{\binom{n}{2}}{m}$ choices are equally likely.

In general, $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$ should behave in a similar way for n large if m is close to the expected number of edges in $\mathcal{G}(n,p)$, namely if $m \approx \frac{n^2 p}{2}$. Again, some rigorous results have been proved in the matter of the relations between the two.

1.2 Games

In this section, we introduce the basic concepts of classical game theory, a branch of mathematics that provides tools to study competitive situations.

In a game, rational individuals, which are called players, interact strategically with the aim of maximizing an outcome that depends on both their action and the choices of the other players.

Definition 8. A *(strategic form) game* is a triple

$$\mathcal{U} = (\mathcal{V}, \mathcal{A}, \{u_i\}_{i \in \mathcal{V}})$$

where $\mathcal{V}$ is the finite set of *players*, $\mathcal{A}$ is the set of *actions* and $\{u_i\}_{i \in \mathcal{V}}$ is the set of the *utility* functions (a.k.a. *reward* or *payoff* functions). For each player $i \in \mathcal{V}$, the function

$$u_i : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$$

returns the outcome $u_i(x)$ of player i when an *action configuration* (or *profile*) $x \in \mathcal{A}^{\mathcal{V}}$ is chosen by the players, namely when every participant $j \in \mathcal{V}$ plays action $x_j \in \mathcal{A}$. The set $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$ of all the possible action profiles is called *configuration space.*

In words, given the set of participants that play in the game and the family of the actions that they are allowed to choose, the strategic interest of the players is quantify through the utility functions, which define how the profit of each player varies according to his action and the choices of the rest of the participants.

Remark 1. Note that the definition mentions a unique action set $\mathcal{A}$ that is the same for all the players. A more general definition with multiple $\mathcal{A}_i$, $i \in \mathcal{V}$, can be found in literature. Since in the thesis we focus on games where all the players have the same action set, we decided to give a more compact definition. Moreover, in the case studies the action set is finite but it can also be continuous, for instance the set of real numbers.

Observation 3. We denote by

$$x_{-i} = x_{|\mathcal{V}\setminus\{i\}}$$

the vector obtained by removing from a configuration $x \in \mathcal{A}^{\mathcal{V}}$ the action of player $i \in \mathcal{V}$. In this way, we can emphasize the fact that the outcome of player i depends, on one side, on the action that he rationally chooses and, on the other, on what the other participants, who play autonomously and with individual aims, decide to do. Therefore, with a slight abuse of notation, we often write

$$u_i(x_i, x_{-i}) = u_i(x) \tag{1.1}$$

A fundamental fact in game theory is that players are assumed to be *rational*, namely each individual chooses the action and the strategy with the aim of maximizing his reward, which varies according to the action configuration of all the players. Therefore, it is reasonable to introduce the idea of best response.

Definition 9. The *best response* (BR) function of player $i \in \mathcal{V}$ is defined as the set-valued function

$$\mathcal{B}_i(x_{-i}) = \text{argmax}_{x_i \in \mathcal{A}}\, u_i(x_i, x_{-i})$$

In simple words, assuming that player $i \in \mathcal{V}$ is aware of the actions of the rest of the players, the best response function returns the best alternatives for player i in the given situation, namely the action, or the actions, that maximizes the utility according to the choices of the other participants. Note that $\mathcal{B}_i(x_{-i})$ can be empty if $\mathcal{A}$ is not finite.

We conclude this section by giving a formal definition of the fundamental concept of Nash equilibrium.

Definition 10. A *(pure strategy) Nash equilibrium* (NE) for the game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is an action configuration $x^* \in \mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \qquad i \in \mathcal{V} \tag{1.2}$$

The main idea behind the definition is that, in a Nash equilibrium, no player has any incentive of changing *unilaterally* his action. Indeed, for any agent $i \in \mathcal{V}$, the chosen action x_i^* belongs to the best response and therefore it is one of the best alternatives in the current profile. In fact, the utility that the player gets is the best possible given the present choices of the other players. Note that other configurations where one, more than one or even all the players have higher outcomes might exist, but they are not reachable unless more than one player changes his action.

We remark that the existence of a Nash equilibrium is in general not guaranteed, as well as its uniqueness. In the thesis, we denote as $\mathcal{N}$ the set of Nash equilibria of a game, which can be empty or include more than one element.

1.2.1 Potential games

We conclude this section by giving the definition of a special class of games that have remarkable behaviors.

Definition 11. A game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is a *potential game* if there exists a function $\Phi : \mathcal{X} \to \mathbb{R}$, which is called *potential function* of the game, such that for any player $i \in \mathcal{V}$ and any $x_{-i} \in \mathcal{A}^{\mathcal{V}\setminus\{i\}}$ it holds that

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) \tag{1.3}$$

for all $x_i, y_i \in \mathcal{A}$.

In words, the variation of the utility incurred by the player i when she switches the action from x_i to y_i, and the rest of the agents keep playing x_{-i}, is equal to the corresponding variation of the potential function.

We remark that the potential function is not indexed: it is the same one for all the players of the game. Here stands the strength of this property. The meaning of finding a potential function is that it is possible to express in one single global function the incentive of all the players to change their action.

One of the most important properties of potential games is that the existence of at least one Nash equilibrium is guaranteed. As stated in the following proposition, a nonempty subset of the Nash equilibria can be found by maximizing the potential function.

Proposition 2. Consider a finite potential game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ with potential function $\Phi(x)$. Then, the game admits at least one Nash equilibrium and, in particular, the set $\mathcal{N}$ of the Nash equilibria contains the set

$$\mathcal{X}^* := \mathrm{argmax}_{x \in \mathcal{X}} \Phi(x)$$

of global maximizers of the potential function.

For the sake of completeness, we specify that the results hold also when the function Φ satisfies a slightly weaker condition than the one in equation (1.3), which is

$$\mathrm{sign}(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})) = \mathrm{sign}(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}))$$

In this case, the Φ is called *ordinal potential function* and the game is referred as *ordinal potential game.*

1.3 Games on graphs

This section is dedicated to the definition of games on networks. The set players of a network game coincide with the set of the nodes of a given graph and, specifically, each agent plays a two-player game with every neighbor. Therefore, we start by describing the properties of two-player games. This simple class of games is useful in the applications and underlies the definition of network games. Then, we give the formal definition of a network game and we make the first observations. In particular, there is a special way to construct a network game that leads for sure to a potential game.

1.3.1 Two-player games

Games having only two players are the simplest example of games. Given the action set $\mathcal{A}$, two-player games are defined by just two utility functions $u_i(r, s)$, $i = 1{,}2$, where we denote as r the action played by agent i, while s is the action chosen by his opponent.

Let us consider the special case where the two utility functions coincide. In this instance, the game is called a *symmetric two-player game* and we can define a unique function $\varphi : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ such that

$$\varphi(r, s) = u_1(r, s) = u_2(r, s), \qquad r, s \in \mathcal{A} \tag{1.4}$$

which gives the utility of a player choosing action r when the opponent picks action s.

If $\mathcal{A}$ is finite, the game can be summarized in a *payoff matrix*. Each row represents a possible action for player 1, while columns correspond the actions of player 2. According to it, each entry contains the comma-separated payoffs of player 1 and 2 given that the first agent plays action r while the second one chooses action s. In formulas, in the $(r-s)$-entry, we find, in order, $u_1(r,s)$ and $u_2(r,s)$. In the case of a symmetric two-player game, the previous values coincide, by definition, with $\varphi(r,s)$ and $\varphi(s,r)$.

The payoff matrix is particularly simple when there are just two possible choices of actions, as shown in Figure 1.1. Two-player games having $|\mathcal{A}| = 2$ are usually called 2×2-*games*.

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

Figure 1.1. Payoff matrix of a 2×2-game, having $\mathcal{A} = \{-1, +1\}$.

Note that every 2×2 game with payoff matrix as in Figure 1.1 is potential. In particular, the potential function is given by

$$\Phi(-1,-1) = a - c, \quad \Phi(+1,+1) = b - d, \quad \Phi(-1,+1) = \Phi(+1,-1) = 0 \tag{1.5}$$

The next two examples are special cases of 2×2 symmetric games and therefore they admit the potential function in (1.5).

Example 1 (Coordination game)**.** Let us consider the 2×2-game in Figure 1.1 having $\mathcal{A} = \{-1, +1\}$. The game is called *coordination game* if

$$a > c \qquad \text{and} \qquad b > d$$

In fact, note that under this assumption

$$\mathcal{B}_1(+1) = \mathcal{B}_2(+1) = +1, \qquad \mathcal{B}_1(-1) = \mathcal{B}_2(-1) = -1$$

In words, in a coordination game, the best response of a player is to copy the action of the other player, namely both players want to coordinate themselves with the opponent.

A straightforward consequence is that the Nash equilibria of the game are $(+1,+1)$ and $(-1,-1)$, which are the two possible action configurations where the players pick the same action.

Note that the two Nash equilibria of the game may be not equally satisfactory for the players. In particular, if $a > b$ (respectively $b > a$), the configuration $(-1, -1)$ (respectively $(+1, +1)$) is called *payoff dominant*.

Example 2 (Anti-coordination game)**.** The *anti-coordination game* is again a special case of a 2×2-game like the one in Figure 1.1, but, conversely to the previous case, we need

$$a < c \qquad \text{and} \qquad b < d$$

This condition implies

$$\mathcal{B}_1(+1) = \mathcal{B}_2(+1) = -1, \qquad \mathcal{B}_1(-1) = \mathcal{B}_2(-1) = +1$$

which means that, in this case, the best response of a player is to choose the opposite action of the other player, namely both players want to differentiate themselves from the opponent.

In the anti-coordination game, the two Nash equilibria are given by $(+1, -1)$ and $(-1, +1)$. Yet again, the two Nash equilibria may give different outcomes to the players.

Example 3 (Discoordination game)**.** Differently from the two previous examples, the payoff matrix of the *discoordination game* is

	-1	+1
-1	a,b	c,d
+1	c,d	a,b

In particular, the entries must be such that

$$a > c \qquad \text{and} \qquad d > b$$

An important observation is that the discoordination game is not symmetric since utility functions of the two players do not coincide. In particular, player 1 goes for a configuration where they match, while player 2 prefers an outcome where they play opposite actions. A consequence is that discoordination games admit no Nash equilibria.

1.3.2 Network games

The main idea in the definition of a network game is that players are identified with nodes of a graph and, therefore, interactions among them are defined according to the graph structure. Indeed, in a network game, the utility of a player depends on his choice and on the action configuration of the out-neighborhood. Note that any game can be seen as a network game over the complete graph. Let us give the formal definition.

Definition 12. A triple $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is a *network game* over the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ (or shortly a $\mathcal{G}$*-game*) if, given any player $i \in \mathcal{V}$, the utility function $u_i : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ satisfies

$$u_i(x) = u_i(y)$$

for all $x, y \in \mathcal{A}^{\mathcal{V}}$ such that $x_j = y_j$, for every $j \in N_i \cup \{i\}$.

Above, we provided the most general definition of a network game. Indeed, the only request is that payoffs do not change according to the actions of players which are not in the out-neighborhood.

We now give a slightly more restrictive definition which also requires that every pair of neighbors plays a two-player game. This definition is more specific since the utility functions are defined as the weighted sum of the payoffs received from every two-player game.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be undirected and with no self-loops. For any two neighbors $i, j \in \mathcal{V}$ we consider a two-player game having utilities $\varphi^{(i,j)}(a, b) : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ and $\varphi^{(j,i)}(a, b) : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$. Recall that $\varphi^{(i,j)}(a, b)$ is the payoff of player i when he plays action a and his opponent plays action b. In this instance, the *network game* $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ over $\mathcal{G}$ has utilities

$$u_i(x) = \sum_j W_{ij} \varphi^{(i,j)}(x_i, x_j) \tag{1.6}$$

The payoff functions $\varphi^{(i,j)}$ and $\varphi^{(j,i)}$, $\{i, j\} \in \mathcal{E}$, are called *interaction utilities*.

The graph is assumed to be undirected for simplicity: in this way, we can say that each couple of neighbors play a two-player game. The definition can be easily generalized to the case of directed graphs by defining an interaction utility for each directed edge (i, j).

Recall that, in the previous chapter, we provide the definition of potential games, a class of games that share important properties. In particular, we point out that if a game has a potential function, then the set of the Nash equilibria is non-empty and contains all the action configurations that maximize the potential function.

The following proposition identifies a type of network games for which it is always possible to find a potential function. In fact, if the underlying two-player game is symmetric, then if the graph is undirected the property transfers to the network game and the the potential can be found as a function of the two-player potential function.

Proposition 3. Consider an undirected $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and assume that for every edge $\{i, j\}$, and therefore for every two neighbors i and j, the interaction utility is such that $\varphi^{(i,j)}(a, b) = \varphi^{(j,i)}(a, b)$, which means the two-player game is symmetric. In addition assume that every two-player game has a potential function $\phi^{\{i,j\}}$. Then, the network game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ having the utilities in (1.6) is also a potential game with potential

$$\Phi(x) = \frac{1}{2} \sum_{i,j\in\mathcal{V}} W_{ij}\phi^{i,j}(x_i, x_j) \tag{1.7}$$

As we shall see, both the network coordination game and the network anti-coordination game satisfy the hypothesis of the theorem and thus they are potential games. This fact implies that they always admit Nash equilibria. However, even though the potential function is a useful tool to look for equilibria, it is sometimes problematic to determine an explicit characterization of all the Nash equilibria of the game. This is usually due to the computational issues arising with complex large-scale networks.

Chapter 2

Heterogeneous network coordination game

In the thesis, we consider examples of network games having binary action set, which are called *binary network games*. Therefore, from now on we consider $\mathcal{A} = \{-1, +1\}$, which is the action set of the game.

Recall that in a network game, agents are influenced by people around them. In this chapter, we present a formal way to model a game where agents have coordinating incentives. In general, a *coordinating* agent prefers an action if enough neighbors are playing it. Furthermore, we introduce a form of *heterogeneity* among the agents. Indeed, every agent is equipped with a value which represent their personal incentive in preferring an action over the other which does not depend on the people around them.

In the first section, we introduce the definition of *heterogeneous network coordination game* and we make the first basic observations. In the literature, network coordination games have been widely studied and a lot of results have been achieved regarding their static properties and dynamics behaviors. This fact is not true for the heterogeneous case, whose behavior is less known. In the second section, we show that the heterogeneous network coordination game is a potential game. We conclude the chapter by presenting an equivalent formulation of the game in terms a network coordination game with stubborn players.

2.1 The heterogeneous network coordination game

Let us consider an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a set of given node weights $\{\alpha_i\}_{i\in\mathcal{V}}$, $\alpha_i \in \mathbb{R}$. We define the *heterogeneous network coordination game* with action set $\mathcal{A} = \{-1, +1\}$ as a network game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ with utilities $\{u_i\}_{i\in\mathcal{V}} : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$

$$u_i(x_i, x_{-i}) = \sum_{j\in\mathcal{V}} W_{ij} x_i x_j - \alpha_i x_i, \qquad i \in \mathcal{V} \tag{2.1}$$

Note that, according to equation (2.1), the payoff of an agent increases as the weighted sum of the neighbors choosing the same action gets larger, while it decreases with the rising of the weighted sum of the neighbors playing the opposite one. Therefore, this game models coordination interests, since agents have a positive incentive in choosing the same action of a neighbor and a disadvantage in picking the opposite one. Namely, the underlying assumption is that agents are interested in adopting a strategy when they interact with other agents who adopt it too.

Furthermore, we introduce a node weight α_i that represents the personal interest of a player in choosing an action over the other. For instance, let us consider the coming of a new technology on the market. If we denote with -1 the old strategy and with $+1$ the new one, the threshold α_i determines how many friends have to adopt the innovation before agent i adopts it too. Observe that if $\alpha_i > 0$ it means that, when the weighted sum of the neighbors playing $+1$ coincides with the weighted sum of the neighbors playing conversely, the preferable action for player i is -1. In the literature, in this case action -1 is called *risk-dominant*. On the other hand, if $\alpha_i < 0$, it conveys that agent i would pick action $+1$ with no external influences. If $\alpha_i = 0$ the two actions are called *risk-neutral*.

Given an agent $i \in \mathcal{V}$ and any choice of actions $x_{-i} \in \mathcal{A}^{\mathcal{V}\setminus\{i\}}$, we have that $+1 \in \mathcal{B}_i(x_{-i})$, which means by definition that $u_i(+1, x_{-i}) \geq u_i(-1, x_{-i})$, if and only if

$$\sum_{j\in\mathcal{V}} W_{ij} x_j - \alpha_i \geq -\sum_{j\in\mathcal{V}} W_{ij} x_j + \alpha_i$$

From which we obtain

$$\sum_{j\in\mathcal{V}} W_{ij} x_j \geq \alpha_i$$

Therefore, the *best response function* for an agent $i \in \mathcal{V}$ is given by:

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j > \alpha_i \\ \{-1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j < \alpha_i \\ \{\pm 1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j = \alpha_i \end{cases} \tag{2.2}$$

The meaning of the value α_i is more evident in this formulation of the game. Recalling that $w = W\mathbb{1}$ denotes the degree vector, we observe that, if $\alpha_i > w_i$, then $\mathcal{B}_i(x_{-i}) = \{-1\}$ for any possible choice of actions $x_{-i} \in \mathcal{A}^{\mathcal{V}\setminus\{i\}}$. Similarly, if $\alpha_i < -w_i$, then $\mathcal{B}_i(x_{-i}) = \{+1\}$ for any action configuration. We will call *stubborn agents* the players having $\alpha_i > w_i$ or $\alpha_i < -w_i$, as their best response function is constantly equal to $\{-1\}$ or $\{+1\}$, respectively, independently of the actions of the other players. In words, stubborn players are not influenced by the choices of the agents around them.

Observation 4. If there are no stubborn players, the two consensus configurations, namely $\mathbb{1}$ and $-\mathbb{1}$, are Nash equilibria in any graph and for any set of node weights. This is a relevant property which is peculiar to the coordination game and does not hold, for instance, for the anti-coordination game.

Note that it is possible, and sometimes useful, to rewrite the conditions of the best response in terms of the degree w_i, which is a given quantity. This permits to find an equivalent definition of the game where the best response depends explicitly only on the total weight of neighbors picking action +1. Therefore, given $x \in \mathcal{X}$, let us define the quantities

$$w_i^+(x) = \sum_{\substack{j\in\mathcal{V} \\ x_j=+1}} W_{ij} \qquad w_i^-(x) = \sum_{\substack{j\in\mathcal{V} \\ x_j=-1}} W_{ij} \tag{2.3}$$

which allow us to write

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) - w_i^-(x) > \alpha_i \\ \{-1\} & \text{if } w_i^+(x) - w_i^-(x) < \alpha_i \\ \{\pm 1\} & \text{if } w_i^+(x) - w_i^-(x) = \alpha_i \end{cases}$$

Recalling that $w_i^+(x) + w_i^-(x) = w_i$, we find that

$$w_i^+(x) - (w_i - w_i^-(x)) > \alpha_i \qquad \Leftrightarrow \qquad w_i^+(x) > \frac{w_i + \alpha_i}{2}$$

and therefore

$$w_i^+(x) > \left(\frac{1}{2} + \frac{\alpha_i}{2w_i}\right) w_i$$

If we introduce the quantity

$$r_i := \frac{1}{2} + \frac{\alpha_i}{2w_i} \tag{2.4}$$

the best response in (2.2) turns into

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) > r_i w_i \\ \{-1\} & \text{if } w_i^+(x) < r_i w_i \\ \{\pm 1\} & \text{if } w_i^+(x) = r_i w_i \end{cases} \tag{2.5}$$

We remark that the two notations are completely equivalent. In fact, the quantities r_i can be uniquely computed from α_i through equation (2.4) as well as $\alpha_i = (2r_i - 1)w_i$.

Note that the quantities r_i represent thresholds. In fact, note that, if the graph is unweighted, the best response function becomes

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > r_i\,|N_i| \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < r_i\,|N_i| \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = r_i\,|N_i| \end{cases} \tag{2.6}$$

where N_i denotes the neighborhood of node i. In this case, thresholds r_i capture the incentive of the agent in preferring one action over the other in terms of the *fraction* of (+1)-neighbors needed in order for the player to prefer action +1.

Furthermore, observe that (−1)-stubborn players, who have by definition $\alpha_i > w_i$, have threshold $r_i > 1$. This is consistent with what we said before. In fact if, for a given player, the fraction of neighbors required to play action +1 in order to coordinate with them is more than 1, then the player will always prefer action −1. Similarly, if $\alpha_i < -w_i$, the threshold r_i becomes negative and therefore he is evidently a (+1)-stubborn player.

Let us make one last observation about thresholds. In the beginning of the section, we said that action −1 is risk-dominant if $\alpha_i > 0$. Note that this is the same as asking $r_i > \frac{1}{2}$.

Example 4 (Network coordination game)**.** Let us consider the case where $r_i = r$ for all players. This means that the only form of heterogeneity among the players is given by the network structure. In this instance, the best response function becomes

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) > rw_i \\ \{-1\} & \text{if } w_i^+(x) < rw_i \\ \{\pm 1\} & \text{if } w_i^+(x) = rw_i \end{cases} \tag{2.7}$$

In the literature, this game is called *network coordination game* and it is defined as a binary network game where every pair of neighbors plays the same given coordination game. Recall that a coordination game is a 2×2-players symmetric game with payoff matrix as in Figure 1.1 having entries $a > c$ and $b > d$.

For instance, let us consider an unweighted graph and a threshold $r = \frac{1}{2}$, which is the risk-neutral case. Note that this corresponds to the case of $\alpha = 0$ and therefore the utilities are simply given by $u_i(x) = \sum_{j \in V} W_{ij} x_i x_j$ for any $i \in \mathcal{V}$. If the graph is undirected, the best response function becomes

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > \frac{|N_i|}{2} \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < \frac{|N_i|}{2} \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = \frac{|N_i|}{2} \end{cases} \tag{2.8}$$

which means that the purpose of every player is to copy the action chosen by the majority of the neighbors. This special example of network coordination game is called *majority game*. Another way to define the majority game is to start from the payoff matrix in Figure 3.1 from which we obtain, since the graph is unweighted,

	-1	+1
-1	1,1	-1,-1
+1	-1,-1	1,1

Figure 2.1. Payoff matrix majority game.

that the utility functions are given by

$$u_i(x) := \sum_{j \in \mathcal{V}} W_{ij} \varphi(x_i, x_j) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j = |\{j \in N_i \mid x_j = x_i\}| - |\{j \in N_i \mid x_j \neq x_i\}|$$

for all $i \in \mathcal{V}$.

We recall that the two consensus configurations are Nash equilibria of the game. Furthermore, it is possible to give a general characterization of all the Nash equilibria of the network coordination game by introducing the concept of *cohesiveness* of a subset. Specifically, a subset S of the vertex set $\mathcal{V}$ is called *q-cohesive* is for any

$i \in S$ the total weight of the out-links towards nodes that belong to the subset S is at least a fraction q of the out-degree of the node. In formulas, $S \subseteq \mathcal{V}$ is q-cohesive if

$$\frac{\sum_{j \in S} W_{ij}}{w_i} \geq q \tag{2.9}$$

for any $i \in S$.

Note that if all the players in a r-cohesive subset of the nodes play action $+1$ then action $+1$ is their best response regardless of the strategies adopted by the rest of the players. Similarly, players in a $(1-r)$-cohesive subset are in equilibrium if they all play action -1.

Therefore, it is not difficult to prove that the set of all the Nash equilibria of a network coordination game with threshold r is given by

$$\mathcal{N} = \left\{ \mathbb{1}_S - \mathbb{1}_{\mathcal{V} \setminus S} \mid \ S \text{ is } r\text{-cohesive and } \mathcal{V} \setminus S \text{ is } (1-r)\text{-cohesive} \right\}$$

We anticipate that the idea of cohesiveness is fundamental for the results achieved in the thesis.

Finally, we recall that coordination games are 2×2 symmetric potential games. Therefore, if the graph is undirected, the network coordination game with threshold r satisfies the hypothesis of Proposition 3 and, therefore, it is a potential game with potential function given by (1.7). For instance, let us consider again a network game where every pair of neighbors plays the two-player symmetric game in Figure 3.1, which leads to a network coordination game with threshold $r = \frac{1}{2}$. Note that $\phi^{\{i,j\}}(x,y) = \phi(x,y)$ for any $i,j \in \mathcal{V}$, $x,y \in \mathcal{A}$ where

$$\phi(+1,+1) = \phi(-1,-1) = 1, \qquad \phi(-1,+1) = \phi(+1,-1) = -1$$

Therefore, the potential function of the game is

$$\Phi(x) = \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij} x_i x_j \tag{2.10}$$

In words, the potential of the game is given by the total weight of the edges where neighbors play the same action minus the total weight of the edges where neighbors play different actions. If the graph is unweighted, as in the majority game, it coincides with the difference between the number of links connecting agents picking identical actions and the number of links connecting players picking opposite actions.

2.2 The potential property

The potential property of the network coordination game over undirected graphs is well known in the literature. In fact, as we observed before, the game satisfies the hypothesis of Proposition 3, which also provides a simple tool to find the potential function. Note that the main assumption of the proposition is that the underlying two-player games are symmetric. This fact is not true for the heterogeneous case where we introduced a form of distinction among the incentives of the players. In this section, we show that the potential property still holds with the addiction of heterogeneous thresholds.

First of all, let us recall one more time the utilities of the heterogeneous network coordination game, which are defined in (2.1). In order to avoid confusion with the next chapter, from now on, we will denote the payoffs of coordinating agents as

$$u_i^c(x_i, x_{-i}) := \sum_{j \in \mathcal{V}} W_{ij} x_i x_j - \alpha_i x_i$$

Proposition 4. The heterogeneous network coordination game with node weights $\{\alpha_i\}_{i \in \mathcal{V}}$ defined over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a potential game with potential function given by

$$\Phi_c(x) = \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij} x_i x_j - \sum_{i \in \mathcal{V}} \alpha_i x_i \tag{2.11}$$

Proof. Consider any player $i \in \mathcal{V}$ and any configuration $x_{-i} \in \mathcal{A}^{\mathcal{V} \setminus \{i\}}$. Let us first observe that

$$\Phi_c(+1, x_{-i}) = \frac{1}{2} \sum_{\substack{k,j \in \mathcal{V} \\ k,j \neq i}} W_{kj} x_k x_j + \sum_{j \in \mathcal{V}} W_{ij} x_j - \sum_{\substack{k \in \mathcal{V} \\ k \neq i}} \alpha_k x_k - \alpha_i$$

$$\Phi_c(-1, x_{-i}) = \frac{1}{2} \sum_{\substack{k,j \in \mathcal{V} \\ k,j \neq i}} W_{kj} x_k x_j - \sum_{j \in \mathcal{V}} W_{ij} x_j - \sum_{\substack{k \in \mathcal{V} \\ k \neq i}} \alpha_k x_k + \alpha_i$$

Therefore, if we compute the difference in the utilities, we obtain

$$\begin{aligned} u_i^c(+1, x_{-i}) - u_i^c(-1, x_{-i}) &= \sum_{j \in \mathcal{V}} W_{ij} x_j - \alpha_i - (-\sum_{j \in \mathcal{V}} W_{ij} x_j + \alpha_i) = \\ &= 2 \sum_{j \in \mathcal{V}} W_{ij} x_j - 2\alpha_i = \\ &= \Phi_c(+1, x_{-i}) - \Phi_c(-1, x_{-i}) \end{aligned}$$

which means that Φ_c is a potential function for the heterogeneous network coordination game. □

As we already know, the potential property guarantees the existence of at least one Nash equilibrium, which is a maximum point of the potential function. This fact is not too surprising for the network coordination game since we already observed that the two consensus configurations $\mathbb{1}$ and $-\mathbb{1}$ are Nash equilibria in any given setting with no stubborn players. In general, an explicit characterization of all the Nash equilibria is not easy to find, although we observe that, if the network is a complete graph and the thresholds are such that there are no stubborn players, the two actions profiles $\mathbb{1}$ and $-\mathbb{1}$ are the only Nash equilibria of the game.

2.3 The network coordination game with stubborn agents

The purpose of this section is to show that the heterogeneous network coordination game defined above is equivalent to a network coordination game with stubborn agents.

Consider a network coordination game with threshold $r = \frac{1}{2}$ for all $i \in \mathcal{V}$ over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For simplicity, we assume $r = \frac{1}{2}$, although we remark that the following observations hold true with slight modifications for any given threshold $\alpha \in \mathbb{R}$.

Recall that the network coordination game is potential. In particular, the utilities of the agents are given by

$$u_i(x) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j, \qquad i \in \mathcal{V}$$

We now change the behavior of some agents in order to introduce *stubborn* agents in the game, which are agents that always prefer a given action regardless of the choices of their neighbors. In particular, let us introduce (+1)-stubborn agents by forcing a subset $\mathcal{V}^+ \subset \mathcal{V}$ of the agents to play action $+1$. Furthermore, let us assume that a subset $\mathcal{V}^- \subset \mathcal{V}$ of the agents always prefer action -1. We remark that this is possible only if $\mathcal{V}^- \cap \mathcal{V}^+ = \emptyset$.

At this point, we denote the set of the remaining nodes as $\tilde{\mathcal{V}}$, namely $\tilde{\mathcal{V}} :=$

$\mathcal{V} \setminus (\mathcal{V}^+ \cup \mathcal{V}^-)$. Let us now consider the payoff of a player $i \in \tilde{\mathcal{V}}$. We find that

$$\begin{aligned} u_i(x) &= \sum_{j\in\mathcal{V}} W_{ij}x_ix_j = \sum_{j\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \sum_{j\in\mathcal{V}^+} W_{ij}x_i - \sum_{j\in\mathcal{V}^-} W_{ij}x_i = \\ &= \sum_{j\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \left(\sum_{j\in\mathcal{V}^+} W_{ij} - \sum_{j\in\mathcal{V}^-} W_{ij}\right) x_i = \\ &= \sum_{j\in\tilde{\mathcal{V}}} W_{ij}x_ix_j - \alpha_i x_i \end{aligned}$$

where we denoted

$$\alpha_i := \sum_{j\in\mathcal{V}^-} W_{ij} - \sum_{j\in\mathcal{V}^+} W_{ij} \tag{2.12}$$

This means that the network coordination game with stubborns over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ can rewritten as a heterogeneous network coordination game with node weights α_i over the induced subgraph $\mathcal{G}[\tilde{\mathcal{V}}] = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, W_{|\tilde{\mathcal{V}}\times\tilde{\mathcal{V}}})$ where $\tilde{\mathcal{E}} = \Big\{\{i,j\} \in \mathcal{E} : i,j \in \tilde{\mathcal{V}}\Big\}$.

Let us now focus on the opposite formulation of the problem. Specifically, given a graph $\tilde{G} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{W})$, we consider the heterogeneous network coordination game with node weights $\alpha_i \in \mathbb{R}$, $i \in \tilde{\mathcal{V}}$, and we want to see if we can write it in the form of a network coordination game with stubborn agents.

Therefore, let us consider a network coordination game defined over the same graph where all the agents have threshold $r = \frac{1}{2}$. Furthermore, let us introduce $n = |\tilde{\mathcal{V}}|$ new agents in the game. In particular, we provide any agent $i \in \tilde{\mathcal{V}}$ having node weight $\alpha_i > 0$ with a (-1)-stubborn neighbor and we set the weight of the interaction as the threshold α_i. Similarly, we add a $(+1)$-stubborn neighbor to the remaining agents having $\alpha_i < 0$. In this second case, we set the weight of the interaction as $-\alpha_i$.

Formally, let us suppose $\tilde{\mathcal{V}} = \{1, \dots, n\}$. We consider an enlarged vertex set $\mathcal{V} = \tilde{\mathcal{V}} \cup \mathcal{V}^+ \cup \mathcal{V}^-$, where $\mathcal{V}^+$ and $\mathcal{V}^-$ are such that $\mathcal{V}^+ \cup \mathcal{V}^- = \{n+1, \dots, 2n\}$ and represents the set of the stubborn agents which, by definition, always prefer action $+1$ or -1 regardless of the rest of the players. We define the new matrix of the weights $W \in \mathbb{R}^{2n\times 2n}$ as

$$W_{ij} = \begin{cases} \tilde{W}_{ij} & \text{if } i,j \in \tilde{\mathcal{V}} \\ \alpha_i & \text{if } i \in \tilde{\mathcal{V}}, j = n+i, \alpha_i > 0 \\ -\alpha_i & \text{if } i \in \tilde{\mathcal{V}}, j = n+i, \alpha_i < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

Note that, if we desire an undirected graph, we need to set $W_{ij} = |\alpha_j|$ for $j \in \tilde{\mathcal{V}}$ and $i = j + n$. We mention it to be consistent with the previous sections but it is not strictly necessary for our purpose since we suppose that players in $\mathcal{V}^+$ are stubborn agents.

Let us now consider the payoff of an agent $i \in \tilde{\mathcal{V}}$ in this new formulation of the game. We have that

$$\begin{aligned} u_i(x) &= \sum_{i\in\mathcal{V}} W_{ij}x_ix_j = \sum_{i\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}^+} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}^-} W_{ij}x_ix_j = \\ &= \sum_{i\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}^+} W_{ij}x_i - \sum_{i\in\mathcal{V}^-} W_{ij}x_i = \sum_{i\in\tilde{\mathcal{V}}} W_{ij}x_ix_j - \alpha_i x_i \end{aligned}$$

In words, we defined a network coordination game with stubborn players where every agents in $\tilde{\mathcal{V}}$ is actually playing a heterogeneous network coordination game with node weights $\{\alpha_i\}_{i\in\tilde{\mathcal{V}}}$.

In conclusion, we have proved that the heterogeneous network coordination game is equivalent to a network coordination game with stubborn agents. This is interesting for many reasons. For instance, the fact that we can study the heterogeneous case as a potential game with stubborn players justify the evidence that the potential property still holds in the non-homogeneous case. This remark is explained in the following observation.

Observation 5. We recall that the potential function of a network coordination game with threshold $r = \frac{1}{2}$ is given by

$$\Phi(x) = \frac{1}{2} \sum_{i,j\in\mathcal{V}} W_{ij}x_ix_j$$

Let us consider the network coordination game with stubborn players derived from the thresholds network coordination game having node weights $\{\alpha_i\}_{i\in\tilde{\mathcal{V}}}$. If we consider the potential function of the network coordination game without stubborn players and we force to +1 the actions of the players that always prefer action +1, we automatically obtain the potential function of the game with stubborns. In formulas

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\sum_{i,j\in\mathcal{V}} W_{ij}x_ix_j = \frac{1}{2}\sum_{i,j\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \frac{1}{2}\sum_{i,j\in\mathcal{V}^+} W_{ij}x_ix_j = \\ &= \frac{1}{2}\sum_{i,j\in\tilde{\mathcal{V}}} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}} \alpha_i x_i = \Phi_c(x)\end{aligned}$$

Furthermore, the observation is useful because it gives a tool to study coordination games with stubborn agents from a more general point of view. This idea is fundamental in the results of the next chapters.

Chapter 3

Heterogeneous network anti-coordination game

In this chapter, we wish to model the opposite situation of the previous section. Indeed, in our assumption, agents have two possible types of interaction with their neighbors. Recall that *coordinating* agents prefer an action if enough neighbors are playing it. On the other hand, the utility of *anti-coordinating* agents in taking an action decreases with the number of neighbors taking the same action.

Since in the previous section we talked about heterogeneous network coordination games, we now move our attention to heterogeneous network anti-coordination games.

In the first section, we give the formal definition of the game. We remark that, even though the definition of the game is apparently similar to the previous case, networks with anti-coordinating agents present quite different behaviors than the previous ones. For instance, consensus configurations are not Nash equilibria in this case and the existence of equilibria is in general not trivial.

On the other hand, some observations that are similar to the previous case can be made also for the anti-coordination case. In fact, in the second section, we show that the heterogeneous network anti-coordination game, as well as the previous example, is a potential game. Therefore, the existence of Nash equilibria is guaranteed over any possible undirected network. Furthermore, we present an analogous way to rewrite the heterogeneous network anti-coordination game as a network anti-coordination game with stubborn agents.

Differently from the previous section, we conclude the chapter by giving a full description of the Nash equilibria of the heterogeneous anti-coordination game in the special case of a complete graph. The analysis is done by taking some ideas from the linear-threshold model [6] that addresses games with coordinating agents.

3.1 The heterogeneous network anti-coordination game

Conversely to the previous chapter, we now wish to model the case of anti-coordinating agents.

Let us consider again an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a set of node weights $\{\alpha_i\}_{i\in\mathcal{V}}$, $\alpha_i \in \mathbb{R}$. The *heterogeneous network anti-coordination game* with action set $\mathcal{A} = \{-1, +1\}$ is a network game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ with utilities $\{u_i\}_{i\in\mathcal{V}} : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$

$$u_i(x_i, x_{-i}) = -\sum_{j\in\mathcal{V}} W_{ij} x_i x_j + \alpha_i x_i, \qquad i \in \mathcal{V} \tag{3.1}$$

Note that, in this case, the payoff of an agent goes down as the weighted sum of the neighbors playing the same action gets larger. Therefore, agents have a negative interest in choosing the same action of a neighbor and an incentive in picking the opposite one. In the anti-coordination game, in fact, we are assuming that players are not in principle interested in taking strategies that are chosen by too many neighbors.

In this case, the node weight α_i has the opposite meaning of the previous one. Indeed, if $\alpha_i > 0$ then action $+1$ is the risk-dominant action for player i, while action -1 becomes risk-dominant if $\alpha_i < 0$. As before, if $\alpha_i = 0$, then the two actions are risk-neutral.

Let us consider an agent $i \in \mathcal{V}$ and any choice of actions $x_{-i} \in \mathcal{A}^{\mathcal{V}\setminus\{i\}}$. In the heterogeneous network anti-coordination game, we have that $+1 \in \mathcal{B}_i(x_{-i})$ if

$$-\sum_{j\in\mathcal{V}} W_{ij} x_j + \alpha_i \geq +\sum_{j\in\mathcal{V}} W_{ij} x_j - \alpha_i$$

which is the same as asking

$$\sum_{j\in\mathcal{V}} W_{ij} x_j \leq \alpha_i$$

Therefore, as predictable, the *best response function* of an agent in the heterogeneous network anti-coordination game takes the opposite form of the one in (2.2), namely

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij} x_j < \alpha_i \\ \{-1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij} x_j > \alpha_i \\ \{\pm 1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij} x_j = \alpha_i \end{cases} \tag{3.2}$$

for any agent $i \in \mathcal{V}$. Note that, in the heterogeneous network anti-coordination game, players with $\alpha_i > w_i$ are (+1)-stubborn players while they always prefer action -1 if $\alpha_i < -w_i$.

Observation 6. Even though the game looks similar to the heterogeneous network coordination game, the two strategic interactions present quite different behaviors. For instance, we recall that the heterogeneous network coordination game admits two configuration that are always Nash equilibria regardless of the node weights and the network structure. This is not true for anti-coordinating agents and in general it is not straightforward to prove that Nash equilibria exist. In the next section, we will show that the existence of at least one Nash equilibrium is guaranteed when the graph is undirected.

If we follow the same method of the previous section, we can express the best response function in terms of the weighted sum of neighbors playing action +1, namely

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) < r_i w_i \\ \{-1\} & \text{if } w_i^+(x) > r_i w_i \\ \{\pm 1\} & \text{if } w_i^+(x) = r_i w_i \end{cases} \tag{3.3}$$

where r_i is again given by equation (2.4).

Furthermore, if the graph is unweighted, the best response function turns into

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < r_i\,|N_i| \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > r_i\,|N_i| \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = r_i\,|N_i| \end{cases} \tag{3.4}$$

where N_i denotes the neighborhood of node i. As in the previous case, thresholds r_i capture the interest of the player in choosing one action over the other in terms of the *fraction* of neighbors playing action +1.

Additionally, note that (+1)-stubborn players, who have $\alpha_i > w_i$, have now threshold $r_i > 1$. Similarly, if $\alpha_i < -w_i$, the threshold r_i becomes negative, which means that the player always prefers action -1.

Example 5 (Network anti-coordination game)**.** Similarly to the previous example, let us consider the case where $r_i = r$ for all $i \in \mathcal{V}$. In this instance, the best response function becomes

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) < rw_i \\ \{-1\} & \text{if } w_i^+(x) > rw_i \\ \{\pm 1\} & \text{if } w_i^+(x) = rw_i \end{cases} \tag{3.5}$$

Clearly, this game is called *network anti-coordination game* and, as before, it can be defined starting from a two-player anti-coordination game, which is a symmetric game with payoff matrix as in Figure 1.1 having entries $a < c$ and $b < d$.

The *minority game*, in particular, is defined according to a 2×2-game with payoff matrix in Figure 3.1. If we consider an unweighted graph and we assume that every

	-1	+1
-1	-1,-1	1,1
+1	1,1	-1,-1

Figure 3.1. Payoff matrix minority game.

pair of neighbors plays the previous two-player symmetric game, we find the utilities

$$u_i(x) := \sum_{j\in\mathcal{V}} W_{ij}\varphi(x_i,x_j) = \sum_{j\in\mathcal{V}} W_{ij}(-x_ix_j) = |\{j \in N_i \mid x_j \neq x_i\}| - |\{j \in N_i \mid x_j = x_i\}|$$

for all $i \in \mathcal{V}$. Therefore the best response is given by

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < \frac{|N_i|}{2} \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > \frac{|N_i|}{2} \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = \frac{|N_i|}{2} \end{cases} \tag{3.6}$$

Note that the minority game is a network anti-coordination game with threshold $r = \frac{1}{2}$, which means $\alpha = 0$. In fact, the goal of the agents is to pick the opposite action of the majority of the neighbors. For this reason, the game is called minority game since agents opt for the action chosen by the minority.

We remark that also anti-coordination games are 2×2 symmetric potential games and therefore we can use again Proposition 3 to prove that the network coordination game with homogeneous thresholds over an undirected graph is a potential game.

For instance, the potential function of the coordination game with threshold $r = \frac{1}{2}$, namely $\alpha = 0$, is given by

$$\Phi(x) = -\frac{1}{2}\sum_{i,j\in\mathcal{V}} W_{ij}x_ix_j \tag{3.7}$$

In contrast with the coordination case, this time links connecting agents playing opposite actions give a positive contribution to the potential function, while links connecting agents picking identical actions remove energy from the game.

3.2 The potential property

Recall that network anti-coordination games, as well as network coordination games, satisfy the hypothesis of Proposition 3 and therefore it is well-known that they are examples of potential games. Furthermore, the proposition also provides an explicit definition of the potential function. On the other hand, when thresholds are heterogeneous the hypothesis of the proposition are not anymore satisfied since the underlying two-player games are not symmetric. In this section, we show that the potential property is preserved with the addiction of heterogeneous thresholds.

Let us recall the utilities of the heterogeneous network anti-coordination game, which are defined in (3.1). We will denote the payoffs of anti-coordinating agents as

$$u_i^a(x_i, x_{-i}) := -\sum_{j\in\mathcal{V}} W_{ij}x_ix_j + \alpha_i x_i$$

Note that, by definition, $u_i^a(x) = -u_i^c(x)$, where we recall that u_i^c denotes the utility function of a coordinating agent defined according to (2.1)

Proposition 5. The heterogeneous network anti-coordination game with node weights $\{\alpha_i\}_{i\in\mathcal{V}}$ defined over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a potential game with potential function given by

$$\Phi_a(x) = -\Phi_c(x) = -\frac{1}{2}\sum_{i,j\in\mathcal{V}} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}}\alpha_i x_i \tag{3.8}$$

where Φ_c is defined according to (2.11).

Proof. The proof for the anti-coordination case is analogous to the proof of Proposition 4. In particular, we can observe that for any $i \in \mathcal{V}$ and any $x, y \in \mathcal{X}$ such that $x_{-i} = y_{-i}$ it holds

$$\begin{aligned} u_i^a(y) - u_i^a(x) &= u_i^c(x) - u_i^c(y) = \\ &= \Phi_c(x) - \Phi_c(y) = \Phi_a(y) - \Phi_a(x) \end{aligned}$$

where $\Phi_a(x) := -\Phi_c(x)$. □

Proposition 4 combined with Proposition 2 allows us to state that the set of the Nash equilibria is nonempty also for the heterogeneous network anti-coordination game defined over any undirected graph. This fact is particularly interesting for the anti-coordination game since the existence of Nash equilibria is not evident as in the coordination case. Anyway, an explicit characterization of all the Nash equilibria is often not easy to find. In the last section we will provide an explicit characterization of all the Nash equilibria of the game with anti-coordinating agents, which is more interesting, when it is defined over the complete graph.

3.3 The network anti-coordination game with stubborn agents

In the previous chapter, we observed that the network heterogeneous coordination game can be considered a network coordination game with stubborn agents and viceversa. The purpose of this section is to prove that the same fact holds true also for the anti-coordination case.

Let us consider a network anti-coordination game with threshold $r = \frac{1}{2}$ for all $i \in \mathcal{V}$ over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In this case $\alpha_i = 0$ for all the agents and the utilities of the agents are given by

$$u_i(x) = -\sum_{j\in\mathcal{V}} W_{ij} x_i x_j, \qquad i \in \mathcal{V}$$

As before, we introduce some *stubborn* agents in the game. In particular, we consider two disjoint subsets $\mathcal{V}^+, \mathcal{V}^- \subset \mathcal{V}$, $\mathcal{V}^- \cap \mathcal{V}^+ = \emptyset$ and we force the actions of the players in the subsets to $+1$ and -1 respectively. If we denote again the set of the remaining nodes as $\tilde{\mathcal{V}}$, we can observe that the payoff of a player $i \in \tilde{\mathcal{V}}$ is given by

$$\begin{aligned} u_i(x) &= -\sum_{j\in\mathcal{V}} W_{ij} x_i x_j = -\sum_{j\in\tilde{\mathcal{V}}} W_{ij} x_i x_j - \left(\sum_{j\in\mathcal{V}^+} W_{ij} - \sum_{j\in\mathcal{V}^-} W_{ij} \right) x_i = \\ &= -\sum_{j\in\tilde{\mathcal{V}}} W_{ij} x_i x_j + \alpha_i x_i \end{aligned}$$

where we denoted

$$\alpha_i := \sum_{j\in\mathcal{V}^-} W_{ij} - \sum_{j\in\mathcal{V}^+} W_{ij}$$

as in equation (2.12). Therefore, the network anti-coordination game over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with threshold $r = \frac{1}{2}$, $i \in \tilde{\mathcal{V}}$, and stubborn players $\mathcal{V}^+$ and $\mathcal{V}^-$ can be rewritten as a heterogeneous network anti-coordination game with node weights in (2.12) over the induced subgraph $\mathcal{G}[\tilde{\mathcal{V}}] = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, W_{|\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}})$ where $\tilde{\mathcal{E}} = \Big\{ \{i, j\} \in \mathcal{E} : i, j \in \tilde{\mathcal{V}} \Big\}$.

Let us now consider the opposite formulation of the problem, which is to check if we can write the heterogeneous network anti-coordination game in the form of a network anti-coordination game with stubborn players.

In this instance, we can apply exactly the same method used for the case of the coordination game, namely we consider the network anti-coordination game with threshold $r = \frac{1}{2}$ over the same graph and we provide any player with a new $(+1)$-stubborn neighbor or a new (-1)-stubborn neighbor depending on the sign of the node weight. Furthermore, we set the weight of the interaction as the the absolute value of the node weight.

Formally, given a set of node weights $\{\alpha_i\}_{i \in \mathcal{V}}$, $\alpha_i \in \mathbb{R}$ and an undirected weighted graph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{W})$ having $\tilde{\mathcal{V}} = \{1, \dots, n\}$, we consider an enlarged vertex set $\mathcal{V} = \tilde{\mathcal{V}} \cup \{n+1, \dots, 2n\}$, where agents in $\mathcal{V}^+ \cup \mathcal{V}^- := \{n+1, \dots, 2n\}$ are either $(+1)$-stubborn agents or (-1)-stubborn agents. Furthermore, we define the new matrix of the weights $W \in \mathbb{R}^{2n \times 2n}$ as in (2.13). If we consider the payoff of a player $i \in \tilde{\mathcal{V}}$, we find that

$$
\begin{aligned}
u_i(x) &= -\sum_{i \in \mathcal{V}} W_{ij} x_i x_j = -\sum_{i \in \tilde{\mathcal{V}}} W_{ij} x_i x_j - \sum_{i \in \mathcal{V}^+} W_{ij} x_i + \sum_{i \in \mathcal{V}^-} W_{ij} x_i = \\
&= -\sum_{i \in \tilde{\mathcal{V}}} W_{ij} x_i x_j + \alpha_i x_i
\end{aligned}
$$

Therefore, we defined a network anti-coordination game with stubborn agents where non-stubborn agents are actually playing a heterogeneous network anti-coordination game with their neighbors.

In conclusion, we showed that the heterogeneous network anti-coordination game is equivalent to a network anti-coordination game with stubborn agents, which is interesting for the same reasons that we pointed out in the previous chapter.

Indeed, recalling that the heterogeneous network anti-coordination game is a potential game, note that its potential function can be derived from the potential function of the network anti-coordination game. Specifically, the potential function can be found with the same idea of the coordination case. First, we find the equivalent formulation of the game in terms of a classical network anti-coordination game with (+1)-stubborn players. We can do it using the method proposed above. Then, we take the potential of the game with homogeneous thresholds and we force to +1 the actions of a subset of the players. In this way, we find a potential function for the game with heterogeneous thresholds, which exactly the one given in Proposition 5.

Furthermore, we will use the observations of this section in the next chapter when we end up studying a network anti-coordination game with stubborn players.

3.4 The heterogeneous anti-coordination game over the complete graph

Let us now consider the special case of the heterogeneous network anti-coordination game on a fully connected population. We recall that, in a complete graph, every node is linked, through unweighted and undirected edges, to all the other nodes. This means that, if the network game is defined over the complete graph, the neighborhood of a player includes all the other participants of the game. In formulas, $N_i = \{i \in \mathcal{V} \setminus \{i\}\}$ and, therefore, $|N_i| = n - 1$ for all $i \in \mathcal{V}$. According to what we just observed, if $\mathcal{G} = \mathcal{K}_n$, $n > 1$, the best response function is given by:

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}| < r_i(n-1) \\ \{-1\} & \text{if } |\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}| > r_i(n-1) \\ \{\pm 1\} & \text{if } |\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}| = r_i(n-1) \end{cases} \tag{3.9}$$

Example 6. As an example, we define a network anti-coordination game over a complete graph with $n = 19$ players having heterogeneous thresholds $r_1 = r_2 = r_3 = 0.1$, $r_4 = r_5 = r_6 = 0.2$, $r_7 = 0.25$, $r_8 = r_9 = 0.3$, $r_{10} = 0.35$, $r_{11} = r_{12} = 0.5$, $r_{13} = r_{14} = r_{15} = 0.6$, $r_{16} = 0.8$, $r_{17} = r_{18} = 0.85$, $r_{19} = 0.9$.

Since the game is potential, we already know that there exists at least one Nash equilibrium. Anyway, we still have many open questions. For instance, we do not know how this equilibrium is defined and if it is unique. We recall that a Nash equilibrium is an action configuration such that $x_i \in \mathcal{B}_i(x_{-i})$ for any player $i \in \mathcal{V}$.

We start by introducing the following notation. Given an action configuration x, we denote as $\mathcal{V}^+(x)$ the set of the players that choose action +1, namely $\mathcal{V}^+(x) = \{i \in \mathcal{V} \mid x_i = +1\}$, and as $n^+(x)$ its cardinality, namely $n^+(x) = |\mathcal{V}^+(x)|$. Similarly, we define $\mathcal{V}^-(x) = \{i \in \mathcal{V} \mid x_i = -1\}$ and $n^-(x) = |\mathcal{V}^-(x)|$.

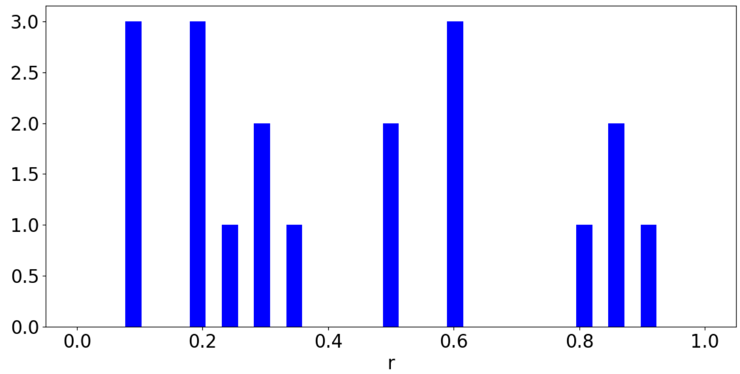


Figure 3.2. Threshold histogram

Those quantities are relevant in the case of a complete graph since the number of neighbors playing action +1, which is a fundamental quantity of the best response, is strongly related to the number of agents playing action +1 in the entire action configuration. This is obviously due to the fact that the neighborhood of a player coincides with the set of all the other participants of the game. Indeed, note that

$$x_i = +1 \quad \Rightarrow \quad |\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}| = |\{j \in \mathcal{V} \mid x_j = +1\}| - 1 = n^+(x) - 1$$
$$x_i = -1 \quad \Rightarrow \quad |\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}| = |\{j \in \mathcal{V} \mid x_j = +1\}| = n^+(x)$$

for any profile $x \in \mathcal{A}^{\mathcal{V}}$. Therefore, according to the best response in (3.9), the conditions for $x \in \mathcal{A}^{\mathcal{V}}$ to be a Nash equilibrium of the game are:

$$x_i = +1 \quad \Rightarrow \quad n^+(x) - 1 \leq r_i(n-1)$$
$$x_i = -1 \quad \Rightarrow \quad n^+(x) \geq r_i(n-1)$$

At this point, we introduce the quantity

$$\tilde{z}(x) := \frac{\{j \in \mathcal{V} \mid x_j = +1\}}{n-1} = \frac{n^+(x)}{n-1} \tag{3.10}$$

which is very close to the fraction of agents playing action +1, denoted in the Linear Threshold Model as $z(x) = \frac{n^+(x)}{n}$. The difference in the normalization is due to the fact that neighborhoods have dimension $n-1$ in the complete graph.

Going back to our problem, a straightforward substitution in the previous requirements leads to the following condition

$$x \in \mathcal{N} \quad \Leftrightarrow \quad \begin{cases} \tilde{z}(x) - \frac{1}{n-1} \leq r_i & i \in \mathcal{V}^+(x) \\ \tilde{z}(x) \geq r_i & i \in \mathcal{V}^-(x) \end{cases} \tag{3.11}$$

where we remark that $\mathcal{N}$ is the set of all the Nash equilibria of the game.

The form of the system on the right of (3.11) motivates the definition of a *threshold cumulative distribution function* which returns the (re-normalized) fraction of the players having threshold less or equal than a given value. Let us recall that, in the Linear Threshold Model, the cumulative distribution function (CDF) of thresholds is defined as

$$F(z) = \frac{1}{n} \left|\{j \in \mathcal{V} \mid r_j \leq z\}\right| \qquad z \geq 0 \tag{3.12}$$

The cumulative distribution is, by definition, non-decreasing, piece-wise constant and continuous to the right with possible discontinuities occurring at points r_j, $j \in \mathcal{V}$.

Note that, in our case, we are interested in a *re-normalized threshold cumulative distribution function* $\tilde{F} : \mathbb{R}^+ \to \left\{0, \frac{1}{n-1}, \dots, \frac{n}{n-1}\right\}$ such that

$$\tilde{F}(z) = \frac{1}{n-1} \left|\{i \in \mathcal{V} \mid r_i \leq z\}\right|, \qquad z \geq 0 \tag{3.13}$$

In particular, we are looking for its complementary $\tilde{S} : \mathbb{R}^+ \to \left\{-\frac{1}{n-1}, 0, \frac{1}{n-1}, \dots, 1\right\}$ defined as

$$\tilde{S}(z) = 1 - \tilde{F}(z) = \frac{1}{n-1} \left|\{i \in \mathcal{V} \mid r_i > z\}\right|, \qquad s \geq 0 \tag{3.14}$$

which returns the (re-normalized) fraction of players having threshold *greater* than a given value.

Even though we defined both the functions with a slightly different normalization, when there is no reason for misunderstanding we might refer to $\tilde{F}$ as threshold cumulative distribution function, or shortly threshold CDF, and to $\tilde{S}$ as threshold complementary cumulative distribution function (CCDF). Figure 3.4 provides an example of the classical threshold CDF and CCDF for the thresholds in Example 6.

Now that we introduced the needed notation, let us consider a Nash equilibrium $x^* \in \mathcal{A}^{\mathcal{V}}$. We recall the first requirement in (3.11), which is $\tilde{z}(x^*) - \frac{1}{n-1} \leq r_i$, $\forall i \in \mathcal{V}^+(x^*)$. Note that this condition permits to find an upper bound for the fraction of agents playing action $+1$. In fact, the number of players choosing $+1$ in the Nash equilibrium is definitely less than the total number of players having

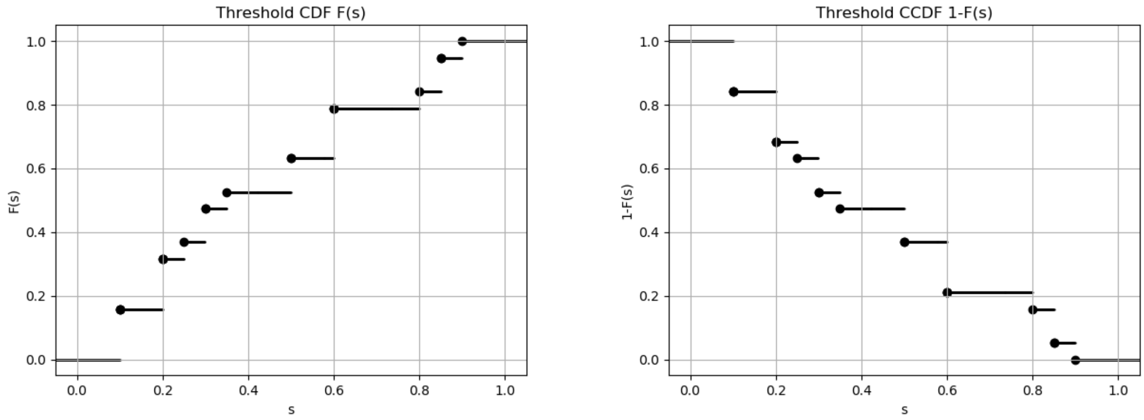


Figure 3.3. The threshold cumulative distribution function (left) and the threshold complementary cumulative distribution function (right) for Example 6.

threshold greater or equal than $\tilde{z}(x^*) - \frac{1}{n-1}$. In formulas, it holds that

$$\begin{aligned}\tilde{z}(x^*) = \frac{1}{n-1}\left|\mathcal{V}^+(x^*)\right| &\leq \frac{1}{n-1}\left|\{j \in \mathcal{V} \mid r_j \geq \tilde{z}(x^*) - 1/(n-1)\}\right| \leq \\ &\leq \frac{1}{n-1}\left|\{j \in \mathcal{V} \mid r_j > \tilde{z}(x^*) - (1+\epsilon)/(n-1)\}\right| = \\ &= 1 - \tilde{F}(\tilde{z}(x^*) - (1+\epsilon)/(n-1))\end{aligned}$$

for any $\epsilon > 0$. Note that we introduced the constant $\epsilon > 0$ with the aim of writing the condition in terms of a strict inequality. In this way, we can use the given definition of complementary CDF. Similarly, the second condition in (3.11), which requires that $\tilde{z}(x^*) \geq r_i$ for all players $i \in \mathcal{V}^-(x^*)$, can be used to find a lower bound again for the quantity $\tilde{z}^*(x)$. Indeed, we have that

$$\begin{aligned}1 - \tilde{z}(x^*) = \frac{1}{n-1}\left|\mathcal{V}^-(x^*)\right| &\leq \frac{1}{n-1}\left|\{i \in \mathcal{V} \mid r_i \leq \tilde{z}(x^*)\}\right| = \\ &= \tilde{F}(\tilde{z}(x^*))\end{aligned}$$

If we combine the two, we find a *necessary condition* for $x \in \mathcal{A}^{\mathcal{V}}$ to be a Nash equilibrium.

Proposition 6. Let $x^* \in \mathcal{A}^{\mathcal{V}}$ be a Nash equilibrium of the network anti-coordination game with thresholds $\{r_i\}_{i\in\mathcal{V}}$, $r_i \in \mathbb{R}$ over $\mathcal{K}_n$, $n > 0$. Consider any $\epsilon > 0$, then

$$1 - \tilde{F}(\tilde{z}(x^*) - (1+\epsilon)/(n-1)) \geq \tilde{z}(x^*) \geq 1 - \tilde{F}(\tilde{z}(x^*)) \tag{3.15}$$

where $\tilde{z} : \mathcal{A}^{\mathcal{V}} \to \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ and $\tilde{F} : \mathbb{R}^+ \to \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ are defined as in (3.10) and (3.13).

In the next lines, we will show that, if x is defined in the proper way, the condition is also *sufficient*.

To begin with, let us suppose that we can find a $(n-1)$-normalized fraction that satisfies (6), namely $z^* \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ such that

$$1 - \tilde{F}(z^* - (1+\epsilon)/(n-1)) \geq z^* \geq 1 - \tilde{F}(z^*) \tag{3.16}$$

for any $\epsilon > 0$. To lighten the notation, we define

$$\begin{aligned} z_1 &:= \frac{1}{n-1} \left|\{j \in \mathcal{V} \mid r_j > z^*\}\right| = 1 - \tilde{F}(z^*) \\ z_2 &:= 1 - \frac{1}{n-1} \left|\left\{j \in \mathcal{V} \mid r_j < z^* - \frac{1}{n-1}\right\}\right| = \frac{1}{n-1} \left|\left\{j \in \mathcal{V} \mid r_j \geq z^* - \frac{1}{n-1}\right\}\right| \\ &= 1 - \tilde{F}(z^* - (1+\epsilon)/(n-1)) \end{aligned} \tag{3.17}$$

for $\epsilon > 0$, sufficiently small. In words, $(n-1)z_1$ is the number of players having threshold greater than the given fraction z^*, while the value $(n-1)z_2$ represents the number of players having threshold greater or equal to $z^* - \frac{1}{n-1}$. By hypothesis, $z_1 \leq z^* \leq z_2$.

At this point, we aim to construct a Nash equilibrium x having $\tilde{z}(x) = z^*$. Note that, if we want to satisfy the conditions in (3.11), we need that all the agents having threshold greater than z^* play action $+1$ and all the agents having threshold less than $z^* - \frac{1}{n-1}$ pick action -1. Therefore, let us consider $x \in \mathcal{A}^{\mathcal{V}}$ such that $x_j = -1$ if $r_j < z^* - \frac{1}{n-1}$ and $x_j = +1$ if $r_j > z^*$. By definition of z_1 and z_2, the number of players that choose action $+1$ in x is at least $(n-1)z_1$ and at most $(n-1)z_2$, namely $z_1 \leq \tilde{z}(x) \leq z_2$. Taking this into account, we observe that it is always possible to set the "free" actions of x, which are the actions of the players having threshold between $z^* - \frac{1}{n}$ and z^*, in such a way that $\tilde{z}(x) = z^*$.

Proposition 7. Consider $n > 0$ and a set of thresholds $\{r_i\}_{i \in \mathcal{V}}$, $r_i \in \mathbb{R}$.
Let $z^* \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ be such that (3.16) is satisfied for any $\epsilon > 0$ and consider $x \in \mathcal{A}^{\mathcal{V}}$ having $x_i = -1$ if $r_i < z^* - \frac{1}{n-1}$ and $x_i = +1$ if $r_i > z^*$ that is such that $\tilde{z}(x) = z^*$.
Any action configuration with the previous structure is a Nash equilibrium of the anti-coordination game with thresholds $\{r_i\}_{i \in \mathcal{V}}$ over $\mathcal{K}_n$. Moreover, these are the only possible Nash equilibria having $\tilde{z}(x) = z^*$.

Proof. Note that, by construction of x, there are exactly $(n-1)z^*$ players that choose action $+1$ in the action configuration, namely $n^+(x) = (n-1)z^*$.
Consider $i \in \mathcal{V}^+(x)$. We can trivially observe that the condition $x_i = -1$ if $r_i < z^* - \frac{1}{n-1}$ implies that if $i \in \mathcal{V}^+(x)$, namely $x_i = +1$, then $r_i \geq z^* - \frac{1}{n-1}$. This means that

$$\frac{|\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}|}{n-1} = \frac{z^*(n-1)-1}{n-1} = z^* - \frac{1}{n-1} \leq r_i$$

and thus $\{+1\} \in \mathcal{B}_i(x_{-i})$ for $i \in \mathcal{V}^+(x)$.
Similarly, if $i \in \mathcal{V}^-(x)$ it holds that $r_i \leq z^*$ and therefore

$$\frac{|\{j \in \mathcal{V} \setminus \{i\} \mid x_j = +1\}|}{n-1} = \frac{z^*(n-1)}{n-1} = z^* \geq r_i$$

Namely, $\{-1\} \in \mathcal{B}_i(x_{-i})$ for $i \in \mathcal{V}^-(x)$.
To conclude the proof, we trivially observe that, if $\tilde{z}(x) = z^*$, $\{+1\} \notin \mathcal{B}_i(x_{-i})$ for $i \in \mathcal{V}$ such that $r_i < z^* - \frac{1}{n-1}$ and $\{-1\} \notin \mathcal{B}_i(x_{-i})$ for $i \in \mathcal{V}$ such that $r_i > z^*$ and this proves the last statement. □

Proposition 6 and 7 give, respectively, a necessary and a sufficient condition for the existence of a Nash equilibrium. We sum up the two results in the following proposition.

Corollary 1. Consider $n > 0$, $n \in \mathbb{N}$ and a set of thresholds $\{r_i\}_{i\in\mathcal{V}}$, $r_i \in \mathbb{R}$.
Let $\tilde{F} : \mathbb{R}^+ \to [0, \frac{n}{n-1}]$ be defined as in (3.13). The network anti-coordination game with thresholds $\{r_i\}_{i\in\mathcal{V}}$ admits at least one Nash equilibrium over $\mathcal{K}_n$ if and only if $\exists z^* \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ such that

$$1 - \tilde{F}(z^* - (1+\epsilon)/(n-1)) \geq z^* \geq 1 - \tilde{F}(z^*)$$

for any $\epsilon > 0$.

In other words, we are looking for a fraction z such that *at least* $z(n-1)$ players have threshold greater or equal to $z - \frac{1}{n-1}$ and *at most* $z(n-1)$ players have threshold greater than z. If we find such a $z \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$, for what we observed in Proposition 7, we find a Nash equilibrium by setting to $+1$ the actions of the players having threshold greater than z, to -1 the actions of the players having threshold less than $z - \frac{1}{n-1}$ and by adjusting the remaining actions in such a way that $\tilde{z}(x) = z$.

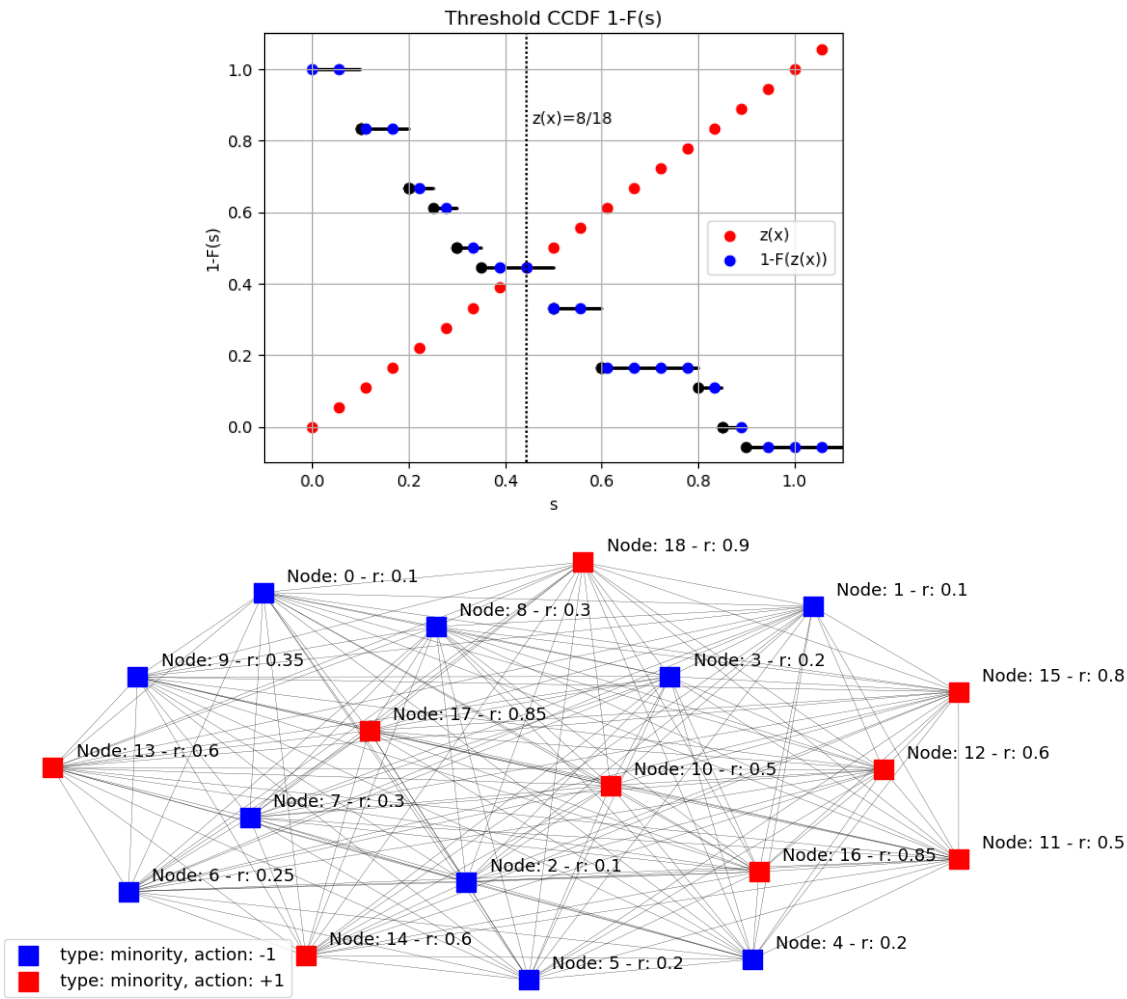
Observation 7. Note that if we find a fixed point of the complementary CDF then the inequalities are automatically satisfied. Formally, consider $z^* \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$

such that $z^* = 1 - \tilde{F}(z^*)$, then z^* satisfies (3.16). Indeed

$$1 - \tilde{F}(z^* - (1+\epsilon)/(n-1) \overset{(1)}{\geq} 1 - \tilde{F}(z^*) = z^*$$

where (1) holds since the $1 - \tilde{F}(z)$ is a non-increasing function.

Figure 3.4. Let us consider the set of thresholds in Example 6. If we write the inequality in (3.16) we find a solution for $z^* = \frac{8}{18}$. In particular, it holds $z^* = 1 - \tilde{F}(z^*)$. The figure below shows the Nash equilibrium than can be found from the solution z^*.



Proposition 8. Consider $n > 0$, $n \in \mathbb{N}$ and a set of thresholds $\{r_i\}_{i\in\mathcal{V}}$, $r_i \in \mathbb{R}$. Consider a network anti-coordination game with thresholds $\{r_i\}_{i\in\mathcal{V}}$ over $\mathcal{K}_n$. If we

let $\tilde{F} : \mathbb{R}^+ \to [0, \frac{n}{n-1}]$ be defined as in (3.13), then $\exists z^* \in \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ such that (3.16) is satisfied for any $\epsilon > 0$.

Remark 2. Note the heterogeneous anti-coordination game is potential and therefore we already know that the existence of at least one Nash equilibrium is guaranteed. Therefore, the proof of the statement is not necessary since we proved that equation 3.16 is satisfied if and only if the Nash equilibrium exists. Anyway, it is still interesting to check the existence of a solution. In fact, the proof gives a method to find the fraction z^* and provides some hints for the exploration of the set of all the Nash equilibria of the game.

Remark 3. In the proof, we ignore the extreme case where $\tilde{F}(0) \geq 1$, namely we will assume $\tilde{F}(0) < 1$, that is $1 - \tilde{F}(0) > 0$. This assumption avoids some tedious technical lines. Moreover, the case is not particularly interesting since it represents a game where all the players (or all the players but one) are $\{-1\}$-stubborns.

Proof. If we find a fraction z^* that satisfies (3.16) then the conclusion comes straightforward from Proposition 7.
We let $z^* \in \left\{\frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ be such that $z^* - \frac{1}{n-1} < 1 - \tilde{F}(z^* - \frac{1}{n-1})$ and $z^* \geq 1 - \tilde{F}(z^*)$, namely we consider the *first* fraction z^* for which the bisector is over (or equal to) the function $1 - \tilde{F}(z)$.
Such a z^* always exists since we assumed $0 < 1 - \tilde{F}(0)$. Indeed, if we suppose that $z < 1 - \tilde{F}(z), \forall z \in \left\{0, \frac{1}{n-1}, \ldots, 1\right\}$, then the condition is satisfied for $z^* = 1 - \tilde{F}(\frac{n}{n-1})$ since $\frac{n}{n-1} > 1 \geq 1 - \tilde{F}(\frac{n}{n-1})$.
Note that $z^* - \frac{1}{n-1} < 1 - \tilde{F}(z^* - \frac{1}{n-1})$, and thus $z^* < 1 - \tilde{F}(z^* - \frac{1}{n-1}) + \frac{1}{n-1}$, implies $z^* \leq 1 - \tilde{F}(z^* - \frac{1}{n-1})$. In particular, this is true since $\text{Im}\, \tilde{F} = \left\{0, \frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$.
Now that we observed that $z^* \leq 1 - \tilde{F}(z^* - \frac{1}{n-1})$, the thesis comes straightforward. Indeed,

$$1 - \tilde{F}(z^* - (1+\epsilon)/(n-1)) \geq 1 - \tilde{F}(z^* - 1/(n-1)) \geq z^* \geq 1 - F(z^*)$$

□

Observation 8. In the proof, we considered $z^* \in \left\{\frac{1}{n-1}, \ldots, 1, \frac{n}{n-1}\right\}$ such that $z^* - \frac{1}{n-1} < 1 - \tilde{F}(z^* - \frac{1}{n-1})$ and $z^* \geq 1 - \tilde{F}(z^*)$ and we proved that this is a possible solution for the inequality (3.16). Note that if it holds

$$z^* < 1 - \tilde{F}(z^* - \epsilon)$$

for any $\epsilon > 0$, then $z^{**} = z^* + \frac{1}{n-1}$ satisfies (3.16) too.
There are at most 2 possible $z \in \left\{0, \frac{1}{n-1}, \dots, 1, \frac{n}{n-1}\right\}$ satisfying (3.16). Indeed, such a z^* is *unique* because of the opposite monotone behaviors of the bisector and $1 - \tilde{F}(z)$.

Proof. On one side, $1 - \tilde{F}(z^* - \epsilon) > z^*$ implies $1 - \tilde{F}(z^* - \epsilon) \geq z^* + \frac{1}{n-1}$ because of the discrete co-domain of $\tilde{F}(z)$. On the other hand

$$z^* + \frac{1}{n-1} > z^* \geq 1 - \tilde{F}(z^*) \overset{(1)}{\geq} 1 - \tilde{F}(z^* + \frac{1}{n-1})$$

where (1) holds since the complementary cumulative function is non-increasing. □

Recall that, from Proposition 7, given $z^* \in \left\{0, \frac{1}{n-1}, \dots, 1, \frac{n}{n-1}\right\}$ such that (3.16) is satisfied for any $\epsilon > 0$, the set

$$\left\{x \in \mathcal{A}^{\mathcal{V}} : x_i = -1 \text{ if } r_i < z^* - \frac{1}{n-1}, x_i = +1 \text{ if } r_i > z^*, \tilde{z}(x) = z^*\right\}$$

contains all the possible Nash equilibria having $\tilde{z}(x) = z^*$.
On the other hand, from Proposition 6, we know that if $x^* \in \mathcal{A}^{\mathcal{V}}$ is a Nash equilibrium, then $z = \tilde{z}(x^*)$ must satisfy (3.16).
Therefore, if we let $\mathcal{Z}$ be the set of all the z that satisfy the condition in (3.16) for every $\epsilon > 0$, then the set $\mathcal{N}$ of the Nash equilibria of the game is given by

$$\mathcal{N} = \left\{x \in \mathcal{A}^{\mathcal{V}} : x_i = -1 \text{ if } r_i < z^* - \frac{1}{n-1}, x_i = +1 \text{ if } r_i > z^*, \tilde{z}(x) = z^*, z^* \in \mathcal{Z}\right\}$$

Given $z^* \in \mathcal{Z}$, let us define z_1 and z_2 as in (3.17). Any Nash equilibrium $x \in \mathcal{A}^{\mathcal{V}}$ such that $\tilde{z}(x) = z^*$ has $z_1(n-1)+(n-z_2(n-1))$ fixed actions. In order to have $\tilde{z}(x) = z^*$, there must be $(z^* - z_1)(n-1)$ more agents that picked action $+1$: those players are chosen among the remaining $n - z_1(n-1) - (n - z_2(n-1)) = (z_2 - z_1)(n-1)$ players whose actions are not fixed.
In other words, the dimension of the set $\mathcal{N}$ is given by the sum of all the possible ways of choosing $(z - z_1)(n-1)$ players (that will play action $+1$) out of $(z_2 - z_1)(n-1)$ agents for any $z \in \mathcal{Z}$.

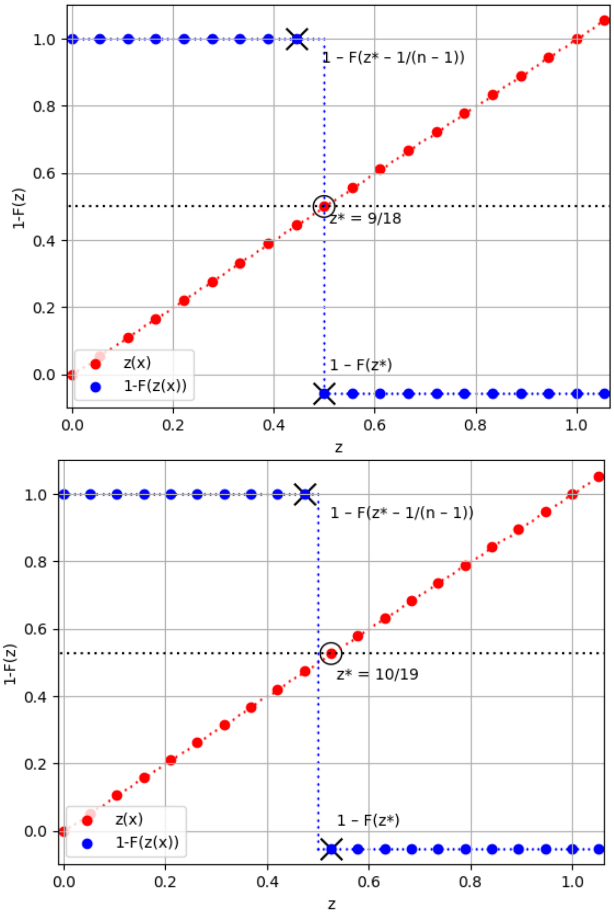
Corollary 2. Consider the game in Proposition 8. $\mathcal{U}_r$ admits exactly

$$\sum_{z \in \mathcal{Z}} \binom{(z_2 - z_1)(n-1)}{(z - z_1)(n-1)} \tag{3.18}$$

Nash equilibria. Note that, for Observation 8, $|\mathcal{Z}| = 1$ or $|\mathcal{Z}| = 2$.

If $|\mathcal{Z}| = 1$ and $z^* = z_1 = 1 - F(z^*)$ or $z^* = z_2 = 1 - F(z^* - (1+\epsilon)/(n-1))$, $\mathcal{U}_r$ admits *exactly one Nash equilibrium*. In this case, indeed, there is just one possible choice for the set of the agents playing action +1. This condition is satisfied, in particular, if the thresholds are sufficiently heterogeneous.

Figure 3.5. Let us consider the pure minority game with $n = 19$ and $n = 20$. Since the minority game over an undirected graph without self-loops is a potential game, we already know that there exists at least one Nash equilibrium. In particular, we find that $z^* = \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$ is a solution of the inequality in (3.16). Note that, in the first case where $n = 19$, $z^{**} = \frac{10}{18}$ also satisfies the condition.



Another approach for Nash equilibria

It is interesting to observe that the same results can be found using a slightly different approach.

Observation 9. Let $x \in \mathcal{A}^{\mathcal{V}}$ be an action configuration and $i^+_{min}(x) = \text{argmin}_{i \in \mathcal{V}^+(x)} r_i$. It holds that:

$$\{+1\} \in \mathcal{B}_i(x_{-i}), \ \forall i \in \mathcal{V}^+(x) \quad \Leftrightarrow \quad \{+1\} \in \mathcal{B}_{i^+_{min}(x)}(x_{-i^+_{min}(x)})$$

This means that players in $\mathcal{V}^+(x)$ are in equilibrium if and only if the player with the minimum threshold r_i is in equilibrium.

Proof. The right implication is trivial since $i^+_{min} \in \mathcal{V}^+(x)$ by definition.
On the other hand, if $\{+1\} \in \mathcal{B}_{i^+_{min}(x)}(x_{-i^+_{min}(x)})$, we have that for $i \in \mathcal{V}^+(x)$

$$n^+(x) - 1 \leq r_{i^+_{min}(x)}(n-1) \leq r_i(n-1)$$

where the second inequality holds by definition of $i^+_{min}(x)$. □

Observation 10. Similarly, given $x \in \mathcal{A}^{\mathcal{V}}$, if we denote $i^-_{max}(x) = \text{argmax}_{i \in \mathcal{V}^-(x)} r_i$, we have that:

$$\{-1\} \in \mathcal{B}_i(x_{-i}), \ \forall i \in \mathcal{V}^-(x) \quad \Leftrightarrow \quad \{-1\} \in \mathcal{B}_{i^-_{max}(x)}(x_{-i^-_{max}(x)})$$

Namely, players in $\mathcal{V}^-(x)$ are in equilibrium if and only if the player with the maximum threshold r_i is in equilibrium.

Proof. Again, the right implication is trivial, while the other one holds since

$$n^+(x) \geq r_{i^-_{max}}(n-1) \geq r_i(n-1)$$

for $i \in \mathcal{V}^-$. □

The two observations allow us to rewrite the conditions for x to be a Nash equilibrium in the following way:

$$\begin{cases} n^+(x) - 1 \leq r_{i^+_{min}(x)}(n-1) \\ n^+(x) \geq r_{i^-_{max}(x)}(n-1) \end{cases} \Leftrightarrow \begin{cases} \frac{n^+(x)-1}{n-1} \leq r_{i^+_{min}(x)} \\ \frac{n^+(x)}{n-1} \geq r_{i^-_{max}(x)} \end{cases} \tag{3.19}$$

Namely, the condition for $x \in \mathcal{A}^{\mathcal{V}}$ to be an equilibrium is

$$r_{i^-_{max}(x)} \leq \tilde{z}(x) \leq r_{i^+_{min}(x)} + \frac{1}{n-1} \tag{3.20}$$

Observation 11. Let $\{r_i\}_{i\in\{1,\dots,n\}}$ be an ordered sequence such that $0 \le r_1 \le \cdots \le r_n \le 1$ and let k be the minimum $k \in \{0,\dots,n-1\}$ such that $\frac{k}{n-1} \ge r_{n-k}$. Since k is the minimum k for the condition to hold, we have that $\frac{k-1}{n-1} \le r_{n-k+1}$. The action configuration $x^k \in \mathcal{A}^{\mathcal{V}}$ with $x_i^k = -1$, for $i \in \{1,\dots,n-k\}$ and $x_i^k = +1$, for $i \in \{n-k+1,\dots,n\}$ is a Nash equilibrium for the minority game with thresholds $\{r_i\}_{i\in\{1,\dots,n\}}$ over K_n.

Observation 11 suggests an algorithm to find a Nash equilibrium that has linear complexity if $\{r_i\}_{i\in\mathcal{V}}$ is already ordered. Indeed, once we have an ordered sequence of thresholds (we can obtain it in time $\mathrm{O}(n\log(n))$), we can find $k \in \{0,\dots,n-1\}$, and thus the equilibrium, with at most n iterations in a very simple way: we start from $k = 0$ and we check the condition $\frac{k}{n-1} \ge r_{n-k}$ until it is satisfied.

Chapter 4

Networks with coordinating and anti-coordinating agents

We begin this chapter with the definition of the heterogeneous mixed coordination anti-coordination game, which is the most general case studied in the thesis. Our first observation is that it is enough to introduce one edge between a coordinating and an anti-coordinating agent in order to lose the potential property. For instance, the discoordination game, defined as one coordinating agent and one anti-coordinating agent linked by a simple edge, is a famous example of a game of this class that does not admit Nash equilibria.

In the second section, we prove the main result of the thesis, which is to provide a sufficient condition for the existence of Nash equilibria. We recall that the condition is based on the idea of cohesiveness introduced by Morris in 1997 [11]. In particular, we show that if the subset of the coordinating agent is sufficiently cohesive then the existence of at least one Nash equilibrium is guaranteed.

We conclude the chapter by studying some cases of mixed coordination anti-coordination games where the previous hypothesis is not satisfied. In particular, we focus on two simple examples having homogeneous thresholds: the complete graph and a simple graph with one anti-coordinating agent. For these instances, we provide a complete analysis that remark that the condition is sufficient but not necessary.

4.1 Definition of the game and preliminary observations

Let us consider an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a given partition of the vertex set $\mathcal{V}_c \cup \mathcal{V}_a = \mathcal{V}$, $\mathcal{V}_c \cap \mathcal{V}_a = \emptyset$. We define the *heterogeneous mixed coordination anti-coordination game* with node weights $\{\alpha_i\}_{i\in\mathcal{V}}$, $\alpha_i \in \mathbb{R}$, as a binary network game with action set $\mathcal{A} = \{-1, +1\}$ and utilities $\{u_i\}_{i\in\mathcal{V}} : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ such that

$$u_i(x_i, x_{-i}) := \begin{cases} \sum_{j\in\mathcal{V}} W_{ij}x_ix_j - \alpha_i x_i & \text{if } i \in \mathcal{V}_c \\ -\sum_{j\in\mathcal{V}} W_{ij}x_ix_j + \alpha_i x_i & \text{if } i \in \mathcal{V}_a \end{cases} \tag{4.1}$$

In other words, in a mixed coordination anti-coordination game, a given subset of the participants plays with coordination incentives with their neighbors, while the rest of the players follows the anti-coordination rule. Therefore, nodes in $\mathcal{V}_c \subseteq \mathcal{V}$ are called *coordinating agents* while nodes in $\mathcal{V}_a = \mathcal{V} \setminus \mathcal{V}_c$ are called *anti-coordinating agents*. Observe that, if we introduce the quantity

$$\delta_i = \begin{cases} +1 & \text{if } i \in \mathcal{V}_c \\ -1 & \text{if } i \in \mathcal{V}_a \end{cases} \tag{4.2}$$

we can write the utility of any player $i \in \mathcal{V}$ in the compact form

$$u_i(x_i, x_{-i}) = \delta_i \left(\sum_{j\in\mathcal{V}} W_{ij}x_ix_j - \alpha_i x_i \right) \tag{4.3}$$

Note that this is a very general formulation of the game where we allow coordinating and anti-coordinating agents to interact over the same network with heterogeneous thresholds. In other words, we are modeling a situation where diversified players interact with different interests over an interconnected system.

Therefore, both the heterogeneous network coordination game and the heterogeneous network anti-coordination game are special cases of the previous game having respectively $\mathcal{V}_c = \emptyset$ and $\mathcal{V}_a = \emptyset$.

Consistently with the definition, the best-response function of a coordinating agent coincides with the BR of the heterogeneous network coordination game given in (2.2). On the other hand, in the case of an anti-coordinating agent, it is equal to the BR of a participant of the anti-coordination version of the game which can be found in (3.2). In formulas,

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j > \alpha_i \\ \{-1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j < \alpha_i \\ \{\pm 1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j = \alpha_i \end{cases} \qquad i \in \mathcal{V}_c$$

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j < \alpha_i \\ \{-1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j > \alpha_i \\ \{\pm 1\} & \text{if } \sum_{j\in\mathcal{V}} W_{ij}x_j = \alpha_i \end{cases} \qquad i \in \mathcal{V}_a$$

We recall that, according to (2.5) and (3.3), the best response function above can be rewritten in the form

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) > r_i w_i \\ \{-1\} & \text{if } w_i^+(x) < r_i w_i \\ \{\pm 1\} & \text{if } w_i^+(x) = r_i w_i \end{cases} \qquad i \in \mathcal{V}_c$$

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } w_i^+(x) < r_i w_i \\ \{-1\} & \text{if } w_i^+(x) > r_i w_i \\ \{\pm 1\} & \text{if } w_i^+(x) = r_i w_i \end{cases} \qquad i \in \mathcal{V}_a$$

where we recall that $w_i^+(x)$ is given by (2.3) and the thresholds r_i are defined according to (2.4).

In the previous chapter, we proved that the heterogeneous network coordination and the heterogeneous anti-coordination games are potential games, which implies that they always admit at least one Nash equilibrium. As we shall see, this is not true for the mixed coordination anti-coordination game.

Example 7 (Discoordination game)**.** Let us consider a very simple case, namely a two-player game with $\mathcal{V}_c = \{1\}$ and $\mathcal{V}_a = \{2\}$ over a line graph where both players have node weights $\alpha = 0$. Note that this is exactly the same as considering the *discoordination game* defined in Example 3, since one agent aims to coordinate with the other while the other wants the opposite outcome.

We recall that the game admits no Nash equilibria. In fact, if the players are coordinating which means that $x = (+1, +1)$ or $x = (-1, -1)$ then player 2 is not in equilibrium, namely $+1 \notin \mathcal{B}_2(+1)$ and $-1 \notin \mathcal{B}_2(-1)$. On the other hand, if they are in a profile with opposite actions then agent 1 is not anymore in equilibrium since $-1 \notin \mathcal{B}_i(+1)$ as well as $+1 \notin \mathcal{B}_i(-1)$. Remind that this game models the *matching pennies* game.

The fact that the set of the Nash equilibria of the discoordination game is empty, as well as the one of the star graph with one anti-coordinating agent in the middle, implies that the game is *not* a *potential* game. Therefore, the mixed coordination anti-coordination game is in general not a potential game. The following proposition proves that the game is never a potential game when coordinating and anti-coordinating agents coexist.

Proposition 9. Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, a partition of the node set $\mathcal{V}_c \cup \mathcal{V}_a = \mathcal{V}$ and a set of node weights $\{\alpha_i\}_{i\in\mathcal{V}}$, $\alpha_i \in \mathbb{R}$. If $\mathcal{V}_c$ and $\mathcal{V}_a$ are such that $\mathcal{V}_c \neq \emptyset$ and $\mathcal{V}_a \neq \emptyset$, the heterogeneous mixed coordination anti-coordination game over $\mathcal{G}$ is not a potential game.

Remark 4. The following proof is based on Corollary 2.9 of [9]. In our notation, it states that a strategic form game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is a potential game if and only if for every two players $i, j \in \mathcal{V}$, every action configuration $x_{-\{i,j\}} \in \mathcal{A}^{\mathcal{V}\setminus\{i,j\}}$ and every two possible actions of player i, which are $x_i, y_i \in \mathcal{A}$, and of player j, namely $x_j, y_j \in \mathcal{A}$, it holds

$$u_i(B) - u_i(A) + u_j(C) - u_j(B) + u_i(D) - u_i(C) + u_j(A) - u_j(D) = 0$$

where

$$A = (x_i, x_j, x_{-\{i,j\}}) \quad B = (y_i, x_j, x_{-\{i,j\}}) \quad C = (y_i, y_j, x_{-\{i,j\}}) \quad D = (x_i, y_j, x_{-\{i,j\}})$$

Note that, since in our case the action set is binary, it is enough to very the condition for $x_i = y_i = +1$, $x_j = y_j = -1$. In the proof, we will show that the property does not hold when we pick a coordinating agent and an anti-coordinating agent.

Proof. Consider a coordinating agent $i_c \in \mathcal{V}_c$ and an anti-coordinating agent $i_a \in \mathcal{V}_a$ such that $W_{i_a,i_c} \neq 0$ which means they interact in the game. Let $x_{-\{i_c,i_a\}} \in \mathcal{A}^{\mathcal{V}\setminus\{i_c,i_a\}}$. Note that

$$u_{i_c}(x_{i_c} = +1, x_{i_a} = +1, x_{-\{i_c,i_a\}}) = \sum_{\substack{j\in\mathcal{V}\\ j\neq i_a}} W_{i_cj}x_j + W_{i_ci_a} - \alpha_{i_c}$$

$$u_{i_c}(x_{i_c} = -1, x_{i_a} = +1, x_{-\{i_c,i_a\}}) = -\sum_{\substack{j\in\mathcal{V}\\ j\neq i_a}} W_{i_cj}x_j - W_{i_ci_a} + \alpha_{i_c}$$

Therefore

$$\begin{aligned}\Delta_{(+1,+1)\to(-1,+1)} &:= u_{i_c}(x_{i_c}=+1, x_{i_a}=+1, x_{-\{i_c,i_a\}}) - u_{i_c}(x_{i_c}=-1, x_{i_a}=+1, x_{-\{i_c,i_a\}}) = \\ &= 2\sum_{\substack{j\in\mathcal{V}\\ j\neq i_a}} W_{i_a j}x_j + 2W_{i_c i_a} - 2\alpha_{i_c}\end{aligned}$$

On the other hand, if we change the action of the anti-coordinating player i_a, we have

$$u_{i_c}(x_{i_c}=-1, x_{i_a}=+1, x_{-\{i_c,i_a\}}) = -\sum_{\substack{j\in\mathcal{V}\\ j\neq i_c}} W_{i_a j}x_j + W_{i_c i_a} + \alpha_{i_a}$$
$$u_{i_c}(x_{i_c}=-1, x_{i_a}=-1, x_{-\{i_c,i_a\}}) = \sum_{\substack{j\in\mathcal{V}\\ j\neq i_c}} W_{i_a j}x_j - W_{i_c i_a} - \alpha_{i_a}$$

from which we obtain

$$\Delta_{(-1,+1)\to(-1,-1)} := -2\sum_{\substack{j\in\mathcal{V}\\ j\neq i_c}} W_{i_c j}x_j + 2W_{i_c i_a} + 2\alpha_{i_a}$$

Similarly, we find

$$\Delta_{(-1,-1)\to(+1,-1)} := -2\sum_{\substack{j\in\mathcal{V}\\ j\neq i_a}} W_{i_a j}x_j + 2W_{i_c i_a} + 2\alpha_{i_c}$$

and

$$\Delta_{(-1,+1)\to(+1,+1)} := -2\sum_{\substack{j\in\mathcal{V}\\ j\neq i_c}} W_{i_c j}x_j + 2W_{i_c i_a} - 2\alpha_{i_a}$$

In conclusion, if we sum all the quantities

$$\Delta_{(+1,+1)\to(-1,+1)}+\Delta_{(-1,+1)\to(-1,-1)}+\Delta_{(-1,-1)\to(+1,-1)}+\Delta_{(+1,-1)\to(+1,+1)} = 8W_{i_a i_c} \neq 0$$

where the last inequality holds since $W_{i_a,i_c} \neq 0$. Therefore, we have proved that the game is not a potential game if there are at least two coordinating and anti-coordinating agents. □

Proposition 9 states that it is enough to introduce one anti-coordinating (resp. coordinating) player in a pure network coordination (resp. anti-coordination) game to lose the potential property. We remark that, even though the potential property is lost, there still can exist Nash equilibria. In the next section, a sufficient condition for the existence of Nash equilibria is provided.

We conclude the section by introducing a relevant example of mixed coordination anti-coordination game that will be recalled in the next sections.

Example 8 (Mixed majority-minority game)**.** Let us consider the special case where the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is unweighted. Furthermore let us assume homogeneous thresholds $r_i = \frac{1}{2}$ for all $i \in \mathcal{V}$. In this case, the utility in (4.3) becomes

$$u_i(x_i, x_{-i}) = \delta_i \sum_{j \in N_i} x_i x_j = \delta_i(|\{i \in N_i \mid x_i = x_j\}| - |\{i \in N_i \mid x_i \neq x_j\}|) \tag{4.4}$$

where δ_i is given by (4.2). We call this game *mixed majority-minority game* since coordinating and anti-coordinating agents are playing respectively a majority game and a minority game with their neighbors. Therefore, in this instance, we introduce the notations

$$\mathcal{V}_{maj} := \mathcal{V}_c \qquad \mathcal{V}_{min} := \mathcal{V}_a$$

The best response function of the mixed majority-minority game is given by

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > \frac{|N_i|}{2} \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < \frac{|N_i|}{2} \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = \frac{|N_i|}{2} \end{cases} \qquad i \in \mathcal{V}_{maj}$$

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| < \frac{|N_i|}{2} \\ \{-1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| > \frac{|N_i|}{2} \\ \{\pm 1\} & \text{if } |\{j \in N_i \mid x_j = +1\}| = \frac{|N_i|}{2} \end{cases} \qquad i \in \mathcal{V}_{min}$$

4.2 Cohesiveness and diffusivity

We are about to begin the main section of the thesis where we derive a sufficient condition for the existence of Nash equilibria in the heterogeneous mixed coordination anti-coordination game. In particular, we will prove that if the set $\mathcal{V}_c$ of the coordinating agents is sufficiently *cohesive*, then the set of the Nash equilibria of the game is surely nonempty.

Even though we have already introduced the idea of cohesiveness in the previous chapter, specifically in (2.9), let us recall the definition of this important property. Furthermore, we present the definition of the opposite concept, which we will call diffusivity.

Consider an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a threshold $0 \leq q \leq 1$, $q \in \mathbb{R}$. A subset of the vertex set $S \subseteq \mathcal{V}$ is called

- *q-cohesive* with respect to the graph $\mathcal{G}$ if for any $i \in S$ it holds

$$\frac{\sum_{j \in S} W_{ij}}{w_i} \geq q$$

- *q-diffusive* with respect to the graph $\mathcal{G}$ if for any $i \in S$ it holds

$$\frac{\sum_{j \in S} W_{ij}}{w_i} \leq q$$

Note that, if the graph is unweighted, then the set S is q-cohesive if for any $i \in S$ it holds

$$\frac{|N_i \cap S|}{|N_i|} \geq q$$

and q-diffusive if

$$\frac{|N_i \cap S|}{|N_i|} \leq q$$

for any $i \in S$.

In other words, in the unweighted case, a subset S is q-cohesive if each node in S has *at least* a fraction q of its neighbors in S. Conversely, S is q-diffusive if each node in S has *at most* a fraction q of its neighbors in S.

Observe that if a set is q-cohesive then it is also q'-cohesive for any $q' \leq q$. Similarly, a q-diffusive subset is also q'-diffusive for any $q' \geq q$.

Now, let us go back to the heterogeneous mixed coordination anti-coordination game. Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, a partition of the vertex set $\mathcal{V}_c \cup \mathcal{V}_a = \mathcal{V}$ and a set of node weights $\{\alpha_i\}_{i \in \mathcal{V}}$. We recall that, according to equation (2.4), thresholds can be written in the form

$$r_i := \frac{1}{2} + \frac{\alpha_i}{2w_i}$$

which permit to study the best response function in terms of the fraction of neighbors playing action $+1$. We introduce the following notations

$$r^c_{max} := \max_{i \in \mathcal{V}_c} r_i \quad r^c_{min} := \min_{i \in \mathcal{V}_c} r_i \quad r^a_{max} := \max_{i \in \mathcal{V}_a} r_i \quad r^a_{min} := \min_{i \in \mathcal{V}_a} r_i \tag{4.5}$$

Proposition 10. Let us consider a mixed coordination anti-coordination game with node weights $\{\alpha_i\}_{i \in \mathcal{V}}$, $\alpha_i \in \mathbb{R}$. If $\mathcal{V}_c$ is r^c_{max}-cohesive and $\mathcal{V}_a$ is $(1 - r^a_{max})$-diffusive, then the set $\mathcal{N}$ of its Nash equilibria is non-empty and contains the action configuration

$$x^*_i = \begin{cases} +1 & \text{if } i \in \mathcal{V}_c \\ -1 & \text{if } i \in \mathcal{V}_a \end{cases} \tag{4.6}$$

Proof. Let us consider any $i \in \mathcal{V}_c$. We have that

$$w^+_i(x^*) = \sum_{\substack{j \in \mathcal{V} \\ x^*_j = +1}} W_{ij} = \sum_{j \in \mathcal{V}^c} W_{ij} \overset{(1)}{\geq} r^c_{max} w_i \geq r_i w_i$$

where (1) comes straightforward from the definition of a r^c_{max}-cohesive subset. This means that $+1 \in \mathcal{B}_i(x^*_{-i})$ for all $i \in \mathcal{V}_c$.
On the other hand, if we consider $i \in \mathcal{V}_a$, it holds

$$\begin{aligned} w_i^+(x^*) &= w_i - w_i^-(x^*) = w_i - \sum_{j\in\mathcal{V}^a} W_{ij} \\ &\geq w_i - (1 - r^c_{max})w_i = r^c_{max} w_i \geq r_i w_i \end{aligned}$$

and thus $-1 \in \mathcal{B}_i(x^*_{-i})$, $i \in \mathcal{V}_a$. In conclusion, we have shown that $x^*_i \in \mathcal{B}_i(x^*_{-i})$ for every $i \in \mathcal{V}$. □

Note that also if $\mathcal{V}_a$ is $(1 - r^c_{min})$-cohesive and $\mathcal{V}_a$ is r^a_{min}-diffusive the set $\mathcal{N}$ of the Nash equilibria is non-empty. In this case, a possible Nash equilibrium is given by

$$x_i^{**} = \begin{cases} -1 & \text{if } i \in \mathcal{V}_c \\ +1 & \text{if } i \in \mathcal{V}_a \end{cases}$$

Proposition 10 points out a sufficient condition for the existence of at least one Nash equilibrium.

Example 9. Consider the mixed majority-minority game over $\mathcal{G} = \mathcal{K}_n$. Observe that, since $r_i = \frac{1}{2}$ for all $i \in \mathcal{V}$, we have

$$r^c_{max} = 1 - r^c_{min} = \frac{1}{2}$$

Let $\mathcal{V}_{min} \cup \mathcal{V}_{maj} = \mathcal{V}$, $\mathcal{V}_{min} \cap \mathcal{V}_{maj} = \emptyset$, be such that $|\mathcal{V}_{min}| \leq |\mathcal{V}_{maj}| - 1$. In this instance, the subset $\mathcal{V}_{maj} \subseteq \mathcal{V}$ is $\frac{1}{2}$-cohesive. In fact, given $i \in \mathcal{V}_{maj}$,

$$\frac{|N_i \cap \mathcal{V}_{maj}|}{|N_i|} = \frac{|\mathcal{V}_{maj}| - 1}{|\mathcal{V}_{min}| + |\mathcal{V}_{maj}| - 1} \geq \frac{|\mathcal{V}_{maj}| - 1}{|\mathcal{V}_{maj}| - 1 + |\mathcal{V}_{maj}| - 1} = \frac{1}{2}$$

Moreover, $\mathcal{V}_{min}$ is $\frac{1}{2}$-diffusive. Indeed

$$\frac{|\mathcal{V}_{min}| - 1}{|\mathcal{V}_{maj}| + |\mathcal{V}_{min}| - 1} \leq \frac{|\mathcal{V}_{min}| - 1}{|\mathcal{V}_{min}| + 1 + |\mathcal{V}_{min}| - 1} < \frac{|\mathcal{V}_{min}|}{2\,|\mathcal{V}_{min}|} = \frac{1}{2}$$

for all $i \in \mathcal{V}_{min}$. Therefore, the mixed majority-minority game over a complete graph having $|\mathcal{V}_{min}| < |\mathcal{V}_{maj}|$ admits the Nash equilibrium x^* defined in (4.6).

This condition is simple and intuitive but stronger than needed. We can move forward thanks to the following simple observation which is strongly related to the previous statement.

Observation 12. Consider a mixed coordination anti-coordination game with heterogeneous thresholds such that $\mathcal{V}_c$ is a r^c_{max}-cohesive subset of $\mathcal{V}$. Let $x \in \mathcal{A}^{\mathcal{V}}$ be an action configuration having $x_i = +1$ for all $i \in \mathcal{V}_c$. Then for any $i \in \mathcal{V}_c$ it holds that

$$x_i \in \mathcal{B}_i(x_{-i})$$

irrespective of the actions x_j, $j \in \mathcal{V}_a$ chosen by the remaining players. Note that the same observation holds true if the set of the coordinating players $\mathcal{V}_c$ is $(1 - r^c_{min})$-cohesive and we set all their actions to -1.

In other words, if the set of the coordinating agents is r^c_{max}-cohesive, then the coordinating players are "in equilibrium" if they all play the same action, *regardless of the choices of the anti-coordinating players.*

According to this consideration, we can change the point of view on the problem. Specifically, consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, a subset of the vertex set $\mathcal{V}_a \subseteq \mathcal{V}$ and a set of given node weights $\alpha_i \in \mathbb{R}$, $i \in \mathcal{V}$. If we assume that $\mathcal{V}_a$ is a r^c_{max}-cohesive subset of the nodes, we can focus on a heterogeneous network anti-coordination game with node weights $\alpha_i \in \mathbb{R}$, $i \in \mathcal{V}_c$ having a set $\mathcal{V}_a$ of $(+1)$-*stubborn players.* Trivially, if this game admits a Nash equilibrium, then the mixed coordination anti-coordination game over $\mathcal{G}$, having the same thresholds and $\mathcal{V}_c$ coordinating players, also admits a Nash equilibrium, which can be directly derived from the other one. If we look at the mixed coordination anti-coordination game in this new terms, we can make further statements since we already studied this problem in the previous section. Indeed, following the same procedure, we can rewrite the utility of an agent $i \in \mathcal{V}_a$ as

$$\begin{aligned} u_i(x_i, x_{-i}) &= -\sum_{j \in \mathcal{V}} W_{ij} x_i x_j + \alpha_i x_i = \\ &= -\sum_{j \in \mathcal{V}_a} W_{ij} x_i x_j - \sum_{j \in \mathcal{V}_c} W_{ij} x_i + \alpha_i x_i = \\ &= -\sum_{j \in \mathcal{V}_a} W_{ij} x_i x_j + \left(\alpha_i - \sum_{j \in \mathcal{V}_c} W_{ij} \right) x_i = \\ &= -\sum_{j \in \mathcal{V}_a} W_{ij} x_i x_j + \tilde{\alpha}_i x_i \end{aligned}$$

where we denoted

$$\tilde{\alpha}_i := \alpha_i - \sum_{j \in \mathcal{V}_c} W_{ij} \tag{4.7}$$

Therefore, we have found that, if the set of coordinating agents is r^c_{max}-cohesive, we can investigate the existence of Nash equilibria by studying a network anti-coordination game defined over the induced subgraph $\mathcal{G}[\mathcal{V}_a] = (\mathcal{V}_a, \mathcal{E}_a, W_{|\mathcal{V}_a \times \mathcal{V}_a})$ where $\mathcal{E}_a = \{\{i,j\} \in \mathcal{E} : i,j \in \mathcal{V}_a\}$ with node weights $\tilde{\alpha}_i$, $i \in \mathcal{V}_a$ given in (4.7). We know from the previous chapter that this game is potential and it has potential function given by (3.8), which means that the existence of at least one Nash equilibrium is guaranteed.

We remark that a similar method can be applied when the set of the coordinating agents is $(1 - r^c_{min})$-cohesive.

Theorem 11. Consider an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, a partition of the vertex set $\mathcal{V}_c \cup \mathcal{V}_a = \mathcal{V}$ and a set of node weights $\{\alpha_i\}_{i \in \mathcal{V}}$, $\alpha_i \in \mathbb{R}$. Assume that $\mathcal{V}_c$ is either

- r^c_{max}-cohesive or
- $(1 - r^c_{min})$-cohesive

where r^c_{max} and r^c_{min} are defined according to (4.5). Then, the mixed coordination anti-coordination game over $\mathcal{G}$ having node weights α_i,$i \in \mathcal{V}$ admits at least one Nash equilibrium.

For instance, let us suppose that $\mathcal{V}_c$ is r^c_{max}-cohesive. Such equilibrium can be found in the following way. Let us denote as $\mathcal{G}_a = \mathcal{G}[\mathcal{V}_a]$ the subgraph induced by $\mathcal{V}_a$ and as $\mathcal{V}_a = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_K$ the partition in connected components of $\mathcal{G}_a$. For each component $j \in \{1, \ldots, K\}$, we define the anti-coordination game with node weights $\tilde{\alpha}_i$ given in (4.7) and we find a Nash equilibrium of the game. We know that this is always possible since the game with only anti-coordinating agents is potential. Given the Nash equilibria $x^1, \ldots, x^K$ of the previous games, the Nash equilibrium of the mixed coordination anti-coordination game with node weights α_i, $i \in \mathcal{V}$ is given by $x^* \in \mathcal{A}^{\mathcal{V}}$ such that

$$x^*_i = \begin{cases} +1 & \text{if } i \in \mathcal{V}_c \\ x^0_i & \text{if } i \in \mathcal{V}_0 \\ \cdots & \\ x^K_i & \text{if } i \in \mathcal{V}_K \end{cases} \tag{4.8}$$

If the set of the coordinating agents is $(1 - r^c_{min})$-cohesive, a similar procedure can be applied. The only difference is that coordinating agents become (-1)-stubborn players and therefore the definition of $\tilde{\alpha}_i$ slightly changes.

Example 10. Let us consider a mixed coordination anti-coordination game with homogeneous threshold $r = \frac{1}{2}$ over a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let us suppose that the set of the coordinating agents is such that the subgraph induced by $\mathcal{V}_c$ is complete. Then, if $|\mathcal{V}_c| > |\mathcal{V}_a|$, the existence of at least one Nash equilibrium is guaranteed by Proposition 11.

For instance, suppose that we have a fully connected population of coordinating agents and we introduce some anti-coordinating agents in order to destabilize it. The previous proposition affirms that the existence of at least one Nash equilibrium is guaranteed until we introduce as many anti-coordinating players as the number of coordinating agents already existing in the network.

4.3 Non-cohesive examples

In the previous section, we observed that if the set of the coordinating players is $r-$*cohesive*, where $r = \max_{i\in\mathcal{V}_c} r_i$, then the existence of at least one Nash equilibrium is guaranteed. Similarly, the set of Nash equilibria is nonempty if coordinating agents form a $1-r^c_{min}$-cohesive subset. We conclude the chapter by investigating the existence of Nash equilibria in some simple examples where the set of the coordinating players is not cohesive enough.

4.3.1 The mixed majority-minority game over a complete graph

Consider a mixed majority-minority game over a complete graph $\mathcal{K}_n$ having $\mathcal{V}_{maj}$ majority players and $\mathcal{V}_{min}$ minority players. If $|\mathcal{V}_{min}| < |\mathcal{V}_{maj}|$, we know from Example 9 that the existence of at least one Nash equilibrium is guaranteed (Proposition 10). On the other hand, the subset $\mathcal{V}_{maj}$ loses the $\frac{1}{2}$-cohesiveness property once the number of minority players reaches the number of majority players. Let us focus on this second case in which $|\mathcal{V}_{min}| \geq |\mathcal{V}_{maj}|$.
Given an action configuration $x^* \in \mathcal{A}^{\mathcal{V}}$, we introduce the following notation

$$\mathcal{V}^+_{maj}(x^*) := \{i \in \mathcal{V}_{maj} \mid x^*_i = +1\}$$
$$\mathcal{V}^-_{maj}(x^*) := \{i \in \mathcal{V}_{maj} \mid x^*_i = -1\}$$

Similarly, we define $\mathcal{V}^+_{min}(x^*)$ and $\mathcal{V}^-_{min}(x^*)$. For simplicity, we denote as $n_{maj} = |\mathcal{V}_{maj}|$ the number of majority players and as $n_{min} = |\mathcal{V}_{min}|$ the number of minority players. Note that $|\mathcal{V}^-_{maj}(x^*)| = |\mathcal{V}_{maj}| - |\mathcal{V}^+_{maj}(x^*)| = n_{maj} - |\mathcal{V}^+_{maj}(x^*)|$ and, analogously, $\mathcal{V}^-_{min}(x^*) = n_{min} - \mathcal{V}^+_{min}(x^*)$.

Let us suppose that $x^* \in \mathcal{A}^{\mathcal{V}}$ is a Nash equilibrium. Without loss of generality, we can assume $\mathcal{V}^+_{maj}(x^*) \neq \emptyset$ and consider $i \in \mathcal{V}^+_{maj}(x^*)$. We have that $+1 \in \mathcal{B}_i(x^*_{-i})$ if and only if

$$\frac{n-1}{2} = \frac{|N_i|}{2} \leq |\{i \in N_i \mid x^*_i = +1\}| = |\mathcal{V}^+_{maj}(x^*)| - 1 + \left|\mathcal{V}^+_{min}(x^*)\right| \tag{4.9}$$

On the other hand, if $\mathcal{V}^+_{maj}(x^*) \neq \mathcal{V}_{maj}$, we can consider $i \in \mathcal{V}^-_{maj}(x^*)$. In this case, we obtain the necessary condition

$$\frac{n-1}{2} \geq |\mathcal{V}^+_{maj}(x^*)| + \left|\mathcal{V}^+_{min}(x^*)\right|$$

Combining the two, we obtain the system

$$\begin{cases} |\mathcal{V}^+_{maj}(x^*)| + \left|\mathcal{V}^+_{min}(x^*)\right| \geq \frac{n+1}{2} \\ |\mathcal{V}^+_{maj}(x^*)| + \left|\mathcal{V}^+_{min}(x^*)\right| \leq \frac{n-1}{2} \end{cases}$$

which has no solution. Therefore, we have found

$$\mathcal{V}^+_{maj}(x^*) = \mathcal{V}_{maj} \tag{4.10}$$

In words, we proved that in a Nash equilibrium all majority players choose the same action. In particular, in the current profile, all majority players choose action +1 since we assumed a configuration where at least one majority agent plays action +1. According to this discovery, we can rewrite the condition in (4.9) in this way

$$\frac{n-1}{2} \leq |\mathcal{V}^+_{maj}(x^*)| - 1 + \left|\mathcal{V}^+_{min}(x^*)\right| = n_{maj} - 1 + \left|\mathcal{V}^+_{min}(x^*)\right|$$

Thus, in this instance, a necessary condition for x^* to be a Nash equilibrium is

$$\left|\mathcal{V}^+_{min}(x^*)\right| \geq \frac{n - 2n_{maj} + 1}{2}$$

If we consider $i \in \mathcal{V}^+_{min}(x^*)$, we find the opposite condition

$$\frac{n-1}{2} \geq n_{maj} + \left|\mathcal{V}^+_{min}(x^*)\right| - 1 \quad \Leftrightarrow \quad \left|\mathcal{V}^+_{min}(x^*)\right| \leq \frac{n - 2n_{maj} + 1}{2}$$

Observe that the last condition is never satisfied if $n_{maj} > n_{min}$, which implies $2n_{maj} > n_{maj} + n_{min} = n$. This proves that if the majority of the agents plays a majority game Nash equilibria are action configurations where all majority players pick the same action, while the minority players choose the opposite one.

Let us go back to the case where most of the agents play the minority game. If we combine the two previous conditions, we trivially find

$$\left|\mathcal{V}_{min}^{+}(x^*)\right| = \frac{n - 2n_{maj} + 1}{2} = \frac{n+1}{2} - n_{maj} \tag{4.11}$$

Note that, since we are dealing with cardinalities, the number of minority players choosing action $+1$ must be a natural number, namely $\left|\mathcal{V}_{min}^{+}(x^*)\right| \in \mathbb{N}$. Therefore, the equation in (4.11) makes senses if only if

1. $\frac{n+1}{2} \in \mathbb{N}$, which means that n must be an *odd* number
2. $n_{maj} \leq \frac{n+1}{2}$, which is satisfied under the assumption that $n_{maj} \leq n_{min}$.

Finally, let us check the condition for the remaining players $i \in \mathcal{V}_{min}^{-}(x^*)$. They are in equilibrium if

$$\frac{n-1}{2} \leq |\mathcal{V}_{maj}^{+}(x^*)| + \left|\mathcal{V}_{min}^{+}(x^*)\right| = n_{maj} + \frac{n+1}{2} - n_{maj} = \frac{n+1}{2}$$

which is satisfied. Therefore, an action configuration satisfying the conditions in (4.10) and (4.11) is a Nash equilibrium of the game if n is odd and $n_{maj} \leq n_{min}$. The two conditions are sufficient and necessary, which means that we have found all the Nash equilibria of the game. Note that, since (4.11) is satisfied only if n is odd, the even case admits no Nash equilibria.
The following proposition summarizes the results.

Proposition 12. Consider a mixed majority-minority game over $\mathcal{K}_n$ with $n_{maj} = |\mathcal{V}_{maj}|$ majority players and $n_{min} = |\mathcal{V}_{min}|$ minority players. We distinct 3 possible cases:

1. If $n_{maj} > n_{min}$, namely if the majority of the participants play the majority game, the existence of at least one Nash equilibrium is guaranteed and in particular

   $$\mathcal{N} = \left\{x, -x \in \mathcal{A}^{\mathcal{V}} \text{ s.t. } \mathcal{V}_{maj}^{+}(x) = \mathcal{V}_{maj} \text{ and } \mathcal{V}_{min}^{-}(x) = \mathcal{V}_{min}\right\}$$

   Note that $|\mathcal{N}| = 2$.

2. If $n_{maj} \leq n_{min}$ and n is odd, the set $\mathcal{N}$ of the Nash equilibria of the game is still non-empty. Specifically

   $$\mathcal{N} = \left\{x, -x \in \mathcal{A}^{\mathcal{V}} \text{ s.t. } \mathcal{V}_{maj}^{+}(x) = \mathcal{V}_{maj} \text{ and } \left|\mathcal{V}_{min}^{-}(x)\right| = \frac{n+1}{2} - n_{maj}\right\}$$

   In this instance, $|\mathcal{N}| = 2\binom{n_{min}}{(n+1)/2 - n_{maj}}$.

3. If $n_{maj} \leq n_{min}$ and n is even, the game admits no Nash equilibria.

4.3.2 The mixed majority-minority game with one anti-coordinating agent

Let us consider the mixed majority-minority game on general simple graphs. In particular, we focus on the simplest case where $|\mathcal{V}_{min}| = 1$. Without loss of generality, we can assume $\mathcal{V}_{min} = \{0\}$. Let us denote the set of the leaves in the neighborhood of the minority node as

$$F := \{i \in N_0 \mid |N_i| = 1\} = \{i \in \mathcal{V} \mid N_i = \{0\}\} \tag{4.12}$$

where the second equality remarks that set F coincides with the set of the nodes in the vertex set which are linked only to node 0. Note that $\mathcal{V}_{maj}$ is $\frac{1}{2}$-cohesive if and only if $F = \emptyset$.

Proposition 13. Consider a mixed majority-minority game over a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}_{min} = \{0\}$. Moreover, let F be defined as in (4.12).

1. If $|F| \leq \frac{|N_0|}{2}$, then the set $\mathcal{N}$ of its Nash equilibria is non-empty and contains the Nash equilibrium $x^* \in \mathcal{A}^{\mathcal{V}}$ such that

$$x_i^* = \begin{cases} +1 & \text{if } i \notin F \cup \{0\} \\ -1 & \text{otherwise} \end{cases}$$

   As a straightforward consequence, $x_-^* = -x^*$ is a Nash equilibrium too.

2. Otherwise, if $|F| > \frac{|N_0|}{2}$, the game admits no Nash equilibria.

Proof. In the first case where $|F| \leq \frac{|N_0|}{2}$, we need to verify that x^* is a Nash equilibrium, that is $x_i^* \in B_i(x_{-i}^*)$, $\forall i \in \mathcal{V}$.
Consider a leaf $i \in F$. We have that $-1 \in \mathcal{B}_i(x_{-i}^*)$ since

$$\frac{\left|\left\{j \in N_i \mid x_j^* = +1\right\}\right|}{|N_i|} = \frac{\left|\left\{j \in \{0\} \mid x_j^* = +1\right\}\right|}{1} = 0 \leq \frac{1}{2}$$

On the other hand, let us consider a majority player which is not a leaf, namely $i \notin F \cup \{0\}$. In this instance, we have

$$\frac{\left|\left\{j \in N_i \mid x_j^* = +1\right\}\right|}{|N_i|} \geq \frac{|N_i| - 1}{|N_i|} \geq \frac{1}{2}$$

where the first inequality is true since $F \cap N_i = \emptyset$, which implies $\left\{ j \in N_i \mid x_j^* = -1 \right\} \subseteq \{0\}$, while the second one holds since $|N_i| > 1$. Therefore, $+1 \in \mathcal{B}_i(x_{-i}^*)$, $\forall i \notin F \cup \{0\}$. Lastly, if $i = 0$,

$$\frac{\left|\left\{ j \in N_0 \mid x_j^* = +1 \right\}\right|}{|N_0|} = \frac{|N_0| - |F|}{|N_0|} = 1 - \frac{|F|}{|N_0|} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

which means $-1 \in \mathcal{B}_0(x_{-0}^*)$, as node 0 plays the minority game.
Now let us suppose $|F| > \frac{|N_0|}{2}$. Conversely to the previous case, this time we want to prove that there are no action configurations which are Nash equilibria. We prove it by contradiction.
Let us suppose that $x^* \in \mathcal{A}^{\mathcal{V}}$ is a Nash equilibrium. Without loss of generality, we can assume $x_0^* = -1$. Note that $\mathcal{B}_i(x_{-i}^*) = \{-1\}$ for all $i \in F$. Indeed, if $i \in F$, then

$$\frac{\left|\left\{ j \in N_i \mid x_j^* = +1 \right\}\right|}{|N_i|} = |\{ j \in \{0\} \mid x_j = +1 \}| = 0 \leq \frac{1}{2}$$

Since x^* is Nash equilibrium, this means that $x_i^* = -1$ for all $i \in F$. Therefore

$$\frac{\left|\left\{ j \in N_0 \mid x_j^* = +1 \right\}\right|}{|N_0|} \leq \frac{|N_0| - |F|}{|N_0|} = 1 - \frac{|F|}{|N_0|} \leq 1 - \frac{1}{2} = \frac{1}{2}$$

Given that node 0 plays the minority game, this last inequality means that $\mathcal{B}_0(x_{-0}^*) = \{+1\}$, which is a contradiction since we assumed $x_0^* = -1$. $\square$

Chapter 5

Conclusions

In the thesis, we provided an overview of games with coordinating and anti-coordinating agents. The motivation behind the interest in modeling these two opposite types of interaction is given by the simplicity of their definition and their wide use in the applications. For instance, games with coordinating agents model the spread of social behaviors, technological innovations or viral infections. On the other hand, results on anti-coordinating agents can be applied in traffic congestion problems or division of labor.

In particular, we focused on heterogeneous models where agents with opposite interests and different preferences interact over the same network. In the applications, this game models heterogeneous populations of conforming and nonconforming agents. We recall that network coordination games and network anti-coordination games where all players have the same behavior have been largely studied in the literature, while much less is known for the heterogeneous cases.

The first important observation of the thesis is that the potential property of the homogeneous symmetric version of the games is preserved when players have the same underling interests but heterogeneous priorities. Potential games share special properties including the existence of Nash equilibria. More in general, this means that population heterogeneity as long as irregular network structure cannot be a cause of nonconvergence issues. A similar result was obtained by Cao et al [12] in 2016 although the authors do not explicitly discuss the potential property and the deriving peculiarities.

We remark that the regular behavior of the heterogeneous network anti-coordination game, in particular, is not trivial. Therefore, it is interesting to observe the aspect of the Nash equilibria in the special case of the complete graph . Although we are dealing with a specific example, the characterization given at the end of Chapter 3 is useful to picture the behavior of the game and permits to get an idea of its properties in the general case. Moreover, the study is done according to the threshold cumulative distribution function generalizing some ideas of the linear-threshold model [6]. As we shall see, this interpretation can be useful to study the dynamics of the game.

Furthermore, the reformulation of the games in terms of network coordination and anti-coordination games with stubborn players highlights some interesting characteristics of the games and permits to see the behavior of the players from a different point of view. For instance, it is remarkable that if one consider the heterogeneous case as a modification of the symmetric one given by the addiction of stubborn players, then the potential property mentioned above comes straightforward. Indeed, as we pointed out in the thesis, the potential function of the game with stubborns can be derived from the potential function of the homogeneous case by simply substituting in the general formula the actions of the stubborn players with one or minus one respectively.

While games with only coordinating or anti-coordinating agents present a quite regular behavior, we proved that it is enough to introduce one single interaction between two players having opposite outcomes to lose the potential property. This is actually the main problem addressed by the thesis. In fact, the main focus of the work is on mixed network coordination anti-coordination games with heterogeneous thresholds.

Given that mixture of coordinating and anti-coordinating agents do not in principle admit Nash equilibria, we provided a sufficient condition for the existence of equilibrium states which is based on the idea of cohesiveness of a subset introduced by Morris [11].

In real applications, we can imagine cohesive subsets of coordinating agents as closed groups of people that are highly connected and have similar thoughts and behaviors. In this case, it is harder to destabilize the system and many nonconformists are needed in order to lose the property of the existence of at least one equilibrium state. What we essentially proved is that if the hypothesis are satisfied, not just the coordinating players are able to find an equilibrium, but also anti-coordinating agents can be set in such a way that they have no interests in changing the actions. In other words, when coordinating agents have no incentives in changing their strategy, anti-coordinating agents adapt themselves to the situation.

The condition is relevant since it is general and includes many different network structures. On the other hand, it is far from being necessary. In the last part of the thesis, we provided some simple examples of games that admit Nash equilibria even if the set of the coordinating agents is not sufficiently cohesive.

In conclusion, the work points out some important characteristics of mixed network coordination anti-coordination games with a special regard to the investigation of the existence of Nash equilibria. More specifically, the thesis provides essential observations for the study of games with only coordinating or anti-coordinating agents and moves a first step in the analysis of the mixed case.

Many questions remain unanswered, but the work highlights some promising directions that can be taken.

For instance, once the game is defined, one can be interested in defining the evolutionary dynamics of the game. In this way, it is possible to investigate the convergence to Nash equilibria. The notion of best-response dynamics is the most intuitive example of *asynchronous dynamics*. Each player is equipped with an independent Poisson clock of rate 1: when it rings, he is allowed to update his strategy according to the best response function. In potential games, the dynamics almost surely converges to the set of Nash equilibria. Therefore, if we assume that in the initial state all coordinating players are in a consensus configuration, then the best-response dynamics of the mixed coordination anti-coordination game will converge to a Nash equilibrium with probability one when the hypothesis of the theorem are satisfied. It could be interesting to investigate the behavior of the system when coordinating agents are not in a consensus configuration at the beginning of the dynamics.

Furthermore, another possible dynamics that might occur on the network is the *synchronous best response dynamics* where each player has a binary state and updates it at discrete time instants according to his best response function. The difference from the previous dynamics is that in this second case all players update their actions at the same time. Let us focus on the anti-coordination game with heterogeneous thresholds on a fully-connected population. The main idea behind the results of the last section of Chapter 3 is to study the behavior of the game over the complete graph in terms of a slightly modified threshold cumulative distribution function. This interpretation of the game is relevant also when studying the synchronous best response dynamics. Indeed, following similar steps, one can find that the dynamical system satisfies a condition which is analogous to the one that we firstly introduced in (3.15). In particular, under some assumptions, the condition becomes an equality and the dynamics can be studied as a simpler dynamical system that depends only on the fraction of players playing action +1. A similar method was proposed to study the linear-threshold model which addresses the coordination case. Therefore, when the population are mixed, it might be possible to find a unique dynamical system that represent the entire synchronous best response dynamics and depends only on the fraction of players picking action +1.

Finally, recalling that the condition for the existence of Nash equilibria is sufficient but not necessary, a challenging problem is to focus on the non-cohesive examples and try to identify more general conditions for the existence on Nash equilibria of the game and eventually study the convergence to these other configurations according to the definition of the dynamics.

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