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Hardy type inequalities on graphs

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Chapter 1

Introduction

The subject of this thesis is the analysis of Hardy type inequalities on graphs. A graph is a pair $\Gamma = (V, E)$ where V denotes the set of vertices and E the set of edges. If $(x, y) \in E$ we say that x and y are neighbours and we write $x \sim y$. In this work we always consider infinite, locally finite connected graphs with the usual discrete metric. This means that V is countably infinite, every vertex has a finite number of neighbours and for every $x, y \in V$ with $x \neq y$ a path from x to y exists.
Given a graph $\Gamma = (V, E)$ and a nonnegative operator P defined on the set $C(V) = \{f : V \to \mathbb{R}\}$, for *Hardy type inequality* on Γ we roughly mean a functional inequality of the form $P \geq CW$ involving a positive weight function W which has to be taken as "large" as possible and a (possibly optimal) positive constant C.

In 1921 Landau proved that

$$\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^2 \geq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|\varphi(n)|^2}{n^2},$$

for all finitely supported $\varphi : \mathbb{N} \to \mathbb{R}$ such that $\varphi(0) = 0$.
This inequality was stated before by Hardy, so it is known as *discrete Hardy inequality.* It represents the simplest example of Hardy inequality on a graph. Since then, Hardy type inequalities have been of great interest to mathematicians in various contexts. In Chapter 3 of the thesis, following [7], we present a historical excursus about the development of the Hardy inequality during the 1920s in both its discrete and continuous settings, showing classical proofs based on elementary calculus inequalities.

Aiming to study Hardy inequalities on more general graphs, we recall that the combinatorial Laplacian on Γ is a linear operator which acts on functions $f : V \to \mathbb{R}$ as follows

$$\Delta f(x) = \sum_{y \sim x} \Big(f(x) - f(y) \Big).$$

It is known that Δ is a symmetric and nonnegative operator and it is a ℓ^2 bounded operator on a graph Γ if and only if Γ has bounded vertex degree. Moreover, as shown in [10], in general Δ is essentially self-adjoint and

$$\langle \Delta f, f \rangle_{\ell^2} = \frac{1}{2} \sum_{\substack{x,y \in V \\ x \sim y}} \Big(f(x) - f(y) \Big)^2$$

holds true for all functions f in $C_0(V)$.
In our study of Hardy weights a crucial role is played by the *Green function* associated to Δ on a graph $\Gamma = (V, E)$ which is defined as follows

$$G(x, y) = \sum_{n=0}^{\infty} p_n(x, y),$$

where $x, y \in V$ and $p_n(x, y)$ are the elements of the n-th power of the transition matrix. One of its main properties is the following: when we fix a vertex $o \in V$ the one variable function $x \mapsto G_o(x) := G(x, o)$ is harmonic outside o. In Chapter 2 of the thesis we illustrate the explicit construction of the Green function on two particular classes of graphs: the homogeneous trees and the bi-regular trees. The first one, denoted by $\mathbb{T}_{q+1}$, has the property that all the vertices have $q+1$ neighbours. In the second one, denoted by $\mathbb{T}_{P,D}$, a vertex has either P or D neighbours if its distance from a reference vertex is, respectively, even or odd. We compute the Green functions by making use of certain geometric properties of these trees.

The core of the present thesis is Chapter 4, where we discuss an alternative approach with respect to those outlined in Chapter 3 to derive Hardy inequalities: the *supersolution method*. Following [1], we show that given a positive superharmonic functions u, the Hardy type inequality

$$\frac{1}{2} \sum_{\substack{x,y \in V \\ x \sim y}} \Big(\varphi(x) - \varphi(y) \Big)^2 \geq \sum_{x \in V} \frac{\Delta u(x)}{u(x)} \varphi^2(x) \tag{1.1}$$

holds for all finitely supported functions φ on V. Subsequently, we apply this technique in $\mathbb{T}_{q+1}$ and in $\mathbb{T}_{P,D}$. More precisely, we determine radial positive superharmonic functions on these trees which yield relevant Hardy inequalities.
Recently, a refinement of the supersolution method has been developed in [6], where the notion of *optimal weight* is introduced. In particular, if a weight W is optimal for the operator P, then the Hardy inequality fails for every $\tilde{W} > W$.
We note that the Hardy weights derived from (1.1) need not to be optimal. The abovementioned theory developed in [6] is based on the use of positive H-superharmonic functions to derive optimal weights for Schrödinger operators, where for a Schrödinger operator on V we mean an operator of the form $H = \Delta + Q$, with Q given potential.

In Section 4.3 we adapt to our context the general results of [6] to derive *optimal* Hardy weights for Δ in $\mathbb{T}_{q+1}$. This is achieved by constructing suitable radial H-superharmonic functions among which the square root of the Green function is a particular case. It is worth mentioning that, as a by-product of our results, in Section 4.4 we obtain a new family of improved Poincaré inequalities of the form

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y) \Big)^2 \geq \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} R(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}).$$

Here $\Lambda_q > 0$ is the bottom of the Laplacian ℓ^2 spectrum on $\mathbb{T}_{q+1}$ and $R \geq 0$ are various optimal reminder terms, corresponding to the weights discussed before in the Hardy type inequality.

Chapter 2

Graphs and combinatorial Laplacian

In this chapter we firstly introduce some preliminary notions about graph theory and we define the functional spaces which are needed in the later discussion; secondly, we provide some basic properties concerning unbounded operators on Hilbert spaces, to then analyse the combinatorial Laplacian on graphs and the bottom of its ℓ^2 spectrum. Finally, we discuss the fundamentals of Markov chains theory and random walk on graphs. We refer to [8] for the unbounded operators section, to [10] for the properties of the Laplacian and to [9] for the results about Markov chains.

2.1 Basic notions

In this section we introduce the notion of graph, the standard metric on it and the functional spaces on a graph which we shall use later on. We finally give some interesting examples of graphs which will be the object of our investigation.

Definition 2.1.1. *A graph Γ is a pair $\Gamma = (V, E)$ where V denotes the set of vertices (also called nodes) and E the set of edges.*

If $(x, y) \in E$ we say that x and y are neighbours or adjacent and we write $x \sim y$, while $[x, y]$ denotes the oriented edge directed from x to y. We use the notation $m(x)$ to indicate the **degree** of x, that is the number of edges that are attached to x.
In this thesis we focus on infinite, locally finite graphs. It means that V is infinite and $m(x) < \infty$ for all $x \in V$.
We define a **path** in Γ a sequence of $k \geq 2$ vertices v_1, v_2, ..., v_k such that consecutive vertices in this sequence are adjacent. A **closed path** is a path where the first and last vertex coincide ($v_1 = v_k$). A **simple path** is a path where the vertices are all distinct one from another. A **cycle** is a closed path with no repeated vertices except for the first and the last one (that coincide) and no repeated edges.
We say that a graph Γ is **connected** if $\forall\, v, w \in V$ with $v \neq w$, a path from v to w exists. If the graph is connected, we can define a metric on Γ, $\rho(x, y)$, that denotes the number of the edges in the shortest path connecting the vertices x and y. In such case we call geodesic path a path from x to y of length $\rho(x, y)$. When we fix a vertex $o \in V$, we denote $|x| = \rho(x, o)$ for all $x \in V$.

Definition 2.1.2. *A **tree** is a connected graph with no cycles.*

Now we define two considerable examples of trees which are needed in the later discussion.

Definition 2.1.3. *A* **homogeneous tree** *is a tree, such that all vertices have the same degree.*

We will denote $\mathbb{T}_{q+1}$ the homogeneous tree in which $m(x) = q + 1$ for all $x \in V$.

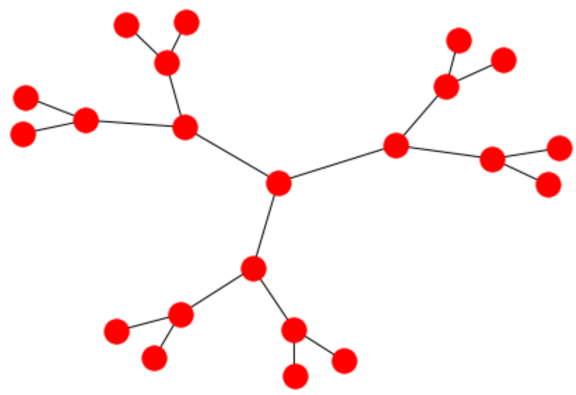

Figure 2.1. Homogeneous tree for $q = 2$.

Definition 2.1.4. *A* **bi-regular tree** *is a tree, such that the vertex degrees are constant in each of two bipartite classes: the vertices at even or at odd distance, respectively, from a reference node.*

We write $\mathbb{T}_{P,D}$ for a bi-regular tree with $m(x) = \begin{cases} P & \text{if } |x| \text{ is even,} \\ D & \text{if } |x| \text{ is odd,} \end{cases}$ where $|x| = \rho(x, o)$ and $o \in \mathbb{T}_{P,D}$ is the reference node.

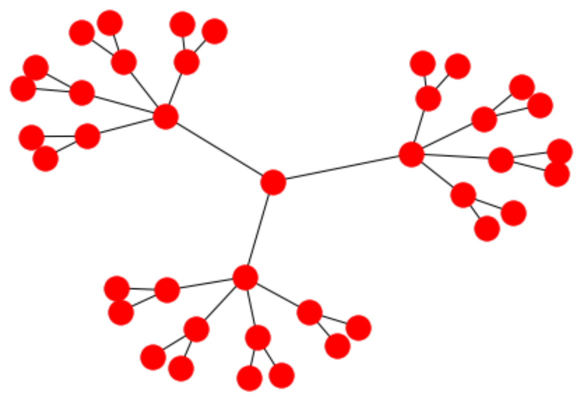

Figure 2.2. Bi-regular tree for $P = 3,\ D = 5$.

We call f a function on a graph Γ if it is a mapping $f : V \to \mathbb{R}$. The set of all functions is denoted by $C(V)$, while $C_0(V)$ is the set of all functions finitely supported in V. These functions are automatically continuous because the discrete topology is induced by the metric ρ.
Finally, we define the space of square summable functions, $\ell^2(V)$ that is

$$\ell^2(V) = \{f \in C(V) \text{ such that } \sum_{x \in V} f(x)^2 < +\infty\}.$$

This is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x),$$

and the induced norm $\|f\|^2 = \langle f, f \rangle$. Similarly we denote with $\ell^2(\tilde{E})$ the space of all square summable functions on oriented edges that satisfy the relation

$$\psi[x, y] = -\psi[y, x]$$

with the inner product

$$\langle \phi, \psi \rangle = \sum_{[x,y] \in \tilde{E}} \phi[x, y]\psi[x, y],$$

where $\tilde{E}$ denotes the set of all oriented edges of Γ.
The function d, with domain $C(V)$ and range in the space of function on $\tilde{E}$, defined as $df[x, y] = f(y) - f(x)$ is called coboundary function.

2.2 Combinatorial Laplacian

The combinatorial Laplacian plays a crucial role in this thesis because, as shown later in Chapter 4, there are different techniques which involve such operator in order to derive Hardy weights on graphs.
Consider a graph $\Gamma = (V, E)$.

Definition 2.2.1. *The* **combinatorial Laplacian** *is the map that acts on $C(V)$ by the formula*

$$\Delta f(x) := \sum_{y \sim x} \Big(f(x) - f(y) \Big) = m(x)f(x) - \sum_{y \sim x} f(y) \quad \text{for all } x \in V. \tag{2.1}$$

Notice that by (2.1) the Laplacian is bounded on ℓ^2 if and only if m is bounded on V. Indeed, the following holds

Theorem 2.2.1. *The combinatorial Laplacian is a bounded operator in $\ell^2(V)$ if and only if there exists $M \in \mathbb{N}$ such that $m(x) \leq M$ for all $x \in V$.*

Proof. We first assume that Δ is a bounded operator on $\ell^2(V)$. By contradiction, if m would not be bounded, there would be a sequence $\{x_n\} \subset V$ such that $\sup_n m(x_n) = +\infty$.
Define

$$\delta_z(x) = \begin{cases} 1 & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\|\delta_z\| = 1$ for all $z \in V$ and $\Delta\delta_z(x) = m(x)\delta_z(x) - \sum_{y\sim x}\delta_z(y)$. Thus

$$\Delta\delta_z(x) = \begin{cases} m(z) & \text{if } x = z \\ -1 & \text{if } x \sim z \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\|\Delta\delta_{x_n}\| = \sqrt{m(x_n)^2 + m(x_n)}.$$

This would imply that the combinatorial Laplacian is not a bounded operator, which contradicts our assumption.
Assume now that $\max_x m(x) = M$. By a direct computation

$$\begin{aligned}\|\Delta f\|^2 &= \sum_{x\in V}\left(m(x)f(x) - \sum_{y\sim x} f(y)\right)^2 \le 2\sum_{x\in V}\left(\left((m(x)f(x)\right)^2 + \left(\sum_{y\sim x} f(y)\right)^2\right) \\ &\le 2M^2\|f\|^2 + 2\sum_{x\in V}\left(\sum_y \chi_x(y)f(y)\right)^2\end{aligned}$$

where $\chi_x(y) = \begin{cases} 1 & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases}$
Using Cauchy-Schwarz's inequality, it follows that

$$\left(\sum_y \chi_x(y)f(y)\right)^2 \le \sum_y \chi_x(y)f(y)^2 \sum_y \chi_x(y) \le M\sum_y \chi_x(y)f(y)^2.$$

Moreover,

$$\sum_{x\in V}\sum_{y\sim x} f(y)^2 \le M\|f\|^2.$$

Hence

$$\|\Delta f\|^2 \le 4M^2\|f\|^2.$$

□

Definition 2.2.2. *A subgraph $\Gamma' = (V', E')$ of a graph $\Gamma = (V, E)$ is a graph such that $V' \subset V$ and $E' \subset E$.*

Now we state an analogue of the Green's theorem.

Theorem 2.2.2. *Let D a finite connected subgraph of V. Then*

$$\begin{aligned}\sum_{x\in D}\Delta f(x)g(x) &= \sum_{[x,y]\in\tilde{E}(D)} df([x,y])dg([x,y]) + \sum_{\substack{x\in D \\ z\sim x,\ z\notin D}} (f(x) - (f(z))g(x) \\ &= \sum_{[x,y]\in\tilde{E}(D)} df([x,y]dg([x,y]) - \sum_{\substack{[x,z] \\ x\in D,\ z\notin D}} df([x,z])g(x).\end{aligned}$$

Proof. Every edge $[x, y]$ with $x, y \in V(D)$ contributes two terms to the sum on the left hand side: $\Delta f(x)g(x)$ with $(f(x) - f(y))g(x)$ and $\Delta f(y)g(y)$ with $(f(y) - f(x))g(y)$. Adding these two terms, we obtain $(f(x) - f(y))(g(x) - g(y)) = df[x, y]dg[x, y]$. The remaining contributions are given from the any vertex $x \in D$ which is connected with a vertex $z \notin D$. These give $(f(x) - f(z))g(x) = df[x, z]g(x)$. □

Definition 2.2.3. *A vertex $x \in \partial D$, i.e. x is in the boundary of D, if there exists a vertex z such that $z \sim x$ and $z \notin D$. If $x \in D$ and $x \notin \partial D$, it is said to be in the interior of D.*

It follows from Theorem 2.2.2 that if either f or g are zero on the complement of the interior of D then

$$\langle \Delta f, g\rangle_{V(D)} = \langle df, dg\rangle_{\tilde{E}(D)} = \langle f, \Delta g\rangle_{V(D)}. \tag{2.2}$$

Moreover, if f and g are functions on V and at least one of them is finitely supported, then

$$\langle \Delta f, g\rangle_{V(G)} = \langle df, dg\rangle_{\tilde{E}(G)} = \langle f, \Delta g\rangle_{V(G)}. \tag{2.3}$$

This show that Δ is a symmetric nonnegative operator on $C_0(V)$.

2.3 Unbounded operators

In this section we provide some basic definitions and theorems useful for analysing unbounded operators. We proved that the combinatorial Laplacian may not be a bounded operator. Moreover, we showed that it is a symmetric but not self-adjoint operator in general. However, we will show that it is essentially self-adjoint, i.e. that it admits a unique self-adjoint extension to $\ell^2(V)$.

Definition 2.3.1. *Let H be a Hilbert space. By an operator in H we mean a linear mapping T whose domain $\mathcal{D}(T)$ is a dense subspace of H and whose range $\mathcal{R}(T)$ lies in H.*

Definition 2.3.2. *The graph of the linear operator T is the set of pairs*

$$\{(\phi,\ T\phi) \mid \phi \in \mathcal{D}(T)\}.$$

The graph of T, denoted by $\mathcal{G}(T)$, is a subset of $H \times H$ which is a Hilbert space with the inner product

$$\langle(\phi_1, \psi_1), (\phi_2, \psi_2)\rangle = \langle\phi_1, \phi_2\rangle + \langle\psi_1, \psi_2\rangle.$$

T is called closed operator if $\mathcal{G}(T)$ is a closed subset of $H \times H$.

We point out that a linear operator $T : H \to H$ is bounded if and only if T has closed graph because of the closed graph theorem. Notice that, in general, an unbounded operator is defined on a dense subspace of H.

Definition 2.3.3. *Let T_1 and T be operators on H. If $\mathcal{G}(T) \subset \mathcal{G}(T_1)$, then T_1 is said to be an extension of T and we write $T \subset T_1$. Notice that T_1 extends T if and only if $\mathcal{D}(T) \subset \mathcal{D}(T_1)$ and $T_1\phi = T\phi$ for all $\phi \in \mathcal{D}(T)$.*

Definition 2.3.4. *An operator T is closable if it has closed extensions. It follows that every closable operator has a smallest closed extension said its closure, which we denote by $\overline{T}$.*

The notion of adjoint operator can be extended to the unbounded case.

Definition 2.3.5. *Let T be a linear operator on H. Let $\mathcal{D}(T^*)$ be the set of $\phi \in H$ such that exists an $\eta \in H$ with*

$$\langle T\psi,\ \phi\rangle = \langle\psi,\ \eta\rangle \qquad for\ all\ \psi \in \mathcal{D}(T).$$

For each such $\phi \in \mathcal{D}(T^*)$ we define $T^*\phi = \eta$. T^* is called the adjoint of T.

Notice that by the Riesz-Frechét theorem, $\phi \in \mathcal{D}(T^*)$ if and only if $|\langle T\psi,\ \phi\rangle| \leq \mathrm{C}\|\psi\|$ for some constant $C > 0$ and for all $\psi \in \mathcal{D}(T)$. Moreover, η is uniquely determined by the previous definition since $\mathcal{D}(T)$ is dense.

Remark 2.3.1. *If $T_1 \subset T$ then $T^* \subset (T_1)^*$ by definition of adjoint operator.*

Theorem 2.3.1. *Let T be a densely defined operator on a Hilbert space H. Then:*

a) T^* is closed.

b) T is closable if and only if $\mathcal{D}(T^*)$ is dense in which case $\overline{T} = T^{**}$.

c) If T is closable, then $(\overline{T})^* = T^*$.

A natural way to obtain a closed extension of an operator T is to consider the closure of $\mathcal{G}(T)$ in $H \times H$. The problem with such operation is that $\overline{\mathcal{G}(T)}$, in general, is not the graph of an operator. However, symmetric operators, which we now define, admit always a closed extension.

Definition 2.3.6. *Let T be an operator defined on $\mathcal{D}(T)$. T is called symmetric if $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T\phi = T^*\phi$ for all $\phi \in \mathcal{D}(T)$. In other words, T is symmetric if and only if*

$$\langle T\phi,\ \psi\rangle = \langle \phi,\ T\phi\rangle \quad for\ all\ \phi, \psi\ \in \mathcal{D}(T).$$

Thus, by Theorem 2.2.2, the combinatorial Laplacian is a symmetric operator with domain $C_0(V)$.

Definition 2.3.7. *T is called self-adjoint if $T = T^*$, that means, if and only if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.*

As we previously said, a symmetric operator is always closable, because $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ is dense in H.

It follows that if T is symmetric, then T^* is a closed extension of T, so the smallest closed extension T^{**} of T must be contained in T^*. Hence for symmetric operators we have

$$T \subset T^{**} \subset T^*$$

while, for closed symmetric operators

$$T = T^{**} \subset T^*$$

and for self-adjoint operators,

$$T = T^{**} = T^*.$$

Definition 2.3.8. *A symmetric operator T is called essentially self-adjoint if its closure $\overline{T}$ is self-adjoint.*

If T is essentially self-adjoint, then it has a unique self-adjoint extension. Let A be a self-adjoint extension of T. Then $T^{**} \subset A$. A is self-adjoint, thus $A = A^* \subset (T^{**})^* = (\overline{T})^* = T^{**}$. It follows that $A = T^{**}$.

Theorem 2.3.2. *Let A a strictly positive symmetric operator, that means, $\langle A\phi,\ \phi\rangle \geq c\langle\phi,\ \phi\rangle$ for all $\phi\ \in \mathcal{D}(A)$ and for some $c > 0$. Then, the following are equivalent:*

a) A is essentially self-adjoint;

b) $Ker\{A^*\} = 0$;

c) $Ran\{A\}$ is dense.

Now we consider the combinatorial Laplacian on a graph with domain $C_0(V)$. Let Δ^* denote the adjoint of Δ with domain $C_0(V)$.

Proposition 2.3.1. *The domain of Δ^* is*

$$\mathcal{D}(\Delta^*) = \{f \in \ell^2(V) \mid \Delta f \in \ell^2(V)\}.$$

Proof. By definition,

$$\mathcal{D}(\Delta^*) = \{f \in \ell^2(V) \mid \exists!\ \eta \in \ell^2(V) \text{ such that } \langle \Delta g,\ f\rangle = \langle g,\ \eta\rangle \quad \forall g \in\ C_0(V)\}.$$

For all $g \in C_0(V)$ and $f \in \mathcal{D}(\Delta^*)$ by formula (2.3)

$$\langle \Delta g,\ f\rangle = \langle g,\ \Delta f\rangle = \langle g,\ \eta\rangle.$$

In particular, for $g = \delta_x$ we obtain that $\Delta f(x) = \eta(x)$ for all $x \in V$. Hence $\Delta^* f(x) = \eta(x) = \Delta f(x)$, as required. $\square$

Theorem 2.3.3. *$\Delta : C_0(V) \to \mathbb{R}$ is essentially self-adjoint.*

Proof. From Theorem 2.3.2 applied to $A = \Delta + I$, it suffices to show that

$$Ker\{A^*\} = Ker\{\Delta^* + I\} = \{0\}$$

or in other words that -1 is not an eigenvalue of Δ^*. Indeed, we now show that if $f \neq 0$ satisfies $\Delta^* f = -f$, then $f \notin \ell^2(V)$.
We can use Theorem 2.3.2 because

$$\langle (\Delta + I)f,\ f\rangle_{\ell^2(V)} = \langle \Delta f,\ f\rangle_{\ell^2(V)} + \langle f,\ f\rangle_{\ell^2(V)}$$

and it holds

$$\langle \Delta f,\ f\rangle_{\ell^2(V)} = \langle df, df\rangle_{\ell^2(\tilde{E})} \geq 0.$$

Thus $\langle Af,\ f\rangle \geq \langle f,\ f\rangle$. By contradiction, suppose there exists $f \in \ell^2(V)$, $f \neq 0$ such that $\Delta^* f(x) = -f(x)$ for all $x \in V$. Then, by the analogue of Green's theorem, $\Delta^* f(x) = \Delta f(x)$ and

$$(m(x) + 1)f(x) = \sum_{x \sim y} f(y).$$

Therefore it must exist a neighbour $y \sim x$ such that $f(y) > f(x)$. This is true of all $x \in V$. It follows that $f \notin \ell^2(V)$. $\square$

2.4 Bottom of the spectrum

Now we show that in general, for a graph $\Gamma = (V, E)$, we can characterize the infimum of the ℓ^2 Laplacian's spectrum λ_0 as

$$\lambda_0 = \inf_{f \in C_0(V)} \frac{\langle df, df\rangle}{\langle f, f\rangle}. \tag{2.4}$$

This means that λ_0 is the best constant for which the **Poincaré inequality** on Γ

$$\langle df, df\rangle \geq \lambda_0 \langle f, f\rangle \tag{2.5}$$

or equivalently

$$\frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \lambda_0 \sum_{x\in V}\varphi^2(x)$$

holds for all $f \in C_0(V)$.

Let $x_0 \in V$ be a vertex and denote $\overline{B}_r = \overline{B}_r(x_0)$ the ball of center x_0 and radius r, $\partial B_r = \partial\overline{B}_r(x_0)$ its boundary and with B_r its interior. We have

$$\overline{B}_r = \{x \in V \ \text{ such that } \ \rho(x_0,x) \leq r\}.$$

Finally let $C(\overline{B}_r, \partial B_r)$ denote functions defined on $\overline{B}_r$ which vanish on ∂B_r and Δ_r the reduced Laplacian which acts in this space. That is,

$$C(\overline{B}_r, \partial B_r) = \{f \in C(\overline{B}_r) \mid f_{|\partial B_r} = 0\},$$
$$\Delta_r f(x) = \begin{cases} \Delta f(x) & \text{if } x \in B_r,\\ 0 & \text{if } x \in \partial B_r.\end{cases}$$

Lemma 2.4.1. *For every $r \geq 1$ Δ_r is a self-adjoint, nonnegative operator on $C(\overline{B}_r, \partial B_r)$.*

Proof. This is an immediate consequence of (2.2) since

$$\langle \Delta_r f, g\rangle_{V(\overline{B}_r)} = \langle df, dg\rangle_{E(\overline{B}_r)} = \langle f, \Delta_r g\rangle_{V(\overline{B}_r)}.$$

□

Notice that Δ_r is also compact because it is a finite range operator. Thus Lemma 2.4.1 implies that all eigenvalues are nonnegative and the existence of an orthonormal basis of eigenfunctions with respect to the ℓ^2-inner product. Denote by $\{\lambda_i^r\}_{i=0}^{k(r)}$ the set of eigenvalues of Δ_r listed in increasing order and choose $\{\phi_i^r\}_{i=0}^{k(r)}$ corresponding eigenfunctions which are an orthonormal basis for $C(\overline{B}_r, \partial B_r)$. This means,

$$\Delta_r \phi_i^r = \lambda_i^r \phi_i^r \quad \text{for } i = 1, ..., k(r),$$
$$\sum_{x\in B_r} \phi_j^r(x)\phi_k^r(x) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k,\\ 0 & \text{if } j \neq k.\end{cases}$$

Next we show that the smallest eigenvalue λ_0^r of Δ_r is the minimum of the problem (2.6) and then letting $r \to \infty$ we prove that $\lambda_0^r \to \lambda_0$.

Lemma 2.4.2. *Let Δ_r denote the reduced Laplacian on $\overline{B}_r$. Then*

$$\gamma = \min_{\substack{f\in C(\overline{B}_r,\partial B_r),\\ f\neq 0}} \frac{\langle df, df\rangle}{\langle f, f\rangle}, \tag{2.6}$$

is the smallest eigenvalue of Δ_r, i.e. $\gamma = \lambda_0^r$. Moreover, if f_0^r is a function in $C(\overline{B}_r, \partial B_r)$ that satisfies

$$\lambda_0^r = \frac{\langle df_0^r, df_0^r\rangle}{\langle f_0^r, f_0^r\rangle}, \tag{2.7}$$

then $\Delta_r f_0^r = \lambda_0^r f_0^r$ and f_0^r can be chosen so that $f_0^r > 0$ in B_r.

Proof. If λ is any eigenvalue of Δ_r with eigenfunction f, then

$$\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \lambda \geq \gamma.$$

If f_0^r satisfies (2.6) and $\{\lambda_i^r\}_{i=0}^{k(r)}$ are the eigenvalues of Δ_r with $\{\phi_i^r\}_{i=0}^{k(r)}$ the corresponding eigenfunctions, which are an orthonormal basis of $C(\overline{B}_r, \partial B_r)$, then

$$f_0^r = \sum_{i=0}^{k(r)} a_i \phi_i^r,$$

with $a_i = \langle f_0^r, \phi_i^r\rangle$. We claim that $a_i = 0$ if $\lambda_i^r \neq \lambda_0^r$.
We first compute

$$\begin{aligned}
0 &= \langle d(f_0^r - \sum_{i=0}^{k(r)} a_i\phi_i^r), d(f_0^r - \sum_{i=0}^{k(r)} a_i\phi_i^r)\rangle \\
&= \langle df_0^r, df_0^r\rangle - 2\sum_{j=0}^{k(r)} a_i\langle f_0^r, \Delta_r\phi_i^r\rangle + \sum_{i,j=0}^{k(r)} a_ia_j\langle \phi_i^r, \Delta_r\phi_j^r\rangle \\
&= \langle df_0^r, df_0^r\rangle - 2\sum_{j=0}^{k(r)} a_i^2\lambda_i^r + \sum_{i,j=0}^{k(r)} a_ia_j\lambda_j^r\langle \phi_i^r, \phi_j^r\rangle \\
&= \langle df_0^r, df_0^r\rangle - \sum_{j=0}^{k(r)} a_i^2\lambda_i^r.
\end{aligned}$$

Thus

$$\langle df_0^r, df_0^r\rangle = \sum_{j=0}^{k(r)} a_i^2\lambda_i^r.$$

But (2.6) implies that

$$\langle df_0^r, df_0^r\rangle = \gamma\langle f_0^r, f_0^r\rangle = \gamma\sum_{j=0}^{k(r)} a_i^2,$$

therefore $a_i = 0$ if $\lambda_i^r \neq \gamma$ and $\gamma = \lambda_0^r$. Notice that

$$\langle f_0^r, f_0^r\rangle = \langle |f_0^r|, |f_0^r|\rangle$$

while

$$\langle df_0^r, df_0^r\rangle \geq \langle d|f_0^r|, d|f_0^r|\rangle$$

because $|a-b| \geq \big||a|-|b|\big|$ for all $a, b \in \mathbb{R}$. It follows that (2.6) can only be decreased by replacing f_0^r with $|f_0^r|$, so we can assume that $f_0^r \geq 0$ in B_r. Moreover, if there exists $\overline{x} \in B_r$ such that $f_0^r(\overline{x}) = 0$ then $\Delta_r f_0^r(\overline{x}) = \lambda_0^r f_0^r(\overline{x}) = 0$ that is

$$\Delta_r f_0^r(\overline{x}) = -\sum_{y\sim\overline{x}} f_0^r(y) = 0.$$

It implies $f_0^r = 0$ because it is nonnegative, hence we get a contradiction. $\square$

It follows from Lemma 2.4.2 that

$$\lambda_0^r \geq \lambda_0^{r+1} > 0,$$

so that we can define

$$\lambda_0 = \lambda_0(\Delta) = \lim_{r \to \infty} \lambda_0^r.$$

We now show that, equivalently, λ_0 can be defined as in (2.4).

Proposition 2.4.1. *Let* $\lambda_0(\Delta) = \lim_{r\to\infty} \lambda_0^r$*, where* $\lambda_0^r = \min\limits_{\substack{f \in C(\overline{B}_r, \partial B_r),\\ f \neq 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$*. Then,*

$$\lambda_0(\Delta) = \inf_{\substack{f \in C_0(V),\\ f \neq 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}. \tag{2.8}$$

Proof. Denote $\gamma = \inf\limits_{\substack{f \in C_0(V),\\ f \neq 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$. Notice that the corresponding eigenfunctions of λ_0^r, denoted by f_0^r, belongs to $C_0(V)$ for all r thus $\gamma \leq \lambda_0(\Delta)$. Let now $\{f_n\}_{n \in \mathbb{N}}$ be a minimizing sequence in (2.8). Then, each of $f_n \in C(\overline{B}_{r_n}, \partial B_{r_n})$ for some r_n depending on n. This means that,

$$\lambda_0(\Delta) \leq \lambda_0^{r_n} \leq \frac{\langle df_n, df_n \rangle}{\|f_n\|^2} \to \gamma.$$

We conclude that $\gamma = \lambda_0(\Delta)$. $\square$

Remark 2.4.1. *The combinatorial Laplacian is a bounded operator on* $\ell^2(\mathbb{T}_{q+1})$*.*

In $\mathbb{T}_{q+1}$ the ℓ^2 spectrum $\sigma_2(\Delta)$ of the Laplacian is known, we refer to [2] for this result. More explicitly, one has that

$$\sigma_2(\Delta) = [(\sqrt{q} - 1)^2, (\sqrt{q} + 1)^2].$$

We denote by $\Lambda_q = \lambda_0(\Delta) = (\sqrt{q} - 1)^2$ the minimum of $\sigma_2(\Delta)$. It follows the Poincaré inequality on $\mathbb{T}_{q+1}$

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1}\\ x \sim y}} \Big(f(x) - f(y)\Big)^2 \geq \Lambda_q \sum_{x \in \mathbb{T}_{q+1}} f(x)^2 \tag{2.9}$$

for all $f \in C_0(V)$.

2.5 Markov chains

In the upcoming section we elucidate some definitions and theorems concerning the Markov chains theory. Notions such as the Green function are crucial for the purpose of later discussions, since the above-mentioned theory will be applied to simple random walks on undirected graphs. We follow in our description [9].

A **Markov chain** is given by a countable state space X and a stochastic transition matrix (or transition operator) $P = (p(x, y))_{x,y\, \in X}$, where $p(x, y)$ is the probability of moving from x to y in one step, also called transition kernel. We also have to specify a starting point. Consequently, we have a X-valued random variables sequence Z_n, $n \geq 0$, where Z_n represents the random position in X at the time n. To model Z_n is usual to choose as probability space the trajectory space $\Omega = X^{\mathbb{N}_0}$, equipped with the product σ-algebra arising from the discrete one X. Z_n is the n-th projection $\Omega \to X$. This characterizes the Markov chain starting at x, when Ω is equipped with the probability measure given by

$$\mathbb{P}_x[Z_0 = x_0, Z_1 = x_1, ..., Z_n = x_n] = \delta_x(x_0)p(x_0, x_1) \cdot ... \cdot p(x_{n-1}, x_n).$$

We define

$$p^{(n)}(x, y) = \mathbb{P}_x[Z_n = y].$$

We assume that (X, P) is irreducible, i.e. $\forall x, y \in X$ there exists some $n \in \mathbb{N}$ such that $p^{(n)}(x, y) > 0$.

Definition 2.5.1. *Given a Markov chain (X, P) for every $x, y \in X$ and $z \in \mathbb{C}$ we define the* **Green function**

$$G(x.y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)z^n.$$

The notion of Green function on a Markov chain is related to the partial differential equations theory. Theorem 2.5.3 explains how they are connected. Before stating it we need some preliminary results.

Lemma 2.5.1. *For real $z > 0$, $G(x, y|z)$ either diverge or converge simultaneously for all $x,\ y \in X$.*

Proof. If $u,\ v,\ x,\ y \in X$, by irreducibility, $\exists\ k,\ l \in \mathbb{N}$ such that $p^{(k)}(u, v) > 0$ and $p^{(l)}(x, y) > 0$. Thus

$$p^{(k+n+l)}(u, v) \geq p^{(k)}(u, x)p^{(n)}(x, y)p^{(l)}(y, v)$$

and, for $z > 0$,

$$G(u, v|z) \geq p^{(k)}(u, x)p^{(l)}(y, v)z^{k+l}G(x, y|z).$$

By the comparison theorem for series, if $G(x, y|z)$ diverges for some $x, y \in V$ then it diverges for all $u, v \in X$. On the other hand, if $G(u, v|z)$ is convergent, then $G(x, y|z)$ is convergent $\forall x, y \in X$. □

Thus, all $G(x, y|z)$ (with $x, y \in X$) have the same radius of convergence $\mathrm{r}(P) = \frac{1}{\eta(P)}$, where

$$\eta(P) = \limsup_{n\to\infty} p^{(n)}(x, y)^{\frac{1}{n}} \in (0,1].$$

This number is called the spectral radius of P.

Definition 2.5.2. *The period of P is the number $d = d(P) = gcd\{n \geq 1 : p^{(n)}(x, x) > 0\}$.*

It is known that d is independent from x by assumption of irreducibility. If $d = 1$ we will say that the chain is aperiodic.

Lemma 2.5.2. *$p^{(n)}(x, x) \leq \eta(P)^n$, and $\lim_{n\to\infty} p^{(nd)}(x, x)^{\frac{1}{nd}} = \eta(P)$.*

Proof. Let $a_n = p^{(nd)}\ (x, x)$. It is true that $0 \leq a_n \leq 1$. If $N(x) = \{n : a_n > 0\}$, then $gcd\ N(x) = 1$. Note that $a_m a_n \leq a_{m+n}$. Initially, we show that $\exists\ n_0$ such that $a_n > 0$ for all $n \leq n_0$. If $m, n \in N(x)$ then $m + n \in N(x)$. It is known that the greatest common divisor of a set of integers can be written as a finite linear combination of elements of that set. For this reason, we can write $1 = \gcd N(x) = n_1 - n_2$, where $n_1,\ n_2\ \in N(x) \cup \{0\}$. If $n_2 = 0$ the proof is done and $n_0 = 1$. Otherwise, set $n_0 = n_2^2$ and decompose $n \geq n_0$ as $n = qn_2 + r = (q-r)n_2 + rn_1$, where $0 \leq r < n_2$. It must be that $q \geq n_2 > r$, so that $n \in N(x)$. Next, fix $m \in N(x)$, let $n \geq n_0$, let $n \geq n_0 + m$, and decompose $n = q_n m + r_n$, where $n_0 \leq r_n < n_0 + m$. Write $b = b(m)$ $= \min \{\ a_r : n_0 \leq r < n_0 + m\ \}$. Then $b > 0$ and $a_n \geq a_m^{q_n} a_{r_n}$, so that $a_m^{\frac{q_n}{n}} b^{\frac{1}{n}}$. If $n \to \infty$ then $\frac{q_n}{n} \to \frac{1}{m}$. Hence,

$$a_m^{\frac{1}{m}} \leq \liminf_{n\to\infty} a_n^{\frac{1}{n}} \leq \eta(P)^d \qquad \forall m \in N(x).$$

This proves the first statement. If we now let $m \to\ \infty$, then

$$\limsup_{m\to\infty} a_m^{\frac{1}{m}} \leq \liminf_{n\to\infty} a_n^{\frac{1}{n}}.$$

Thus $a_n^{\frac{1}{n}}$ converges and this ends the proof. $\square$

It is useful to define the stopping time and the hitting probabilities in order to provide an easier way to compute the Green function on a Markov chain.

Definition 2.5.3. *We define the stopping time*

$$s^y = \min_n \{n \geq 0 \mid Z_n = y\}$$

where this minimum is $+\infty$ if the set is empty, the hitting probabilities and the associated functions

$$f^{(n)}(x,\ y) = \mathbb{P}_x[s^y = n],$$
$$F(x,y|z) = \sum_{n=0}^{\infty} f^{(n)}(x,y)z^n.$$

Similarly, we define $t^x = \min_n\ \{n \geq 1 : Z_n = x\}$ and the associated function

$$U(x,x|z) = \sum_{n=0}^{\infty} \mathbb{P}_x[t^x = n]z^n.$$

The next lemma is one of the most important tool for dealing with Green functions. We will apply this result many times in order to prove useful properties and to compute Green functions on particular trees.

Lemma 2.5.3. *Let (X, P) a Markov chain, G the Green function and F, U the above defined functions. Then,*

a) $G(x,x|z)\ = \frac{1}{1-U(x,x|z)}$,

b) $G(x,y|z)\ = F(x,y|z)G(y,y|z)$,

c) $U(x,x|z)\ = \sum_y p(x,y)zF(y,x|z)$ *and*

d) if $y \neq x$, $F(x,y|z)\ = \sum_w\ p(x,w)zF(w,y|z)$.

Proof. Part *a*) follows from the identity

$$p^{(n)}(x,x) = \sum_{k=0}^{n} \mathbb{P}_x[t^x = k] p^{(n-k)}(x,x), \qquad if\ n \geq 1,$$

while $p^{(0)}(x,x) = 1$ and $\mathbb{P}_x[t^x = 0] = 0$.
Indeed,

$$\begin{aligned} G(x,x|z) &= \sum_{n=0}^{\infty} p^{(n)}(x,x) z^n = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \mathbb{P}_x[t^x = k] p^{(n-k)}(x,x) z^n + p^{(0)}(x,x) \\ &= 1 + \sum_{k=0}^{\infty} z^k p^{(k)}(x,x) \Big(\sum_{n=0}^{\infty} \mathbb{P}_x[t^x = n] z^n \Big) = 1 + G(x,x|z) U(x,x|z). \end{aligned}$$

Using the same argument we can prove *b*):
for $x \neq y$,

$$\begin{aligned} G(x,y|z) &= \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbb{P}_x[s^y = k] p^{(n-k)}(y,y) z^n \\ &= \sum_{k=0}^{\infty} p^{(k)}(y,y) z^k \Big(\sum_{n=0}^{\infty} \mathbb{P}_x[s^y = n] z^n \Big) = G(y,y|z) F(x,y|z). \end{aligned}$$

Note that if $x \neq y$, then $f^{(0)}(x,y) = 0$. Finally, by factoring through the first step we obtain *c*) and *d*):

$$\mathbb{P}_x[t^x = n] = \sum_y p(x,y) \mathbb{P}_y[t^x = n-1] \quad for\ n \geq 1,$$

and for $x \neq y$:

$$\mathbb{P}_x[s^y = n] = \sum_y p(x,y) \mathbb{P}_y[s^y = n-1] \quad for\ n \geq 1.$$

Thus,

$$\begin{aligned} U(x,x|z) &= \sum_{n=1}^{\infty} \mathbb{P}_x[t^x = n] z^n = \sum_{n=1}^{\infty} \sum_y p(x,y) \mathbb{P}_y[t^x = n-1] z^n \\ &= \sum_y p(x,y) \sum_{n=1}^{\infty} \mathbb{P}_y[t^x = n-1] z^n = \sum_y p(x,y) z \sum_{k=0}^{\infty} \mathbb{P}_y[t^x = k] z^k \\ &= \sum_y p(x,y) z F(y,x|z). \end{aligned}$$

If $x \neq y$:

$$\begin{aligned} F(x,y|z) &= \sum_{n=0}^{\infty} \mathbb{P}_x[s^y = n] z^n = \sum_{n=1}^{\infty} \sum_w p(x,w) \mathbb{P}_w[s^y = n-1] z^n \\ &= \sum_w p(x,w) z \sum_{k=0}^{\infty} [s^y = k] z^k = \sum_w p(x,w) z F(w,y|z). \end{aligned}$$

$\square$

We will write $G(x,y)$, $F(x,y)$, $U(x,x)$ respectively for $G(x,y|1)$, $F(x,y|1)$, $U(x,x|1)$. It can be proved that $G(x,y)$ represents the expected number of visits of $(Z_n)_{n\geq 0}$ to y when $Z_0 = x$. Analogously, $F(x,y)$ is the probability of ever reaching y when starting at x and $U(x,x)$ is the probability of ever returning after starting at x.

Definition 2.5.4. *The Markov chain (X,P) is said* **recurrent** *if $G(x,y) = \infty$ for some (that means for all) $x, y \in X$. Equivalently, X is recurrent if $U(x,x) = 1 \;\; \forall x \in \; X$. Otherwise (X,P) is said* **transient***.*

We now consider P acting on functions $f : X \to \mathbb{R}$ by the formula

$$Pf(x) = \sum_y p(x,y)f(y).$$

Then it is natural to define the weighted Laplacian $\tilde{\Delta}$.

Definition 2.5.5. *Let $\tilde{\Delta}$ be the operator acting on functions on X by the formula*

$$\tilde{\Delta}f(x) = (I - P)f(x).$$

Definition 2.5.6. *Assume that $P|f|$ is finite. Then we say that f is superharmonic if $\tilde{\Delta}f \geq 0$ pointwise, and harmonic if $\tilde{\Delta}f = 0$.*

Proposition 2.5.1. *If f is superharmonic then $P^n f$ is superharmonic for all $n \geq 1$.*

Proof. Consider $n = 2$. Computing $P^2 f = P(Pf)$ we obtain

$$\begin{aligned} P^2 f(x) &= \sum_y p(x,y)Pf(y) = \sum_y p(x,y) \sum_z p(y,z)f(z) = \sum_z \sum_y p(x,y)p(y,z)f(z) \\ &= \sum_z p^{(2)}(x,z)f(z). \end{aligned}$$

Thus, iterating this argument, we can define P^n as the operator which acts on functions by the formula

$$P^n f(x) = \sum_y p^{(n)}(x,y)f(y).$$

We assume by hypothesis that

$$Pf(x) \leq f(x) \quad \text{ for all } x \in X.$$

Then,

$$p(x,y)Pf(y) \leq p(x,y)f(y) \quad \text{ for all } y \in X$$

and

$$P^2 f(x) \leq Pf(x) \leq f(x).$$

We can iterate this argument for all $n \geq 1$. □

Theorem 2.5.1 (Minimum principle)**.** *If f is superharmonic and there is $x \in X$ such that $f(x) = min_X f$ then f is constant.*

Proof. By Proposition 2.5.1 $f(x) \geq \sum_y p^{(n)}(x,y)f(y)$. Hence it is impossible that $f(y) > f(x)$ for any y such that $p^{(n)}(x,y) > 0$. Indeed, it holds $\sum_y p^{(n)}(x,y) = 1$ and if there exists $\bar{y} \in X$ such that $f(\bar{y}) > f(x)$ then

$$P^n f(x) = \sum_y p^{(n)}(x,y)f(y) \geq f(\bar{y})p^{(n)}(x,\bar{y}) + \sum_{y \neq \bar{y}} p^{(n)}(x,y)f(x) > f(x)\sum_y p^{(n)}(x,y) = f(x).$$

This is impossible because of our assumption. Finally, (X,P) is irreducible so $f \equiv f(x)$. □

Then it follows the discrete analogue of Liouville Theorem.

Theorem 2.5.2. *Let f be an harmonic function. If there exists $x \in X$ such that $f(x) = max_X f$ then f is constant.*

Proof. By assumption $f(x) = max_X f$. It implies $-f(x) = min_X(-f)$ and $-f$ is harmonic (hence superharmonic). Then by Theorem 2.5.1 $-f$ is constant and so f is constant. □

Next theorem yields the main property of the discrete Green function on a Markov chain and explains how it is linked to the continuous version.

Theorem 2.5.3. *Let (X,P) be a transient Markov chain and fix $z \in X$. Then the function $f : X \to \mathbb{R}$ such that $f(x) = G(x,z)$ is superharmonic. Moreover, $\tilde{\Delta} f(x) = \delta_z(x)$.*

Proof.

$$Pf(x) = \sum_{y \in X} p(x,y)G(y,z) = \sum_{y \in X} p(x,y)F(y,z)G(z,z).$$

If $x \neq z$:

$$Pf(x) = G(z,z)\sum_{y \in X} F(y,z)p(x,y) = F(x,z)G(z,z) = G(x,z) = f(x).$$

Where the last two identities hold because of Lemma 2.5.3 b), d).
If $x = z$:

$$Pf(z) = \sum_{y \in X} p(z,y)G(y,z) = \sum_{y \in X} p(z,y)F(y,z)G(z,z) = G(z,z)U(z,z) < G(z,z) = f(z).$$

This is true because X is transient if and only if $U(x,x) < 1$ for all $x \in X$. In particular

$$f(z) - Pf(z) = G(z,z)(1 - U(z,z)) = 1,$$

for part a) of Lemma 2.5.3. □

2.6 Green function on trees

The thesis now proceeds with the purpose of associating a Green function to every transient graph. The idea is to obtain a Markov chain from a graph and then to compute the Green function on this chain. For this reason, we introduce the notion of simple random walk on a graph. Subsequently, we state and prove useful theorems essential to then find the Green function on a tree. Finally, we present explicit computations of Green function in two particular cases. We anticipate that the Green function has great importance in this work since it can be used to obtain Hardy weights, that is the main goal of Chapter 4.

The simple random walk on a locally finite graph $\Gamma = (V, E)$ is the Markov chain with state space $X = V$ and transition probabilities

$$p(x,y) = \begin{cases} 1/m(x) & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

The graph is called recurrent (transient) if the simple random walk has this property.
Note that it is equivalent to say that f is superharmonic if $\Delta f(x) \geq 0$ or $\tilde{\Delta} f(x) \geq 0$ because

$$\Delta f(x) = m(x)f(x) - \sum_{y\sim x} f(y) = m(x)(f(x) - \sum_{y\sim x} p(x,y)f(y)) = m(x)\tilde{\Delta} f(x).$$

Now we focus on trees. We recall that in a tree for every pair of vertices $x, y \in V$ there exists a unique path (said geodesic path) $\pi(x,y)$ of length $\rho(x,y)$ connecting the two.

Theorem 2.6.1. *If $w \in \pi(x,y)$ then $F(x,y|z) = F(x,w|z)F(w,y|z)$.*

Proof. Because of the tree structure, the random walk must pass through w on the way from x to y. Conditioning with respect to the first visit in w,

$$f^{(n)}(x,y) = \sum_{k=0}^{n} f^{(k)}(x,w) f^{(n-k)}(w,y).$$

Hence, by a direct computation

$$\begin{aligned} F(x,y|z) = &\sum_{n=0}^{\infty} f^{(n)}(x,y)z^n = f^{(0)}(x,w)\Big[f^{(0)}(w,y) + f^{(1)}(x,w)z + f^{(2)}(w,y)z^2 + ...\Big] + \\ & f^{(1)}(x,w)z\Big[f^{(0)}(w,y) + f^{(1)}(x,w)z + f^{(2)}(w,y)z^2 + ...\Big] + \dots \\ & = \sum_{n=0}^{\infty} z^n f^{(n)}(x,w)\left[\sum_{n=0}^{\infty} f^{(n)}(w,y)z^n\right] = F(x,w|z)F(w,y|z). \end{aligned}$$

□

Notice that in a homogeneous tree if $x \sim y$ obviously $F(x,y|z) = F(z)$, so that Theorem 2.6.1 implies $F(v,w|z) = F(z)^{\rho(v,w)}$.

Theorem 2.6.2. *For the simple random walk on $\mathbb{T}_{q+1}$, one has*

$$G(x,y|z) = \frac{2q}{q-1+\sqrt{(q+1)^2-4qz^2}} \left(\frac{q+1-\sqrt{(q+1)^2-4qz^2}}{2qz}\right)^{\rho(x,y)}.$$

In particular, $\eta(P) = \frac{2\sqrt{q}}{q+1}$.

Proof. Consider two neighbours x, y. Using Lemma 2.5.3 d) we obtain

$$F(z) = F(x,y|z) = \sum_{w\sim x} \frac{1}{q+1} zF(z)^{\rho(x,y)} = \frac{1}{q+1}z + \frac{q}{q+1}zF(z)^2.$$

Computing the solutions of the second order equation, we choose

$$F(z) = \frac{1}{2qz}\left(q+1-\sqrt{(q+1)^2-4qz^2}\right)$$

as the right solution because $F(0) = 0$. Now apply Lemma 2.5.3 c) and

$$U(x,x|z) = \sum_y p(x,y)zF(y,x|z) = \sum_{y\sim x} \frac{1}{q+1} zF(z) = zF(z)$$

then, use Lemma 2.5.3 a) and b)

$$\begin{aligned}
G(x,x|z) &= \frac{1}{1-zF(z)}, \\
G(x,y|z) &= F(x,y|z)G(y,y|z) = \frac{1}{1-zF(z)}F(z)^{\rho(x,y)} \\
&= \frac{1}{1-\frac{1}{2q}\left(q+1-\sqrt{(q+1)^2-4qz^2}\right)} \frac{1}{2qz}\left(q+1-\sqrt{(q+1)^2-4qz^2}\right)^{\rho(x,y)} \\
&= \frac{2q}{q-1+\sqrt{(q+1)^2-4qz^2}}\left(q+1-\sqrt{(q+1)^2-4qz^2}\right)^{\rho(x,y)}.
\end{aligned}$$

Finally, by Pringsheim theorem, $G(x,y|z)$ is a non-negative coefficients power series, thus its radius of convergence $r(P) = \frac{1}{\eta(P)}$ is its smallest positive singularity. So it is the value of $z > 0$ where the term under square root is zero. □

It follows that the simple random walk on $\mathbb{T}_{q+1}$ is transient for every $q \geq 2$. In this thesis we will study the homogeneous tree $\mathbb{T}_{q+1}$ always assuming that $q \geq 2$.

Corollary 2.6.1. *If we fix a vertex o in $\mathbb{T}_{q+1}$ and $q \geq 2$, then $\Delta G(y,o) := \Delta G_o(y) = (q+1)\delta_o(y)$ for every $y \in \mathbb{T}_{q+1}$.*

Proof. Take $y \neq o$:

$$\begin{aligned}
\Delta G_o(y) &= (q+1)G_o(y) - \sum_{z\sim y} G_o(z) \\
&= \frac{2q}{2(q-1)}\left(\frac{1}{q}\right)^{\rho(y,o)}\left(q+1-q\left(\frac{1}{q}\right)-q\right) = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta G_o(o) &= (q+1)G_o(o) - \sum_{z\sim o} G_o(z) \\
&= (q+1)\frac{q}{q-1} - (q+1)\frac{q}{q-1}\left(\frac{1}{q}\right) \\
&= \frac{q^2-1}{q-1} = q+1.
\end{aligned}$$

□

Now we focus on the simple random walk on the bi-regular tree.

Theorem 2.6.3. *For the simple random walk on $\mathbb{T}_{P,D}$ one has*

$$G(x,y) = \begin{cases} \Big(F_P F_D\Big)^{\rho(x,y)/2} \frac{1}{1-F_P} & \text{if } |x| \text{ and } |y| \text{ are both even,} \\ \Big(F_P F_D\Big)^{\rho(x,y)/2} \frac{1}{1-F_D} & \text{if } |x| \text{ and } |y| \text{ are both odd,} \\ \Big(F_P\Big)^{(\rho(x,y)+1)/2} \Big(F_D\Big)^{(\rho(x,y)-1)/2} \frac{1}{1-F_P} & \text{if } |x| \text{ is odd and } |y| \text{ is even,} \\ \Big(F_P\Big)^{(\rho(x,y)-1)/2} \Big(F_D\Big)^{(\rho(x,y)+1)/2} \frac{1}{1-F_D} & \text{if } |x| \text{ is even and } |y| \text{ is odd,} \end{cases}$$

with $F_D = D/[(D-1)P]$, $F_P = P/[(P-1)D]$ constants.

Proof. Consider $x \sim y$. It is clear that if $|x|$ is odd then $F(x,y) = F_P$ is constant for any $x \sim y$. Similarly, if $|x|$ is even then $F(x,y) = F_D$. Now we want to determine these constants.
If $|x|$ is even and $x \sim y$, using Lemma 2.5.3 d) and Theorem 2.6.1 we can write

$$\begin{aligned} F_D = F(x,y) &= \sum_w p(x,w)F(w,y) = \frac{1}{m(x)} \sum_{w\sim x} F(w,y) = \frac{1}{P}\Bigg[\Bigg(\sum_{\substack{w\sim x\\ w\neq y}} F(w,y)\Bigg) + F(y,y)\Bigg] \\ &= \frac{1}{P}\Bigg[\Bigg(\sum_{\substack{w\sim x,\\ w\neq y}} F(w,x)F(x,y)\Bigg) + 1\Bigg] = \frac{1}{P}\Bigg((P-1)F_D F_P + 1\Bigg). \end{aligned}$$

Moreover,

$$F_P = F(y,x) = \sum_w p(y,w)F(w,x) = \frac{1}{m(y)} \sum_{w\sim y} F(w,x) = \frac{D-1}{D} F_D F_P + \frac{1}{D}.$$

These two equations provide the system

$$\begin{cases} F_D = \frac{1}{P}(P-1)F_D F_P + \frac{1}{P} \\ F_P = \frac{1}{D}(D-1)F_D F_P + \frac{1}{D}. \end{cases}$$

The only admissible solution is

$$\begin{cases} F_D = \frac{D}{(D-1)P} \\ F_P = \frac{P}{(P-1)D}. \end{cases}$$

Now use Lemma 2.5.3 c) and compute

$$U(x,x) = \sum_y p(x,y)F(y,x) = \begin{cases} F_P & \text{if } |x| \text{ is even,} \\ F_D & \text{if } |x| \text{ is odd,} \end{cases}$$

and by Lemma 2.5.3 a) it holds

$$G(x,x) = \frac{1}{1-U(x,x)} = \begin{cases} \frac{1}{1-F_P} & \text{if } |x| \text{ is even,} \\ \frac{1}{1-F_D} & \text{if } |x| \text{ is odd.} \end{cases}$$

Theorem 2.6.1 yields

$$F(x,y) = \begin{cases} (F_P F_D)^{\rho(x,y)} & \text{if } |x|, |y| \text{ are both even or odd,} \\ F_D^{(\rho(x,y)+1)/2} F_P^{(\rho(x,y)-1)/2} & \text{if } |x| \text{ is even and } |y| \text{ is odd,} \\ F_P^{(\rho(x,y)+1)/2} F_D^{(\rho(x,y)-1)/2} & \text{if } |x| \text{ is odd and } |y| \text{ is even.} \end{cases}$$

Finally, from Lemma 2.5.3 *b*)

$$G(x,y) = F(x,y)G(y,y)$$

we can deduce our thesis. □

Corollary 2.6.2. *If we fix a vertex o in $\mathbb{T}_{P,D}$ and $P, D \geq 3$, then $\Delta G(x,o) = P\delta_o(x)$.*

Proof. Let $o \in \mathbb{T}_{P,D}$.
If $|x|$ is even, and $x \neq o$,

$$\begin{aligned} \Delta G(x,o) &= P(F_D F_P)^{|x|/2} \frac{1}{1-F_P} - (P-1) F_P^{(|x|+2)/2} F_D^{|x|/2} \frac{1}{1-F_P} - F_P^{|x|/2} F_D^{(|x|-2)/2} \frac{1}{1-F_P} \\ &= (F_P F_D)^{|x|/2} \frac{1}{1-F_P} \Big(P - (P-1)F_P - F_D^{-1} \Big) \\ &= G(x,o) \left(P - \frac{P}{D} - \frac{(D-1)P}{D} \right) = 0. \end{aligned}$$

If $|x|$ is odd,

$$\begin{aligned} \Delta G(x,o) &= D(F_D)^{(|x|+1)/2} (F_D)^{(|x|-1)/2} \frac{1}{1-F_P} + \\ &\quad - (D-1) F_P^{(|x|+1)/2} F_D^{(|x|+1)/2} \frac{1}{1-F_P} - F_P^{(|x|-1)/2} F_D^{(|x|-1)/2} \frac{1}{1-F_P} \\ &\quad = G(x,o) \left(\frac{DP - D + D - PD}{P} \right) = 0. \end{aligned}$$

Moreover,

$$\Delta G(o,o) = \left(\frac{P}{1-F_P} - P \frac{F_P}{1-F_P} \right) = P.$$

□

Chapter 3

Classical Hardy inequalities

In this chapter we shall present the historical development of the Hardy inequality, that is the main subject of this work. We consider both the discrete and continuous version of the inequality, introducing contributions of Hardy and other well-known mathematicians. For this section we refer to [7], which we follow in our analysis.
More precisely the statements of the Hardy inequality we will consider are:

Discrete Hardy inequality
If $p > 1$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty} a_n^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p. \tag{3.1}$$

Continuous Hardy inequality
If $p > 1$, f is a nonnegative function in $(0, +\infty)$ and $f \in L^p(0, +\infty)$, then f is integrable over the interval $(0, x)$ for each positive x and

$$\int_0^{\infty} f(x)^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^{\infty} \left(\frac{1}{x}\int_0^{x} f(t)dt\right)^p dx. \tag{3.2}$$

In the following pages we show in chronological order the results that yield these two forms of the Hardy inequality. Finally, in Theorem 3.2.2 we will also state and prove the N-dimensional version of (3.2).

3.1 Discrete Hardy inequality

We start analyzing the "history" of the proof of (3.1). Notice that Theorem 3.1.1 and Theorem 3.1.2 can be seen as weak versions of the discrete Hardy inequality.

Theorem 3.1.1 (Hardy, 1919)**.** *The convergence of the series $\sum_{n=1}^{\infty} a_n^2$ with $a_n \geq 0$ implies the convergence of $\sum_{n=1}^{\infty}(A_n/n)^2$, where $A_n = \sum_{k=1}^{n} a_k$.*

Proof.

$$\left(\frac{A_n}{n}\right)^2 = \left(a_n + \frac{A_n}{n} - a_n\right)^2 \leq 2a_n^2 + 2\left(\frac{A_n}{n} - a_n\right)^2 = 4a_n^2 + 2\left(\frac{A_n}{n}\right)^2 - 4\frac{a_n A_n}{n}.$$

This implies

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 \leq \sum_{n=1}^{N} 4a_n^2 + 2\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 - 4\sum_{n=1}^{N}\frac{a_nA_n}{n} \tag{3.3}$$

for each $N \in \mathbb{N}$. Then,

$$\begin{aligned}
-2a_nA_n &= -\sum_{k=1}^{n} a_k^2 - \sum_{\substack{j,k=1\\ j\neq k}}^{n} a_ja_k + \sum_{k=1}^{n-1} a_k^2 + \sum_{\substack{j,k=1\\ j\neq k}}^{n-1} a_ja_k \\
&= -\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n-1} a_k\right)^2 - a_n^2 \\
&= -(A_n^2 - A_{n-1}^2) - a_n^2 \leq -(A_n^2 - A_{n-1}^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
-2\sum_{n=1}^{N}\frac{a_nA_n}{n} &\leq -\sum_{n=1}^{N}\frac{A_n^2 - A_{n-1}^2}{n} \\
&= -\frac{A_1^2}{1\cdot 2} - \frac{A_2^2}{2\cdot 3} - \cdots - \frac{A_{N-1}}{(N-1)\cdot N} - \frac{A_N^2}{N} \\
&\leq -\sum_{n=1}^{N}\frac{1}{n(n+1)}A_n^2.
\end{aligned}$$

The last inequality holds because $-(A_N^2)/N \leq (-A_N^2)/[N(N+1)]$.
By substituting the last estimate in (3.3), we get

$$\begin{aligned}
\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 &\leq 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 - 2\sum_{n=1}^{N}\frac{1}{n(n+1)}A_n^2 \\
&= 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\frac{(n+1)A_n^2 - nA_n^2}{n^2(n+1)} \\
&= 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\frac{1}{n^2(n+1)}A_n^2,
\end{aligned}$$

which yields

$$\sum_{n=1}^{N}\left(1 - \frac{2}{n+1}\right)\left(\frac{A_n}{n}\right)^2 \leq 4\sum_{n=1}^{N} a_n^2. \tag{3.4}$$

When $N \to \infty$, by the limit comparison we obtain the statement. □

An important generalization of this result was proved by Riesz one year after.

Theorem 3.1.2 (Riesz, 1920)**.** *If $p > 1$, $a_n \geq 0$, and $\sum_{n=1}^{\infty} a_n^p$ is convergent, then $\sum_{n=1}^{\infty}(A_n/n)^p$ is convergent, where $A_n = \sum_{k=1}^{n} a_k$.*

Proof. Let $\Phi_n := n^{-p} + (n+1)^{-p} + \dots$.
Then, defining $A_0 = 0$, we get,

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p &= \sum_{n=1}^{N} A_n^p(\Phi_n - \Phi_{n+1}) = \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n - A_N^p\Phi_{N+1} \\ &\leq \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n \end{aligned} \tag{3.5}$$

Notice that

$$A_n^p - A_{n-1}^p \leq p a_n A_n^{p-1}. \tag{3.6}$$

Indeed, since the function $f : \mathbb{R}^+ \to \mathbb{R},\ \ f(x) = x^p$ is convex for $p \geq 1$, for every fixed $x_0 \in \mathbb{R}^+$

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \qquad \forall\ x \in \mathbb{R}^+.$$

Choosing $x_0 = A_n$ and $x = A_{n-1}$ we obtain

$$A_{n-1}^p \geq A_n^p + pA_n^{p-1}(-a_n)$$

which implies (3.6). By (3.5) and (3.6) we get

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \leq p\sum_{n=1}^{N} a_n A_n^{p-1}\Phi_n.$$

Moreover,

$$\Phi_n < n^{-p} + \int_n^{\infty} x^{-p}dx = n^{-p} + \frac{n^{-(p-1)}}{p-1} \leq \frac{p}{p-1}n^{-(p-1)}.$$

The last inequality holds because

$$\begin{aligned} n^{-p} + \frac{n^{-(p-1)}}{p-1} = n^{-p}\left(1 + \frac{n}{p-1}\right) = n^{-p}\left(\frac{p-1+n}{p-1}\right) \leq \frac{p}{p-1}n^{-(p-1)} \\ \iff \frac{p-1+n}{p-1} \leq \frac{p}{p-1}n \iff (p-1)(n-1) \geq 0, \end{aligned}$$

that is true for all $n \in \mathbb{N}$ and $p > 1$. Finally, recall the Hölder's inequality,

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}$$

where $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$.
Then we can conclude

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p &\leq \frac{p^2}{p-1}\sum_{n=1}^{N} a_n\left(\frac{A_n}{n}\right)^{p-1} \\ &\leq \frac{p^2}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p}\left[\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{(p-1)q}\right]^{1/q}.\end{aligned}$$

It is trivial to show

$$q(p-1) = \left(\frac{p-1}{p}\right)^{-1}(p-1) = p.$$

Hence

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{1-1/q} \leq \frac{p^2}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p},$$

which is equivalent to

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{p} \leq \left(\frac{p^2}{p-1}\right)^{p}\left(\sum_{n=1}^{N} a_n^p\right).$$

□

Theorem 3.1.2 yields more then what Hardy formulated in Theorem 3.1.1. Hardy noticed that the constant $(p^2/(p-1))^p$ could be improved by $(p/(p-1))^p$ yielding the inequality (3.1).

Theorem 3.1.3 (Landau-Hardy, 1921)**.** *Let $p > 1$, $a_n \geq 0$, and $A_n = \sum_{k=1}^{n} a_k$. Then the inequality*

$$\sum_{n=1}^{N} a_n^p \geq \left(\frac{p-1}{p}\right)^{p}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{p} \tag{3.7}$$

holds for $N \in \mathbb{N}$ fixed or $N = \infty$. Moreover, for $N = \infty$ the constant $\left(\frac{p-1}{p}\right)^p$ is sharp.

Proof. We divide the proof in three steps.

Step 1. Consider $c_n = 1 - 2/(n+1)$ and for $m = 2, 3, \dots$ let

$$a_1 = a_2 = \dots = a_m = b_1, a_{m+1} = a_{m+2} = \dots = a_{2m} = b_2, \dots,$$
$$a_{(N-1)m+1} = a_{(N-1)m+2} = \dots = a_{Nm} = b_N.$$

From inequality (3.4) with Nm in place of N we obtain

$$4m\sum_{n=1}^{N} b_n^2 \geq (c_1 + \dots + c_m)\left(\frac{B_1}{1}\right)^2 + (c_{m+1} + \dots + c_{2m})\left(\frac{B_2}{2}\right)^2 + \dots$$
$$+ (c_{(N-1)m+1} + \dots + c_{Nm})\left(\frac{B_N}{N}\right)^2,$$

where $B_n = \sum_{k=1}^{n} b_k$. Dividing by m and letting $m \to \infty$, we find that

$$(c_1 + c_2 + \dots + c_m)/m \to 1,$$
$$(c_{m+1} + \dots + c_{2m})/m \to 1, \qquad etc.$$

A possible way for proving these limits is the following: it is known that $\sum_{n=1}^{m} 1/n \sim C log(m)$. Indeed,

$$log(m+1) = \int_1^{m+1} dt/t \leq \sum_{n=1}^{m}\frac{1}{n} \leq 1 + \int_1^{m} dt/t = 1 + log(m),$$

thus
$$\frac{\sum_{n=1}^{m} c_n}{m} \leq \frac{m - 2(log(m+1) - log(2))}{m} \to 1 \ for \ m \to \infty.$$
This implies
$$\sum_{n=1}^{N} \left(\frac{B_n}{n}\right)^2 \leq 4 \sum_{n=1}^{N} b_n^2,$$
which yields (3.1) with $p = 2$.

Step 2. For $y \geq 0$ it holds
$$y^p - py + p - 1 \geq 0. \tag{3.8}$$
In order to prove this inequality, we could study the first derivative and note that
$$p(y^{p-1} - 1) \geq 0 \iff y \geq 1,$$
and
$$\begin{array}{ll} for \ y = 1 & 1 - p + p - 1 = 0, \\ for \ y = 0 & p - 1 > 0. \end{array}$$
So (3.8) is verified. Then, let $y = \frac{y_1}{y_2}$, with $y_1 \geq 0, y_2 > 0$, and obtain
$$y_1^p - p y_1 y_2^{p-1} + (p-1) y_2^p \geq 0.$$
Now, for a nonnegative $\{b_n\}_n$ choose $y_1 = b_n$ and $y_2 = (p-1)B_n/(pn)$, where $B_n = \sum_{k=1}^{n} b_k$, and obtain
$$b_n^p - p b_n \left(\frac{p-1}{p}\frac{B_n}{n}\right)^{p-1} + (p-1)\left(\frac{p-1}{p}\frac{B_n}{n}\right)^p \geq 0,$$
so
$$\sum_{n=1}^{N} b_n^p - \left(\frac{p-1}{p}\right)^{p-1} \sum_{n=1}^{N} p b_n \left(\frac{B_n}{n}\right)^{p-1} + (p-1)\left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} \left(\frac{B_n}{n}\right)^p \geq 0. \tag{3.9}$$
Moreover, we verified in the proof of Theorem 3.1.2 that
$$p b_n B_n^{p-1} = p B_n^{p-1}(B_n - B_{n-1}) \geq B_n^p - B_{n-1}^p,$$
thus
$$\begin{aligned} \sum_{n=1}^{N} p b_n \left(\frac{B_n}{n}\right)^{p-1} &\geq \sum_{n=1}^{N} (B_n^p - B_{n-1}^p)\frac{1}{n^{p-1}} = \sum_{n=1}^{N-1} B_n^p \left(\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}}\right) + B_N^p \frac{1}{N^{p-1}} \\ &\geq \sum_{n=1}^{N} B_n^p \left(\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}}\right) \geq (p-1) \sum_{n=1}^{N} B_n^p \frac{1}{(n+1)^p}. \end{aligned} \tag{3.10}$$
The last inequality holds because
$$\begin{aligned} &\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}} \geq \frac{p-1}{(n+1)^p} \iff \frac{(n+1)^p}{n^{p-1}} - (n+1) \geq p - 1 \\ &\iff \left(\frac{n+1}{n}\right)^p - \left(\frac{n+1}{n}\right) \geq (p-1)\frac{1}{n}. \end{aligned}$$

which is verified choosing $y = (n+1)/n$ in (3.8). Combining the two inequalities (3.9) (3.10) we discover that

$$\sum_{n=1}^{N} b_n^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} B_n^p \left(\frac{p}{(n+1)^p} - \frac{p-1}{n^p}\right) = \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} c_n \left(\frac{B_n}{n}\right)^p,$$

where $c_n = p(1+\frac{1}{n})^{-p} - p + 1 \to 1$ when $n \to \infty$. Now, as we did in Step 1, consider the sequence

$$b_1 = b_2 = \ldots = b_m = a_1,\ b_{m+1} = b_{m+2} = \cdots = b_{2m} = a_2, \ldots,$$
$$b_{(N-1)m+1} = b_{(N-1)m+1} = b_{(N-1)m+2} = \cdots = b_{Nm} = a_N,$$

replacing N with Nm we conclude

$$m\left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} a_n^p \geq (c_1 + c_2 + \cdots + c_m)\left(\frac{A_1}{1}\right)^p + (c_{m+1} + c_{m+2} + \cdots + c_{2m})\left(\frac{A_2}{2}\right)^p +$$
$$+ \cdots + (c_{(N-1)m+1} + c_{(N-1)m+2} + \cdots + c_{Nm})\left(\frac{A_N}{N}\right)^p.$$

Dividing by m and letting $m \to \infty$ we observe that $(c_1 + c_2 + \cdots + c_m)/m \to 1$, $(c_{m+1} + c_{m+2} + \cdots + c_{2m})/m \to 1$, etc., which means that (3.7) holds for all finite N and in particular it is still valid when $N \to \infty$.

Step 3.We prove that $\Big(p/(p-1)\Big)^p$ is the sharp constant for $N = \infty$. Choose $a_n = n^{-1/p-\epsilon}$ $(0 < \epsilon < 1 - 1/p)$. Then

$$\begin{aligned} A_n &= \sum_{k=1}^{n} k^{-1/p-\epsilon} > \int_1^n x^{1/p-\epsilon} dx \\ &= \frac{1}{1-1/p-\epsilon}(n^{1-1/p-\epsilon} - 1) > \frac{p}{p-1}(n^{1-1/p-\epsilon} - 1), \end{aligned}$$

implying that

$$\begin{aligned} \left(\frac{A_n}{n}\right)^p &> \left(\frac{p}{p-1}\right)^p \frac{(n^{1-1/p-\epsilon} - 1)^p}{n^p} = \left(\frac{p}{p-1}\right)^p n^{-1-\epsilon p}\left(1 - \frac{1}{n^{1-1/p-\epsilon}}\right)^p \\ &\geq \left(\frac{p}{p-1}\right)^p n^{-1-\epsilon p}\left(1 - \frac{p}{n^{1-1/p-\epsilon}}\right) \\ &= \left(\frac{p}{p-1}\right)^p n^{-1-\epsilon p}\left(1 - \frac{1}{n^{1-1/p-\epsilon}}\right)^p \\ &= \left(\frac{p}{p-1}\right)^p (n^{-1-\epsilon p} - pn^{-2+1/p+\epsilon-\epsilon p}). \end{aligned} \tag{3.11}$$

Put $y = 1/(n^{1-1/p-\epsilon})$, then (3.11) holds because it is equivalent to $y^p - py + p - 1 \geq 0$ with $y \geq 0$. Furthermore, the above inequality yields

$$\begin{aligned} \sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p &> \left(\frac{p}{p-1}\right)^p \left(\sum_{n=1}^{N} a_n^p - p\sum_{n=1}^{N} \frac{1}{n^{2-1/p-\epsilon+\epsilon p}}\right) \\ &= \left(\frac{p}{p-1}\right)^p \left(\sum_{n=1}^{N} a_n^p - pC_{N,\epsilon}\right), \end{aligned}$$

where $C_{N,\epsilon} \to C$ as $N \to \infty$ for any $\epsilon > 0$ because $2 - 1/p - \epsilon + \epsilon p > 1$. Thus

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p \Big/ \sum_{n=1}^{N} a_n^p > \left(\frac{p}{p-1}\right)^p \left(1 - pC_{N,\epsilon} \Big/ \sum_{n=1}^{N} a_n^p\right) \to \left(\frac{p}{p-1}\right)^p,$$

since $\sum_{n=1}^{N} a_n^p = \sum_{n=1}^{N} n^{-1-\epsilon} \to \infty$ as $N \to \infty$ and $\epsilon \to 0^+$. So the sharpness is established. □

Finally we show an easier proof of (3.1) given by Elliot in 1926 and based on the following

Proposition 3.1.1 (Young's inequality)**.** *Given two nonnegative numbers a,b and p, q such that $1/p + 1/q = 1$ with p, $q > 1$, then*

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}.$$

Proof. Let $\lambda = 1/p$, so $1 - \lambda = 1/q$. It is known that the logarithm is a concave function. Then

$$\log\left(a^q(1-\lambda) + b^p\lambda\right) \geq \lambda p \log b + (1-\lambda) q \log a = \log a + \log b = \log(ab).$$

□

Then it follows an **alternative proof of** (3.1).

Proof. Let $\alpha_n = A_n/n$ and $\alpha_0 = 0$. Note that

$$a_n = A_n - A_{n-1} = A_n + \frac{1-n}{n-1} A_{n-1} = n\left(\frac{A_n}{n} - \frac{A_{n-1}}{n-1}\right) + \frac{A_{n-1}}{n-1}.$$

Using Young's inequality:

$$\begin{aligned}
\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1} a_n &= \alpha_n^p - \frac{p}{p-1}\Big(n\alpha_n - (n-1)\alpha_{n-1}\Big)\alpha_n^{p-1} \\
&= \alpha_n^p \left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1} \\
&\leq \alpha_n^p \left(1 - \frac{np}{p-1}\right) + \frac{(n-1)}{p-1}\Big((p-1)\alpha_n^p + \alpha_{n-1}^p\Big) \\
&= \frac{1}{p-1}\Big((n-1)\alpha_{n-1}^p - n\alpha_n^p\Big).
\end{aligned}$$

Summing from 1 to N yields

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p - \frac{p}{p-1} \sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^{p-1} a_n \leq -\frac{N\alpha_N^p}{p-1} \leq 0,$$

and from Hölder inequality

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p \leq \frac{p}{p-1} \sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^{p-1} a_n \leq \frac{p}{p-1} \left(\sum_{n=1}^{N} a_n^p\right)^{1/p} \left(\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p\right)^{1/q}.$$

Division by the last factor yields the statement. □

3.2 Continuous Hardy inequality

In 1925, Hardy formulated and proved his famous integral inequality. The main idea, which links the discrete and integral version, is the following: define $f(x) = a_k$ for $k-1 \leq x \leq k$ and $k \geq 1$ with $\{a_n\}$ a nonnegative decreasing sequence. We infer that the function

$$\frac{1}{x}\int_0^x f(t)dt = \frac{\sum_{k=1}^{n-1} a_k + a_n(x-n+1)}{x}$$

is decreasing in $[n-1,\ n]$; indeed we have

$$\frac{d}{dx}\left(\frac{1}{x}\int_0^x f(t)dt\right) = -\frac{\sum_{k=1}^{n-1} a_k - a_n(n-1)}{x^2} = -\frac{\sum_{k=1}^{n} a_k - a_n n}{x^2} \leq 0,$$

which is equivalent to $a_n \leq \frac{\sum_{k=1}^n a_k}{n}$, that holds because a_n is decreasing. Hence, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{\sum_{k=1}^n a_k}{n}\right)^p &\leq \sum_{n=1}^{\infty}\int_{n-1}^{n}\left(\frac{\sum_{k=1}^{n-1} a_k + a_n(x-n+1)}{x}\right)^p dx \\
&= \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx \\
&= \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.
\end{aligned}$$

In line with such idea, Hardy stated the following result:

Theorem 3.2.1 (Hardy, 1926). *Let $p > 1$ and let $f \geq 0$ be p-integrable on $(0,\infty)$. Then $F(x) = \int_0^x f(t)dt < \infty$ for every $x > 0$ and*

$$\int_0^\infty f^p(x)dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \left(\frac{F(x)}{x}\right)^p dx.$$

Furthermore, the constant $(p/(p-1))^p$ is sharp in the sense that the previous inequality cannot hold with a constant $C > \left(\frac{p-1}{p}\right)^p$.

Remark 3.2.1. *If $f \in L^p(\mathbb{R})$, then $F' = f$ in the weak sense. Hence, the inequality*

$$\int_0^\infty \Big(F'(x)\Big)^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \left(\frac{F(x)}{x}\right)^p dx$$

is equivalent to (3.2).

Proof. Using integration by parts and the identity $d/dx(F(x)^p) = pF(x)^{p-1}f(x)$ which holds for almost every x in $(0,\infty)$, for arbitrary $0 < \alpha < A < \infty$ we get:

$$\begin{aligned}
\int_\alpha^A \left(\frac{F(x)}{x}\right)^p dx &= -\frac{1}{p-1}\int_\alpha^A F^p(x)\frac{d}{dx}(x^{1-p})dx \\
&= \frac{\alpha^{1-p}}{p-1}F^p(\alpha) - \frac{A^{1-p}}{p-1}F^p(A) + \frac{1}{p-1}\int_\alpha^A x^{1-p}\frac{d}{dx}(F^p(x))dx \\
&\leq \frac{\alpha^{1-p}}{p-1}F^p(\alpha) + \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx.
\end{aligned}$$

Recall the continuous version of the Hölder's inequality

$$\int_{\alpha}^{A}\left(\frac{F(x)}{x}\right)^{p-1}f(x)dx \le \left(\int_{\alpha}^{A}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\alpha}^{A}\left(\frac{F(x)}{x}\right)^{p}dx\right)^{(p-1)/p},$$

because $(p-1)q = (p-1)p/(p-1) = p$ and $1/q = (p-1)/p$ if $1/p + 1/q = 1$.
Taking β such that $\alpha \le \beta \le A$ and applying the previous two inequalities to $F(x) - F(\alpha)$ instead of $F(x)$, we find that

$$\begin{aligned}&\int_{\alpha}^{A}\left(\frac{F(x)-F(\alpha)}{x}\right)^{p}dx\\ &\le \frac{p}{p-1}\int_{\alpha}^{A}\left(\frac{F(x)-F(\alpha)}{x}\right)^{p-1}f(x)dx\\ &\le \frac{p}{p-1}\left(\int_{\alpha}^{A}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\alpha}^{A}\left(\frac{F(x)-F(\alpha)}{x}\right)^{p}dx\right)^{(p-1)/p}.\end{aligned}$$

Thus

$$\left(\int_{\alpha}^{A}\left(\frac{F(x)-F(\alpha)}{x}\right)^{p}dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_{0}^{\infty}f^p(x)dx\right)^{\frac{1}{p}},$$

and a fortiori,

$$\left(\int_{\beta}^{A}\left(\frac{F(x)-F(\alpha)}{x}\right)^{p}dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_{0}^{\infty}f^p(x)dx\right)^{\frac{1}{p}}.$$

Then let $\alpha \to 0^+$ and observe that $F(x) - F(\alpha) \to F(x)$. To conclude, let $A \to \infty$ and $\beta \to 0^+$. For proving the sharpness of the constant, it suffices to use a variant of the argument of Theorem 3.1.3. □

The previous proof is close to Hardy's original idea but it contains some simplifications suggested by Pólya. Next we state the Minkowski inequality which is useful for an alternative proof of (3.2).

Proposition 3.2.1 (Minkowski inequality)**.** *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a positive measurable function and $1 \le p \le \infty$. Then*

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}f(x,y)dy\right)^{p}dx\right)^{\frac{1}{p}} \le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}f^p(x,y)dx\right)^{\frac{1}{p}}dy.$$

Now we present **Ingham's proof of** (3.2).

Proof. Put

$$Hf(x) = \frac{1}{x}\int_{0}^{x}f(t)dt = \int_{0}^{1}f(tx)dt,$$

By Minkowski's inequality

$$\begin{aligned}\left(\int_{0}^{\infty}\Big(H(fx)\Big)^{p}dx\right)^{\frac{1}{p}} &= \|Hf\|_p = \left\|\int_{0}^{1}f(t)dt\right\|_p\\ &\le \int_{0}^{1}\|f(t)\|_p dt = \int_{0}^{1}\left(\int_{0}^{\infty}f^p(tx)dx\right)^{\frac{1}{p}}dt\\ &= \int_{0}^{1}\left(\int_{0}^{\infty}f^p(s)ds/t\right)^{\frac{1}{p}}dt = \frac{p}{p-1}\left(\int_{0}^{\infty}f^p(s)ds\right)^{\frac{1}{p}}.\end{aligned}$$

□

We conclude this section with the N-dimensional version of (3.2), namely the Hardy inequality in $\mathbb{R}^N$. The natural functional space where the inequality can be stated is the Sobolev space $W^{1,p}(\mathbb{R}^N)$, which we now recall.
Given a multi-index α in $\mathbb{N}^N$ and a function $f:\Omega\subset\mathbb{R}^N\to\mathbb{R}$ for some open set Ω we denote

$$D^\alpha f(x)=\frac{\partial^{|\alpha|}f(x)}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_N}^{\alpha_N}}.$$

Definition 3.2.1. *For $p>1$ and Ω an open subset of $\mathbb{R}^N$ let*

$$\begin{aligned}W^{1,p}(\Omega)&=\{f\in L^p(\Omega) \text{ such that } D^\alpha f\in L^p(\Omega) \text{ for all } |\alpha|=1\},\\ W^{1,p}_{loc}(\Omega)&=\{f\in L^p(\Omega) \text{ such that } D^\alpha f\in L^p_{loc}(\Omega) \text{ for all } |\alpha|=1\},\end{aligned}$$

where D^α is the weak derivative.

Theorem 3.2.2. *Assume $1<p<N$ and $u\in W^{1,p}(\mathbb{R}^N)$. Then $u/|x|\in L^p(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N}|\nabla u(x)|^p dx\geq((N-p)/p)^p\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^p}dx. \tag{3.12}$$

Furthermore, the constant $((N-p)/p)^p$ is sharp.

Proof. A density argument allows us to consider only smooth functions $u\in C_c^\infty(R^N)$. Under this assumption we have that

$$|u(x)|^p=-\int_1^\infty\frac{d}{d\lambda}|u(\lambda x)|^p d\lambda=-p\int_1^\infty|u(\lambda x)|^{p-2}u(\lambda x)\langle x,\nabla u(x\lambda)\rangle d\lambda.$$

By using Hölder's inequality, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^p}dx&=-p\int_1^\infty\int_{\mathbb{R}^N}\frac{|u(\lambda x)|^{p-2}u(\lambda x)}{|x|^{p-1}}\langle\frac{x}{|x|},\nabla u(x\lambda)\rangle dxd\lambda\\&=-p\int_1^\infty\frac{d\lambda}{\lambda^{N+1-p}}\int_{\mathbb{R}^N}\frac{|u(y)|^{p-2}u(y)}{|y|^{p-1}}\frac{\partial u(y)}{\partial r}dy\\&=\frac{p}{N-p}\int_{\mathbb{R}^N}\frac{|u(y)|^{p-2}u(y)}{|y|^{p-1}}\frac{\partial u(y)}{\partial r}dy\\&\leq\frac{p}{N-p}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|y|^p}dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}\left|\frac{\partial u(y)}{\partial r}\right|^p dy\right)^{1/p}.\end{aligned}$$

Then we conclude that

$$\int_{\mathbb{R}^N}\left|\frac{u(x)}{x}\right|^p dx\leq\left(\frac{p}{N-p}\right)^p\int_{\mathbb{R}^N}|\nabla u(x)|^p dx.$$

Following the idea of Hardy used in the discrete case, we show the $C_{N,p}$ is the best constant. Given $\varepsilon>0$, consider the radial function $U(r)=\begin{cases}A_{N,p,\varepsilon} & \text{if } r\in[0,1],\\ A_{N,p,\varepsilon}r^{(p-N)/p-\varepsilon} & \text{if } r>1,\end{cases}$
where $A_{N,p,\varepsilon}=p/(N-p+p\varepsilon)$. It is easy to check that

$$U'(r)=\begin{cases}0 & \text{if } r\in[0,1],\\ -r^{-N/p-\varepsilon} & \text{if } r>1.\end{cases}$$

By a direct computation we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{U^p(x)}{|x|^p} dx &= \int_{B_1} \frac{U^p(x)}{|x|^p} dx + \int_{\mathbb{R}^N \setminus B_1} \frac{U^p}{|x|^p} dx \\ &= A^p_{N,p,\varepsilon} \omega_n \left(\int_0^1 r^{N-1-p} dr + \int_1^\infty r^{-(1+p\varepsilon)} dr \right) \\ &= A^p_{N,p,\varepsilon} \omega_n \int_0^1 r^{N-1-p} dr + A^p_{N,p,\varepsilon} \int_{\mathbb{R}^N} |\nabla u(x)|^p dx,\end{aligned}$$

where ω_n is the measure of the $N-1$ dimensional unit sphere. We conclude letting $\varepsilon \to 0$. □

Chapter 4

The supersolution construction of Hardy weights

By a Hardy type inequality for a nonnegative operator P we mean that the inequality $P \geq CW$ holds for some positive weight function W, called Hardy weight, and a positive constant C.
In the previous chapter classical proofs of the Hardy inequality on $\mathbb{N}$ and on $\mathbb{R}^N$ have been illustrated. They are all based on the clever exploitation of elementary inequalities such as Young's or Minkowski's inequalities.
In this chapter we present an alternative approach which is based on the use of positive supersolutions; i.e. to every positive supersolution v of $Pu = 0$, we associate the Hardy weight $W := Pv/v$. More precisely, in Section 4.1 and 4.2 we show an application of this technique when $P = \Delta$ by means of suitable positive radial superharmonic functions.
In general, this approach gives no information about the optimality of the Hardy weight Pv/v, i.e., it does not allow to say whether the obtained weight is "the largest possible" in a suitable sense. More recently, in [6], a refinement of the supersolution method has been developed in the context of general elliptic operators. In particular, it has been proved that optimal weights can be derived in terms of special superharmonic functions such as the Green function of the operator.
In Section 4.3 we present an application of these results in our framework which, in some case, allows to prove the optimality of the weights previously derived.
Finally, in Section 4.4, as a by-product of our results we derive optimal improvements of the Poincaré inequality.

4.1 Continuous setting

We start by illustrating the continuous version of the supersolution method in the case $P = -\Delta$. For the following theorem we refer to [3].

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let Δ be the Laplace operator on $L^2(\Omega)$. Suppose that there is a positive continuous function $u \in W^{1,2}_{loc}(\Omega)$ and a positive potential $W \in L^1_{loc}(\Omega)$ such that*

$$-\Delta u \geq Wu$$

in weak sense, namely

$$\int_\Omega \nabla u(x)\cdot\nabla\phi(x)\ dx \geq \int_\Omega W(x)u(x)\phi(x)\ dx$$

for all $0\leq\varphi\in C_c^\infty(\Omega)$. *Then the quadratic form inequality*

$$\int_\Omega |\nabla f(x)|^2 dx \geq \int_\Omega W(x)|f(x)|^2 dx$$

holds for all $f\in C_c^\infty(\Omega)$.

Proof. If $f\in C_c^\infty(\Omega)$ we may consider $f = ug$ where $g\in W_c^{1,2}(\Omega)$ and obtain

$$\begin{aligned}
\int_\Omega \sum_{i=1}^N \left(\frac{\partial f(x)}{\partial x_i}\right)^2 dx &= \int_\Omega \sum_{i=1}^N \left(\frac{\partial u(x)}{\partial x_i} g + u(x)\frac{\partial g(x)}{\partial x_i}\right)^2 dx \\
&\geq \int_\Omega \sum_{i=1}^N \left[\left(g(x)\frac{\partial u(x)}{\partial x_i}\right)^2 + 2u(x)g(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_i}\right] dx \\
&= \int_\Omega \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i}\frac{\partial}{\partial x_i}\Big(u(x)(g(x))^2\Big)\ dx \\
&= \int_\Omega \nabla u(x)\cdot\nabla(u(x)g^2(x))\ dx \\
&\geq \int_\Omega W(x)\big(u(x)g(x)\big)^2\ dx = \int_\Omega W(x)|f(x)|^2\ dx,
\end{aligned}$$

where in the last step we exploit the fact that ug^2 can be approximated by a sequence of $0\leq u_n\in C_c^\infty$ by using a mollifier. □

More in general, let P be a symmetric and nonnegative second-order linear elliptic operator defined on a domain $\Omega\subset\mathbb{R}^N$ and let q be the associated quadratic form. A Hardy type inequality with a weight $W\geq 0$ has the form

$$q(\varphi)\geq C\int_\Omega W(x)\varphi^2(x)\ dx \tag{4.1}$$

for all $\varphi\in C_c^\infty(\Omega)$, with $C>0$.

Definition 4.1.1. *A symmetric and nonnegative operator P is said* ***critical*** *in Ω if (4.1) holds true if and only if $W=0$. Conversely, when (4.1) is valid for a non trivial weight W the operator is called* ***subcritical*** *in Ω.*

It has been shown in [4] that $P-W$ is critical if and only if there exists a unique positive solution to $(P-W)u=0$, which is called Agmond **ground state**. In [4] the authors introduce the notion of optimal weight that we now recall.

Definition 4.1.2. *The weight W is said an* **optimal** *Hardy weight for the operator P if*

- *the operator $P-W$ is critical in Ω;*
- *$P-W\geq\lambda W$ fails to hold on $C_c^\infty(\Omega\setminus K)$ for all $\lambda>0$ and all $K\subset\Omega$ compact;*

- *the ground state v is not an eigenfunction, i.e. $v \notin L^2_W(\Omega)$, namely*

$$\int_\Omega v^2(x)W(x)\ dx = +\infty.$$

The next theorem provides a method to obtain an optimal weight for a operator P by means of the associated Green function see [5] and references therein. We refer to [4] for this result.

Theorem 4.1.2. *Let P be a symmetric subcritical operator in Ω, and let $G(x) := G_P^\Omega(x,0)$ be its minimal positive Green function with a pole at $0 \in \Omega$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying*

$$\lim_{x\to\infty} \frac{G(x)}{u(x)} = 0,$$

where ∞ is the ideal point in the one-point compactification of Ω. Consider the supersolution $v = \sqrt{Gu}$. Then

$$W = \frac{Pv}{v}$$

is an optimal Hardy weight with respect to P and the punctured domain $\Omega^* = \Omega \setminus \{0\}$.

Example 4.1.1. *As an application of the above theorem, consider $P = -\Delta$ the Laplace operator on $\Omega^* = \mathbb{R}^N \setminus \{0\}$, where $N \geq 3$, and denote by $G(x) = |x|^{2-N}$ the corresponding positive minimal Green function with pole at 0 see [3] Chapter 1.1.8. Consider the positive superharmonic function in Ω^**

$$v(x) = \sqrt{G(x)1} = \sqrt{G(x)} = |x|^{(2-N)/2}.$$

By a direct computation we infer

$$\begin{aligned} -\Delta v(x) &= \left(\frac{2-N}{2}\right)\frac{N}{2}|x|^{-N/2-1} - \frac{(N-1)(2-N)}{2|x|}|x|^{-N/2} = \frac{(2-N)^2}{2^2|x|^2}|x|^{1-N/2} \\ &= Wv(x). \end{aligned}$$

Hence, by Theorem 4.1.2,

$$\frac{-\Delta v(x)}{v(x)} = \left(\frac{2-N}{2|x|}\right)^2$$

is an optimal weight for $-\Delta$ and we reobtain the classical Hardy inequality

$$\int_{\mathbb{R}^N} \frac{f^2(x)}{|x|^2} dx \leq \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla f(x)|^2 dx \qquad \forall f \in C_c^\infty(\Omega).$$

It is worth noting that v is the unique positive solution of $-\Delta v = Wv$ and $v \notin L^2_W(\mathbb{R}^N)$. Indeed,

$$\int_{\mathbb{R}^N} |x|^{(2-N)} \frac{1}{|x|^2} = \int_0^\infty r^{-N} r^{N-1}\ dr = +\infty.$$

Hence, as expected, v is the ground state but it is not a eigenfunction.

4.2 Discrete setting

Now consider a graph $\Gamma = (V, E)$. We state the analogue of Theorem 4.1.1 and Theorem 4.1.2 for functions on graphs. For the next theorem we refer to [1], Proposition 3.1.

Theorem 4.2.1. *Let Δ be the combinatorial Laplacian and let u be a positive function on V such that*

$$\Delta u(x) \geq W(x)u(x) \qquad \forall x \in V$$

with $W \geq 0$. Then for all $f \in C_0(V)$ the following holds:

$$\langle f, \Delta f\rangle_{\ell^2} = \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}} (f(x) - f(y))^2 \geq \sum_{x\in V} W(x)f(x)^2. \tag{4.2}$$

Proof. Take any $f \in C_0(V)$. Then

$$\begin{aligned}
\langle f, Wf\rangle_{\ell^2} &\leq \sum_{x\in V} f(x)\frac{\Delta u(x)}{u(x)}f(x) = \sum_{x\in V}\sum_{y\in V}\chi_x(y)f^2(x)\left(\frac{u(x)-u(y)}{u(x)}\right)\\
&= \sum_{x\in V}\sum_{y\in V}\chi_x(y)\left(f^2(x) - \frac{u(y)}{u(x)}f^2(x)\right)\\
&= \sum_{x\in V}\sum_{y\in V}\chi_x(y)\left[f(x)^2 - \frac{1}{2}\left(\frac{u(y)}{u(x)}f^2(x) + \frac{u(x)}{u(y)}f^2(y)\right)\right]\\
&\leq \sum_{x\in V}\sum_{y\in V}\chi_x(y)\Big(f^2(x) - f(x)f(y)\Big)\\
&= \frac{1}{2}\sum_{x\in V}\sum_{y\in V}\chi_x(y)\Big(f(x) - f(y)\Big)^2 = \langle f, \Delta f\rangle_{\ell^2},
\end{aligned}$$

where the last inequality holds because it is equivalent to the Young's inequality. $\square$

Notice that the previous theorem is also valid if $W \leq 0$ but, in such case, the inequality (4.2) is trivial. We also point out that if u provides the weight W, then the function cu gives the same inequality for all $c \in \mathbb{R} \setminus \{0\}$.
The approach of [4] has been applied in the context of Schrödinger operators on weighted graphs.

Definition 4.2.1. *Given a graph $\Gamma = (V, E)$ and a potential $Q : V \to \mathbb{R}$ we define the formal Schrödinger operator H on V by*

$$H := \Delta + Q.$$

The associated bilinear form h of H on $C_0(V) \times C_0(V)$ is given by

$$h(\phi, \psi) := \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}} (\phi(x) - \phi(y))(\psi(x) - \psi(y)) + \sum_{x\in V} Q(x)\phi(x)\psi(x).$$

We denote by $h(\phi) := h(\phi, \phi)$ the associated quadratic form on $C_0(V)$.
A Hardy type inequality with a weight $W \geq 0$ has the form

$$h(\phi) \geq C\sum_{x\in V} W(x)\phi^2(x)$$

for all $\phi \in C_0(V)$, with $C > 0$.
We need the following definitions in order to introduce the notion of optimal weight in a graph.

Definition 4.2.2. *Let h be a quadratic form associated with the Schrödinger operator H, such that $h \geq 0$ on $C_0(V)$. The form h is called* **subcritical** *in V if there is a nonnegative $W \in C_0(V)$ different from the identically zero function such that $h - W \geq 0$ on $C_0(V)$. A positive form h which is not subcritical is called* **critical** *in V.*

In [6] it is shown that the criticality of h is equivalent to the existence of a unique positive function which is H-harmonic. Analogously to the continuous case, such a function is called a ground state.

Definition 4.2.3. *Let h be a quadratic form associated with the Schrödinger operator H. We say that a positive function $W : V \to [0, \infty)$ is an optimal Hardy weight for h in V if*

- *$h - W$ is critical in V,*
- *$h - W \geq \lambda W$ fails to hold on $C_0(V \setminus K)$ for all $\lambda > 0$ and all finite $K \subset V$.*
- *The ground state Ψ is not an eigenfunction, i.e. $\Psi \notin \ell^2_W$, namely*

$$\sum_{x \in V} \Psi^2(x) W(x) = +\infty.$$

Definition 4.2.4. *A function $u : V \to \mathbb{R}$ is called proper on $W \subset V$ if $u^{-1}(I)$ is compact (or, equivalently, finite) for all compact $I \subset u(W) := \{u(x) | x \in W\}$.*

In [6] the authors present a theorem which provides an optimal inequality for a Schrödinger operator H under the assumption of the existence of positive H-superharmonic functions that are H-harmonic outside a compact set.

Theorem 4.2.2. *Let $\Gamma = (V, E)$ be a connected graph and let Q be a given potential. Let u, v be positive H superharmonic functions that are H-harmonic outside of a finite set. Let $u_0 := u/v$, and assume that*

- *a) $u_0 : V \to (0, \infty)$ is proper.*
- *b) $\sup\limits_{\substack{x,y \in V \\ x \sim y}} \frac{u_0(x)}{u_0(y)} < \infty$.*

Then the function

$$W := \frac{H[(uv)^{1/2}]}{(uv)^{1/2}}$$

is an optimal Hardy weight. Moreover

$$W(x) = 1/2 \sum_{y \sim x, y \in V} \left[\left(\frac{u(y)}{u(x)} \right)^{1/2} - \left(\frac{v(y)}{v(x)} \right)^{1/2} \right]^2,$$

for all $x \in V$ satisfiyng $Hu(x) = Hv(x) = 0$.

The following lemma is useful to derive from Theorem 4.2.2 the next two theorems where $H = \Delta$.

Lemma 4.2.1. *Given a graph $\Gamma = (V, E)$ assume that $m(x) \leq C$ for all $x \in V$, and let u be strictly positive Δ-superharmonic on $U \subset V$. Then*

$$\sup_{\substack{x \sim y \\ x \in U}} \frac{u(x)}{u(y)} \leq C.$$

Proof. Since $u > 0$ and $\Delta u(y) \geq 0$, we get for $x \sim y$

$$u(x) \leq \sum_{z \sim y} u(z) \leq m(y)u(y) \leq Cu(y)$$

where $C > 0$ does not depend on $x \in U$. □

The next theorem is the discrete analogue of Theorem 4.1.2 when P is the combinatorial Laplacian. We refer to [6].

Theorem 4.2.3. *Let a transient connected graph with a bounded vertex degree be given, and let $o \in V$ be a fixed reference vertex. Let $G_o : V \to (0, \infty)$ be the Green function, and assume that $G = G_o$ is proper. Then the following inequality holds true*

$$\frac{1}{2} \sum_{\substack{x,y \in V \\ x \sim y}} (\phi(x) - \phi(y))^2 \geq \sum_{x \in V} W(x)\phi^2(x)$$

for all finitely supported functions ϕ on V, where

$$W(x) := \frac{\Delta\big(G(x)^{1/2}\big)}{G(x)^{1/2}}$$

is an optimal Hardy weight in V, and for all $x \neq o$

$$W(x) = \frac{1}{2G(x)} \sum_{y \sim x} \big(G(x)^{1/2} - G(y)^{1/2}\big)^2.$$

Proof. We apply Theorem 4.2.2 with $v = G(\cdot, o)$ and $u = 1$. In particular, assumption (a) of Theorem 4.2.2 is satisfied for $u_0 = 1/G(\cdot, o)$. Furthermore, assumption (b) follows from Lemma 4.2.1. Hence the statement follows. □

More in general in [6] it is also proved that it is possible to construct an optimal weight by means of functions that are harmonic outside a compact (i.e. finite) set.

Theorem 4.2.4. *Let $G = (V, E)$ be a connected graph with bounded vertex degree and let K be a finite subset of V. Let $u : V \to (0, \infty)$ be a positive function which is harmonic and proper on $V \setminus K$ and such that $u = 0$ on K. Then the following Hardy-type inequality holds true on K*

$$\frac{1}{2} \sum_{\substack{x,y \in V \\ x \sim y}} (\phi(x) - \phi(y))^2 \geq \sum_{x \in V \setminus K} W(x)\phi^2(x)$$

for all finitely supported functions ϕ with support in $V \setminus K$, where the weight function W is given by

$$W(x) = \frac{1}{2u(x)} \sum_{y \sim x} (u(x)^{1/2} - u(y)^{1/2})^2 \qquad x \in V \setminus K.$$

Moreover, W is optimal in $V \setminus K$.

Proof. Let $h_{V\setminus K}$ be the restriction of the form h to the space $C_0(V \setminus K)$. Then the operator $H_{V\setminus K}$ acts as

$$H_{V\setminus K}\varphi(x) = \sum_{y\in V\setminus K, y\sim x} (\varphi(x) - \varphi(y)) + Q(x)\varphi(x),$$

with $Q(x) := \#\{z \in K | z \sim x\}$. Hence $v = 1$ is $H_{V\setminus K}$-superharmonic in $V \setminus K$ and $H_{V\setminus K}$-harmonic outside of the combinatorial neighborhood of K. Moreover, as $\Delta = H_{V\setminus K}$ for functions supported on $V \setminus K$, the restriction of u to $V \setminus K$ is $H_{V\setminus K}$ harmonic.
Assumption (a) of Theorem 4.2.2 is satisfied for $u_0 = u$. Furthermore, assumption (b) follows from Lemma 4.2.1. Hence we obtain for $\varphi \in C_0(V \setminus K)$,

$$\frac{1}{2}\sum_{x\sim y}(\varphi(x) - \varphi(y))^2 = h(\varphi) \geq \sum_{x\in V\setminus K} W(x)\varphi^2(x)$$

with optimal weight W given by

$$W(x) = \frac{H_{V\setminus K}u^{1/2}}{u^{1/2}}(x) = \frac{1}{2u(x)}\sum_{y\sim x}\left(u^{1/2}(x) - u^{1/2}(y)\right)^2$$

for $x \in V \setminus K$. □

4.3 Hardy inequalities on some particular graphs

In this section we apply the results presented in the previous section in particular classes of trees, namely in $\mathbb{N}$, in the homogeneous tree $\mathbb{T}_{q+1}$ and in the biregular tree $\mathbb{T}_{P,D}$.

4.3.1 Hardy inequalities on $\mathbb{N}$

When $\Gamma = \mathbb{N}$, not only it is possible to recover the classical Hardy inequality but it is also possible to improve it. Notice that the following inequality is consequence of (3.1), and it could be considered the discrete analogous of (3.2.1). For the next theorem we refer to [6].

Theorem 4.3.1. *Consider the graph in which the vertices are the non-negative integers and two nodes n_1, n_2 are neighbours if and only if $|n_1 - n_2| = 1$. Then for all finitely supported function $\varphi : \mathbb{N}_0 \to \mathbb{R}$ with $\varphi(0) = 0$ we have*

$$\sum_{n=0}^{\infty}\left(\varphi(n) - \varphi(n+1)\right)^2 \geq \sum_{n=1}^{\infty} W(n)\varphi^2(n)$$

with the Hardy weight W given by

$$W(n) = \sum_{k=1}^{\infty}\binom{4k}{2k}\frac{1}{(4k-1)2^{4k-1}}\frac{1}{n^{2k}}$$

for $n \geq 2$ and $W(1) = 2 - \sqrt{2}$. In particular, $W(n) > 1/(4n^2)$ $\forall n \geq 1$.

Proof. Consider the combinatorial Laplacian in $\mathbb{N}$

$$\Delta\varphi(n) = \sum_{m\sim n}\left(\varphi(n) - \varphi(m)\right) = 2\varphi(n) - \varphi(n-1) - \varphi(n+1) \qquad \text{for all } n \geq 1.$$

If we choose $u(n) = n$, clearly $\Delta u(n) = 0$ if $n \geq 1$. Then using Theorem 4.2.4 with such a function we get

$$W(n) = \frac{1}{2n}((n+1)^{1/2} - n^{1/2})^2 + (n^{1/2} - (n-1)^{1/2})^2$$
$$= 2 - \left[\left(1 - \frac{1}{n}\right)^{1/2} + \left(1 + \frac{1}{n}\right)^{1/2}\right].$$

Employing the Taylor expansion of the square root at 1, we infer

$$\left(1 \pm \frac{1}{n}\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{\pm 1}{n}\right)^k = 1 \pm \frac{1}{2n} - \frac{1}{8n^2} \pm \frac{1}{16n^3} - \frac{5}{128n^4} \pm \dots,$$

which yields the result. □

4.3.2 Hardy inequalities on $\mathbb{T}_{q+1}$

Consider the homogeneous tree $\Gamma = \mathbb{T}_{q+1}$ with $q \geq 2$. We initially apply Theorem 4.2.3 to get an optimal weight. Subsequently we apply Theorem 4.2.1 using a family of radial positive superharmonic functions on Γ. Finally, we obtain another optimal weight for Δ by means of Theorem 4.2.2.

Theorem 4.3.2. *For all $\varphi \in C_0(\mathbb{T}_{q+1})$ the following inequality holds:*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x \in \mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x),$$

where

$$W_{opt}(x) = \begin{cases} \Lambda_q + q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ \Lambda_q & \text{if } |x| \geq 1. \end{cases}$$

is optimal for Δ and $\Lambda_q = (q^{1/2} - 1)^2$ is the bottom of the ℓ^2 combinatorial Laplacian's spectrum.

Proof. Consider the function $\tilde{u}(x) = \sqrt{G(x,o)}$, where G is the Green function on $\mathbb{T}_{q+1}$. By Theorem 4.2.3 we only need to show that

$$\frac{\Delta \tilde{u}(x)}{\tilde{u}(x)} = W_{opt}(x).$$

In the case of homogeneous tree, for $q \geq 2$, is known explicitly the Green function, see Chapter 2. Hence,

$$\tilde{u}(x) = \sqrt{\frac{q}{q-1}\left(\frac{1}{q}\right)^{|x|}}.$$

Now, for $x \neq o$, we compute

$$\frac{\Delta \tilde{u}(x)}{\tilde{u}(x)} = \left(q + 1 - \left(\frac{1}{q}\right)^{1/2} - q\left(\frac{1}{q}\right)^{-1/2}\right)$$
$$= \left((q+1) - q^{1/2} - q^{1/2}\right)$$
$$= \left(q^{1/2} - 1\right)^2 = \Lambda_q > 0.$$

While for $x = o$ we have,

$$\begin{aligned}\frac{\Delta\tilde{u}(o)}{\tilde{u}(o)} &= \left[(q+1)\left(\frac{q}{q-1}\right)^{1/2} - (q+1)\left(\frac{1}{(q-1)^{1/2}}\right)\right]\left(\frac{q-1}{q}\right)^{1/2}\\ &= q+1-\frac{q+1}{q^{1/2}} = \Lambda_q + q^{1/2} - q^{-1/2} > \Lambda_q.\end{aligned}$$

□

Remark 4.3.1. *Notice that $G^{1/2}$ is the ground state of $h_\Delta - W_{opt}$. Indeed it solves the equation $(\Delta - W_{opt})v = 0$ and*

$$\sum_{x\in\mathbb{T}_{q+1}} G(x)W_{opt}(x) = +\infty.$$

In fact,

$$\sum_{x\in\mathbb{T}_{q+1}} G(x)W_{opt}(x) = W_{opt}(o)\frac{q}{q-1} + \sum_{|x|\geq 1}\Lambda_q\frac{q}{q-1}q^{-|x|}.$$

Then we compute

$$\sum_{N\geq|x|\geq 1} q^{-|x|} = (q+1)q^{-1} + (q+1)qq^{-2} + \cdots + (q+1)q^{N-1}q^{-N} = (q+1)q^{-1}N,$$

which goes to $+\infty$ when $N \to +\infty$. It follows that

$$\sum_{x\in\mathbb{T}_{q+1}} G(x)W_{opt}(x) = +\infty.$$

Theorem 4.3.3. *For all $0 \leq \beta \leq \log_2 q^{1/2}$ and $\gamma \in [q^{-1/2}, q^{-1/2} + q^{1/2} - 2^\beta]$ it holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} W_{\beta,\gamma}(x)\varphi^2(x)$$

for all $\varphi \in C_0(\mathbb{T}_{q+1})$, where $W_{\beta,\gamma} \geq 0$ is defined as follows

$$W_{\beta,\gamma}(x) = \begin{cases}(q+1)(1-q^{-1/2}/\gamma) & \text{if } |x| = 0,\\ q+1-q^{1/2}(2^\beta+\gamma) & \text{if } |x| = 1,\\ q+1-q^{1/2}[(1+1/|x|)^\beta + (1-1/|x|)^\beta] & \text{if } |x|\geq 2.\end{cases}$$

Proof. It suffices to apply Theorem 4.2.1 with $u = u_{\beta,\gamma}$, defined as follows

$$u_{\beta,\gamma}(x) = \begin{cases} q^{-1/2|x|}|x|^\beta & \text{if } |x| \geq 1,\\ \gamma & \text{if } |x| = 0.\end{cases}$$

Indeed,

$$\frac{\Delta u(o)}{u(o)} = q+1-(q+1)\frac{q^{-1/2}}{\gamma} = (q+1)\left(1-\frac{q^{-1/2}}{\gamma}\right),$$

which is nonnegative if $\gamma \geq q^{-1/2}$.
For x such that $|x| = 1$

$$\begin{aligned} \frac{\Delta u(x)}{u(x)} &= q + 1 - q\frac{q^{-1}2^{\beta}}{q^{-1/2}} - \frac{\gamma}{q^{-1/2}} \\ &= q + 1 - q^{1/2}(2^{\beta} + \gamma) \end{aligned}$$

which is nonnegative if $\gamma \leq q^{1/2} + q^{-1/2} - 2^{\beta}$. The restriction $\beta \leq 1/2 \log_2 q$ comes out to make consistent $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^{\beta}$.
Finally, for x such that $|x| \geq 2$

$$\begin{aligned} \frac{\Delta u(x)}{u(x)} &= q + 1 - q\frac{q^{-1/2(|x|+1)}(|x|+1)^{\beta}}{q^{-1/2|x|}|x|^{\beta}} - \frac{q^{-1/2(|x|-1)}(|x|-1)^{\beta}}{q^{-1/2|x|}|x|^{\beta}} \\ &= q + 1 - q^{1/2}(1 + 1/|x|)^{\beta} - q^{1/2}(1 - 1/|x|)^{\beta} \geq 0, \end{aligned}$$

is equivalent to

$$q^{1/2} + q^{-1/2} \geq (1 + 1/|x|)^{\beta} + (1 - 1/|x|)^{\beta}.$$

If $\beta \leq 1$, the function $f : \mathbb{R}^+ \to \mathbb{R}$, such that $f(x) = x^{\beta}$ is concave. It follows that

$$f(1/2(1 + 1/|x|) + 1/2(1 - 1/|x|)) = f(1) \geq 1/2 f(1 + 1/|x|) + 1/2 f(1 - 1/|x|)$$

that is equivalent to

$$2 \geq (1 + 1/|x|)^{\beta} + (1 - 1/|x|)^{\beta}.$$

Then,

$$\frac{\Delta u(x)}{u(x)} = q + 1 - q^{1/2}[(1 + 1/|x|)^{\beta} + (1 - 1/|x|)^{\beta}] \geq q + 1 - 2q^{1/2} = \Lambda_q > 0.$$

If $\log_2 q^{1/2} \geq \beta > 1$, notice that the function $h : [2, +\infty) \to \mathbb{R}$ such that $h(x) = (1 + 1/x)^{\beta} + (1 - 1/x)^{\beta}$ is decreasing. Indeed,

$$h'(x) = -\frac{1}{x^2}\beta((1 + 1/x)^{\beta-1} - (1 - 1/x)^{\beta-1}) < 0.$$

In our computation, $|x| \geq 2$, then h reaches its maximum for $|x| = 2$. Thus it suffices to study

$$\left(\frac{3}{2}\right)^{\beta} + \left(\frac{1}{2}\right)^{\beta} \leq q^{1/2} + q^{-1/2}.$$

Notice that

$$\frac{d}{d\beta}\left[\left(\frac{3}{2}\right)^{\beta} + \left(\frac{1}{2}\right)^{\beta}\right] = 2^{-\beta}(3^{\beta} log(3/2) - log(2)) \geq 0$$

holds true for all $\beta > 1$. Hence it suffices to study

$$\left(\frac{3}{2}\right)^{\log_2 q^{1/2}} + \left(\frac{1}{2}\right)^{\log_2 q^{1/2}} \leq q^{1/2} + q^{-1/2}.$$

It is easy to check the validity of the last inequality because

$$\left(\frac{3}{2}\right)^{\log_2 q^{1/2}} \leq 2^{\log_2 q^{1/2}} = q^{1/2},$$
$$\left(\frac{1}{2}\right)^{\log_2 q^{1/2}} = 2^{\log_2 q^{-1/2}} = q^{-1/2}.$$

□

In the graph below we plot the optimal Hardy weight W_{opt} obtained by means of the Green function in Theorem 4.3.2 and $W_{\beta,\gamma} = W_{-1/2,\beta,\gamma}$ defined in Theorem 4.3.3.

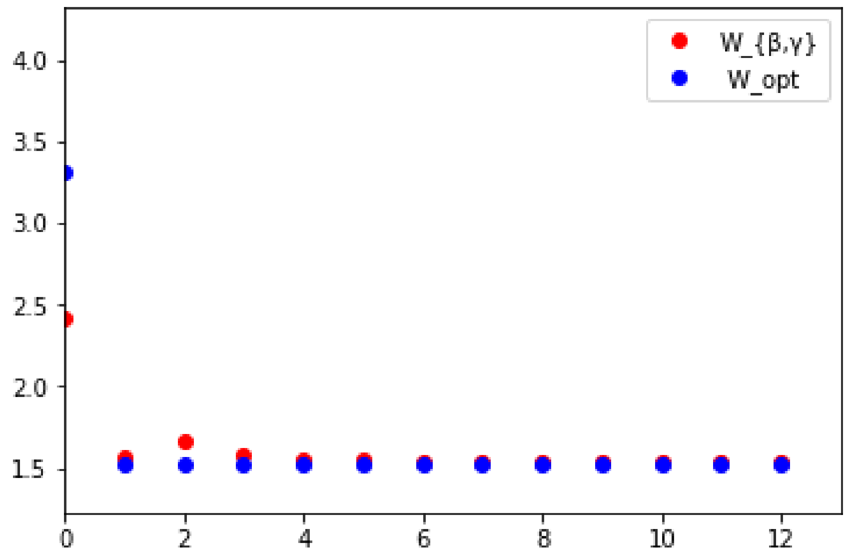

Figure 4.1. Plot of $W_{\beta,\gamma}$ and W_{opt} for $q = 5$, $\beta = 0.3$ and $\gamma = 3/4$.

Remark 4.3.2. *Note that the statement of Theorem 4.3.2 can be enriched by considering the family of radial functions*

$$u_{\alpha,\beta,\gamma}(x) = \begin{cases} q^{\alpha|x|}|x|^{\beta} & \text{if } |x| \geq 1, \\ \gamma & \text{if } |x| = 0, \end{cases}$$

with $\alpha \in \mathbb{R}$ and β and γ as in Theorem 4.3.3.
Indeed, a straightforward computation shows that for $|x| \geq 2$

$$W_{\alpha,\beta,\gamma}(x) = \frac{\Delta u_{\alpha,\beta,\gamma}(x)}{u_{\alpha,\beta,\gamma}(x)} = q + 1 - q^{\alpha+1}(1 + 1/|x|)^{\beta} - q^{-\alpha}(1 - 1/|x|)^{\beta}.$$

Nevertheless, there holds

$$W_{\alpha,\beta,\gamma}(x) = q + 1 - q^{1+\alpha} - q^{-\alpha} + o(1), \qquad \text{as } |x| \to +\infty$$

which is maximum for $\alpha = -1/2$. Indeed, let $t = q^\alpha$ and $g(t) = -qt - 1/t$. Computing the derivative we get

$$g'(t) = 0 \iff -q + 1/t^2 = 0 \iff t = q^{-1/2},$$
$$g''(q^{-1/2}) < 0.$$

Therefore, the choice $\alpha = -1/2$ turns out to be the best to get a weight as large as possible at ∞. It is also worth noting that for $\alpha = -1/2$

$$W_{-1/2,\beta,\gamma}(x) = \Lambda_q + o(1).$$

Remark 4.3.3. *Notice that if we choose $\beta = 0$ and $\gamma = 1$ in Theorem 4.3.3 we obtain*

$$W_{0,1}(x) = W_{opt}(x).$$

Hence we get the same inequality of Theorem 4.3.2.

Remark 4.3.4. *For $0 \le \beta \le 1/2 \log_2 q$ and $q^{-1/2} \le \gamma \le q^{1/2} + q^{-1/2} - 2^\beta$ consider the above defined weights $W_{\beta,\gamma}$. We have*

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{\beta(1-\beta)}{|x|^2} + o(\frac{1}{|x|^2}) \qquad as\ |x| \to \infty.$$

Now we want to maximize the behavior of these weights when $|x|$ tends to infinity; since

$$\max_\beta \beta(1-\beta) = 1/4,$$

which is reached for $\beta = 1/2$. For such a choice of α, β, when $|x| \to \infty$, we have

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{1}{4|x|^2} + o(1/|x|^2).$$

Remarks 4.3.2, 4.3.4 suggest that best choice of parameters in Theorem 4.3.3 to have a largest weight at infinity is $\alpha = -1/2$ and $\beta = 1/2$. This intuition is somehow confirmed by the statement below.

Theorem 4.3.4. *The following holds:*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}),$$

where

$$\overline{W_{opt}}(x) = \begin{cases} 0 & \text{if } |x| = 0,\\ (q+1) - \sqrt{2}q^{1/2} - 1 & \text{if } |x| = 1,\\ (q+1) - q^{1/2}\Big[(1+1/|x|)^{1/2} + (1-1/|x|)^{1/2}\Big] & \text{if } |x| \ge 2,\end{cases}$$

with $\overline{W_{opt}}$ optimal Hardy weight for the Hardy inequality associated with Δ.

Proof. Consider the Schrödinger operator $H = \Delta + Q$, with

$$Q(x) = \begin{cases} q(q+1) & \text{se } |x| = 0, \\ q^{1/2}(q+1) & \text{se } |x| = 1, \\ -\Lambda_q & \text{se } |x| \geq 2. \end{cases}$$

We firstly show that the positive functions

$$u = q^{-1/2|x|},$$
$$v = \begin{cases} q^{-1} & \text{if } |x| = 0, \\ |x| q^{-1/2|x|} & \text{otherwise,} \end{cases}$$

are H-superharmonic on $\mathbb{T}_{q+1}$ and H-harmonic outside of a finite set.
Indeed,

$$Hu(o) = (q+1)(1 - q^{-1/2}) + q(q+1) = (q+1)(q+1-q^{-1/2}),$$

which is clearly positive because $q \geq 2$.

$$\begin{aligned} Hu(|x| = 1) &= (q+1)q^{-1/2} - qq^{-1} - 1 + q^{-1/2}q^{1/2}(q+1) \\ &= (q+1)(q^{-1/2}+1) - 2 \geq 1 + 3\sqrt{2}. \end{aligned}$$

If $|x| \geq 2$, then

$$\begin{aligned} Hu(x) &= (q+1)q^{-1/2|x|} - qq^{-1/2(|x|+1)} - q^{-1/2(|x|-1)} - \Lambda_q q^{-1/2|x|} \\ &= q^{-1/2|x|}(q+1-2q^{1/2} - \Lambda_q) = 0. \end{aligned}$$

Moreover,

$$Hv(o) = (q+1)(q^{-1} - q^{-1/2}) + q(q+1)q^{-1} = (q+1)(1 + q^{-1} - q^{-1/2});$$

notice that $1 \geq q^{-1/2}$, thus $Hv(o)$ is positive.

$$\begin{aligned} Hv(|x| = 1) &= (q+1)q^{-1/2} - q2q^{-1} - q^{-1} + (q+1)q^{1/2}q^{-1/2} \\ &= (q+1)(q^{-1/2}+1) - 2 - q^{-1} = q - 1 + q^{1/2} + q^{-1/2} - q^{-1} \end{aligned}$$

is positive because $q - 1 - q^{-1} \geq 1/2$.
Finally, for $|x| \geq 2$ then

$$\begin{aligned} Hv(x) &= (q+1)|x|q^{-1/2|x|} - (|x|+1)qq^{-1/2(|x|+1)} - (|x|-1)q^{-1/2(|x|-1)} - \Lambda_q q^{-1/2|x|} \\ &= |x|q^{-1/2|x|}(q+1-2q^{1/2} - \Lambda_q) = 0. \end{aligned}$$

Then, in order to apply Theorem 4.2.2, we need to show that

$$u_0(x) := u(x)/v(x) = \begin{cases} q & \text{if } |x| = 0, \\ 1/|x| & \text{otherwise,} \end{cases}$$

is proper and

$$\sup_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \frac{u_0(x)}{u_0(y)} < \infty.$$

The function u_0 is proper because

$$\lim_{|x|\to\infty} u_0(|x|) = 0$$

and $u_0(|x|) > u_0(|x|+1) > 0$ for all $|x| \geq 1$, thus $u_0^{-1}(K)$ is finite for all compact set $K \subset (0,\infty)$. Now consider $x \sim y$,

$$\frac{u_0(x)}{u_0(y)} = \begin{cases} q & \text{if } |x| = 0, \\ 1/q & \text{if } |y| = 0 \text{ and } |x| = 1, \\ 1 + 1/|x| & \text{if } |y| = |x|+1 \text{ and } |x| \geq 1, \\ 1 - 1/|x| & \text{if } |y| = |x|-1 \text{ and } |x| \geq 2. \end{cases}$$

Thus

$$\sup_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \frac{u_0(x)}{u_0(y)} = q.$$

We deduce from Theorem 4.2.2 that the weight

$$\begin{aligned} \tilde{W}(x) := \frac{H[(uv)^{1/2}](x)}{(uv)^{1/2}(x)} &= \frac{\Delta(uv)^{1/2}}{(uv)^{1/2}} + Q(x) \\ &= \begin{cases} q(q+1) & \text{if } |x| = 0, \\ (q+1) - \sqrt{2}q^{1/2} - 1 + q^{1/2}(q+1) & \text{if } |x| = 1 \\ (q+1) - (1+1/|x|)^{1/2}q^{1/2} - q^{1/2}(1-1/|x|)^{1/2} - \Lambda_q & \text{if } |x| \geq 2 \end{cases}, \end{aligned}$$

is an optimal weight for H. Notice that the optimal Hardy inequality obtained considering the quadratic form h associated with H

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 + \sum_{x\in\mathbb{T}} Q(x)\varphi^2(x) \geq \sum_{x\in\mathbb{T}_{q+1}} \left(\frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} + Q(x)\right)\varphi^2(x)$$

is equivalent to the Hardy inequality associated to Δ

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)}\varphi^2(x) \qquad \forall\varphi \in C_0(\mathbb{T}_{q+1}).$$

Moreover,

$$\overline{W_{opt}}(x) := \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} = \begin{cases} 0 & \text{if } |x| = 0, \\ (q+1) - \sqrt{2}q^{1/2} - 1 & \text{if } |x| = 1, \\ (q+1) - (1+1/|x|)^{1/2}q^{1/2} - q^{1/2}(1-1/|x|)^{1/2} & \text{if } |x| \geq 2, \end{cases}$$

is nonnegative. The optimality of $\tilde{W}$ implies that it does not exist a nonnegative function f different from the identically zero function such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 - \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \geq \sum_{x\in\mathbb{T}_{q+1}} f(x)\varphi^2(x),$$

or, equivalently, $h_\Delta - \overline{W_{opt}}$ is critical.
It follows that

$$z(x) = (u(x)v(x))^{1/2}$$

is the ground state of $h_\Delta - \overline{W_{opt}}$. Notice that

$$\begin{aligned} \overline{W_{opt}}(x) &> W_{opt}(x) \qquad \text{if } |x| \geq 2, \\ z(x) &> G^{1/2}(x) \qquad \text{if } |x| \geq 2, \end{aligned}$$

with W_{opt} and $G^{1/2}$ such as in Theorem 4.3.2.
Then

$$\sum_{x\in\mathbb{T}_{q+1}} z^2(x)\overline{W_{opt}}(x) = +\infty$$

because of Remark 4.3.1.
Finally, by contradiction, suppose there exist $\overline{\lambda} > 0$ and $K \subset \mathbb{T}_{q+1}$ compact set such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x)-\varphi(y)\Big)^2 - \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \geq \overline{\lambda} \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \tag{4.3}$$

for all $\varphi \in C_0(\mathbb{T}_{q+1}\backslash K)$. Then, (4.3) holds true on $C_0(\mathbb{T}_{q+1}\backslash(K\cup B_2(o)))$ with $B_2(o) = \{x \in \mathbb{T}_{q+1}$ such that $|x| < 2\}$. Notice that $W_{opt}\varphi^2(x) \leq \overline{W_{opt}}(x)\varphi^2(x)$ for all $\varphi \in C_0(\mathbb{T}_{q+1} \setminus (K \cup B_2(o)))$.
It follows

$$\begin{aligned} &\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x)-\varphi(y)\Big)^2 - \sum_{x\in\mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x) \\ &\geq \frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x)-\varphi(y)\Big)^2 - \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \\ &\geq \overline{\lambda} \sum_{x\in\mathbb{T}_{q+1}} \overline{W_{opt}}(x)\varphi^2(x) \geq \overline{\lambda} \sum_{x\in\mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x) \end{aligned}$$

for all $\varphi \in C_0(\mathbb{T}_{q+1} \setminus (K \cup B_2(o)))$. This is a contradiction because W_{opt} is optimal.

□

4.3.3 Hardy inequalities on $\mathbb{T}_{P,D}$

In the same way we proceeded for the homogeneous tree now we apply Theorem 4.2.1 to $\Gamma = \mathbb{T}_{P,D}$.

Proposition 4.3.1. *Let $\Gamma = \mathbb{T}_{P,D}$ be the bi-regular tree of degree P, D and fix $o \in \Gamma$. Then, for all $\varphi \in C_0(\Gamma)$, it holds the following inequality*

$$\frac{1}{2}\sum_{\substack{x,y\in\Gamma\\ x\sim y}} \Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\Gamma} W(x)\varphi^2(x)$$

where

$$W(x)=\begin{cases} 1/2\Big((P-1)(1-F_P^{1/2})^2+(1-F_D^{-1/2})^2\Big) & \textit{if } |x|>0 \textit{ is even,}\\ 1/2\Big((D-1)(1-F_D^{1/2})^2+(1-F_P^{-1/2})^2\Big) & \textit{if } |x| \textit{ is odd,}\\ P(1-F_P^{1/2}) & \textit{if } x=o,\end{cases}$$

with $F_D = D/(PD-P)$ and $F_P = P/(PD-D)$ for any $P, D \geq 3$. Moreover, W is optimal.

Proof. Choose $u(x)=\sqrt{G(x,o)}$ with G the Green function on $\mathbb{T}_{P,D}$. Subsequently compute the ratio $\Delta u/u$. Notice that u is radial.
If $|x|>0$ is even,

$$\begin{aligned}\frac{\Delta u(|x|)}{u(|x|)} &= P-(P-1)\frac{u(|x|+1)}{u(|x|)}-\frac{u(|x|-1)}{u(|x|)}\\ &= P-(P-1)F_P^{1/2}-F_D^{-1/2}\\ &= \frac{1}{2}\Big(2P-2(P-1)F_P^{1/2}-2F_D^{-1/2}\Big)\\ &= \frac{1}{2}\Big(P-1+P+1-2(P-1)F_P^{1/2}-2F_D^{1/2}\Big)\\ &= \frac{1}{2}\Big((P-1)(1-2F_P^{1/2})+1+P-2F_D^{-1/2}\Big)\\ &= \frac{1}{2}\Big((P-1)(1-2F_P^{1/2}+F_P)-(P-1)F_P+1+P-2F_D^{-1/2}\Big)\\ &= \frac{1}{2}\Big((P-1)(1-2F_P^{1/2}+F_P)+1+F_D^{-1}-2F_D^{-1/2}\Big)\\ &= \frac{1}{2}\Big((P-1)(1-F_P^{1/2})^2+(1-F_D^{-1/2})^2\Big).\end{aligned}$$

If $|x|$ is odd,

$$\begin{aligned}\frac{\Delta u(|x|)}{u(|x|)} &= D-(D-1)\frac{u(|x|+1)}{u(|x|)}-\frac{u(|x|-1)}{u(|x|)}\\ &= D-(D-1)F_D^{1/2}-F_P^{-1/2}\\ &= \frac{1}{2}\Big(2D-2(D-1)F_D^{1/2}-2F_P^{-1/2}\Big)\\ &= \frac{1}{2}\Big(D-1+D+1-2(D-1)F_D^{1/2}-2F_P^{1/2}\Big)\\ &= \frac{1}{2}\Big((D-1)(1-2F_D^{1/2})+1+D-2F_P^{-1/2}\Big)\\ &= \frac{1}{2}\Big((D-1)(1-2F_D^{1/2}+F_D)-(D-1)F_D+1+D-2F_P^{-1/2}\Big)\\ &= \frac{1}{2}\Big((D-1)(1-2F_D^{1/2}+F_D)+1+F_P^{-1}-2F_P^{-1/2}\Big)\\ &= \frac{1}{2}\Big((D-1)(1-F_D^{1/2})^2+(1-F_P^{-1/2})^2\Big).\end{aligned}$$

If $x = o$,

$$\begin{aligned}\frac{\Delta u(|x|)}{u(|x|)} &= P - P\frac{u(1)}{u(0)} \\ &= P(1 - F_P^{1/2}).\end{aligned}$$

□

4.4 Improved Poincaré inequalities

In this section we present three examples of improved Poincaré inequality derived by means of Theorem 4.3.3, Theorem 4.3.2 and Theorem 4.3.4. We recall that the Poincaré inequality on $\mathbb{T}_{q+1}$ states

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \Lambda_q \sum_{x\in\mathbb{T}_{q+1}}\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}).$$

Notice that $\Lambda_q = (q^{1/2}-1)^2$ is the best constant by definition in the sense that the previous inequality cannot hold with a constant $\Lambda > \Lambda_q$.
For improved Poincaré inequality we mean an inequality of the form

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}}\Lambda_q\varphi^2(x) + \sum_{x\in\mathbb{T}_{q+1}}R(x)\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}),$$

for some $R \geq 0$ in $\mathbb{T}_{q+1}$.
The following improved Poincaré inequality is an immediate consequence of Theorem 4.3.2:

Corollary 4.4.1. *It holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}}\Lambda_q\varphi^2(x) + \sum_{x\in\mathbb{T}_{q+1}}R_{opt}\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}),$$

where $R_{opt}(x) = \begin{cases}(q-1)/q^{1/2} & \text{if } |x|=0,\\ 0 & \text{otherwise.}\end{cases}$

From Theorem 4.3.3 it follows a family of improved Poincaré inequality.

Theorem 4.4.1. *For all* $0 \leq \beta \leq \log_2(3/2 - 1/(2q))$ *and* $1/2 + 1/(2q) \leq \gamma \leq 2 - 2^\beta$, *it holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}}\Lambda_q\varphi^2(x) + \sum_{x\in\mathbb{T}_{q+1}}R_\beta\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}),$$

where $0 \leq R_\beta(x) = \begin{cases} q^{-1/2}(2/3q - 1/\gamma) & \text{if } |x|=0,\\ 2-2^\beta-\gamma & \text{if } |x|=1,\\ q^{1/2}\Big(2-(1+1/|x|)^\beta-(1-1/|x|)^\beta\Big) & \text{if } |x|\geq 2.\end{cases}$

Proof. Consider the family of positive radial functions $u_{\beta,\gamma}$ defined by

$$u_{\beta,\gamma}(x) = \begin{cases} q^{-1/2|x|}|x|^{\beta} & \text{if } |x| \geq 1, \\ \gamma & \text{if } |x| = 0, \end{cases}$$

with $\gamma \geq 1/2 + 1/(2q)$, $0 \leq \beta \leq log_2(3/2 - 1/(2q))$.
Then $\Delta u_{\beta,\gamma}/u_{\beta,\gamma}$ provides weights larger than Λ_q in $\mathbb{T}_{q+1}$. Indeed, computing

$$W_{\beta,\gamma}(x) = \frac{\Delta u_{\beta,\gamma}(x)}{u_{\beta,\gamma}(x)},$$

we get

$$W_{\beta,\gamma}(x) = W_{\beta,\gamma}(|x|) = \begin{cases} q + 1 - (q+1)q^{-1/2}/\gamma & \text{if } |x| = 0, \\ q + 1 - q^{1/2}(2^{\beta} + \gamma) & \text{if } |x| = 1, \\ q + 1 - q^{1/2}\Big((1 + 1/|x|)^{\beta} + (1 - 1/|x|)^{\beta}\Big) & \text{if } |x| \geq 2. \end{cases}$$

Next, it is easy to check that $W_{\beta}(0), W_{\beta}(1)$ are larger than Λ_q for our choices of parameters. In fact

$$q + 1 - (q+1)q^{-1/2}/\gamma \geq q + 1 - 2q^{1/2} \iff 1/2 + 1/(2q) \leq \gamma$$

and

$$q + 1 - q^{1/2}(2^{\beta} + \gamma) \geq q + 1 - 2q^{1/2} \iff 2^{\beta} \leq 2 - \gamma.$$

It follows $1/2 + 1/(2q) \leq \gamma \leq 2 - 2^{\beta}$ and $\beta \leq \log_2(3/2 - 1/(2q))$. Notice that for this choice of γ and β it follows $\beta \leq log_2(3/2) < 1$. In Theorem 4.3.3 we proved that for $0 \leq \beta < 1$ it holds $W_{\beta,\gamma}(x) \geq \Lambda_q$ for $|x| \geq 2$. □

In the next theorem we show an improved Poincaré inequality in the complement of the ball of radius 2 by means of Theorem 4.3.4.

Theorem 4.4.2. *The following holds:*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} \overline{R_{opt}}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1} \setminus B_2(o)), \tag{4.4}$$

where

$$\overline{R_{opt}}(x) = q^{1/2}\Big[2 - (1 + 1/|x|)^{1/2} - (1 - 1/|x|)^{1/2}\Big] \qquad \text{if } |x| \geq 2,$$

and $B_2(o) = \{x \in \mathbb{T}_{q+1}$ *such that* $|x| < 2\}$.
Moreover, the constant $q^{1/2}$ *is sharp in the sense that the inequality cannot hold with a reminder term* $C\Big[2 - (1 + 1/|x|)^{1/2} - (1 - 1/|x|)^{1/2}\Big]$ *with* $C > q^{1/2}$.

Proof. We know from Theorem 4.3.4 that the optimal weight $\overline{W_{opt}}(x)$ is larger than Λ_q for $|x| \geq 2$. Then we can define

$$\overline{R_{opt}}(x) = \overline{W_{opt}}(x) - \Lambda_q \qquad \forall x \in \mathbb{T}_{q+1} \setminus B_2(o).$$

and it follows (4.4). The sharpness of $q^{1/2}$ is consequence of the optimality of $\tilde{W}$ for H where $\tilde{W}, H$ are chosen such as in the proof of Theorem 4.3.4. □

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