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ON A CENTRALITY MAXIMIZATION GAME

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Contents

Introduction

Whether we are aware of it or not, we are surrounded by networks through which we interact and whose structure is modified by our actions or decisions. For example, our friendships and relationships are part of a bigger network in which people influence each others' opinions while the roads we travel every day are part of a wider traffic network. The web-pages we consult form part of the World Wide Web (WWW) while our social media profiles are immersed in a broader content-sharing environment. Moreover, by meeting a new person, by adding a new hyperlink on our website, by commenting or liking a post on someone's Facebook page, we actively form and modify the shape of such networks.
Hence, in the last decades, an increasing interest has been paid to measures of the importance of certain node positions in a network. Such measures are called *centrality measures* and can describe, for instance, how much visible a web page is on the internet or how influential a person's opinion is in a community. Definitions of centrality under this heading include the ones of P. Bonacich [1], L. Katz [2] and the PageRank centrality, first defined by S. Brin and L. Page [3].
In the light of the above considerations, strategies to increase a node's centrality are of non-negligible importance.
This work intends to examine the global effects on the topological structure of a network when each node chooses to direct its out-links in order to maximize its PageRank centrality. We cast the problem into a Centrality Maximization Game, a natural game theoretic setting where players are the nodes, actions are the configurations of the out-links and utilities are the corresponding centralities. Such an object of study can be ascribed to the category of Network Formation Dynamics, which includes work as the ones from Erdős-Rényi [4], Watts and Strogatz [5] and Barabási-Albert [6], and more precisely to the class of Network Formation Games, in the wake of Jackson and Wolinsky [7] and Jackson and Watts [8] researches.

In this thesis we have considered a special instance of the problem in which each node has the same number m of out-links. In particular, we present rigorous results for the case when $m = 1$ and $m = 2$.
For the case $m = 1$, we exhibit that the best strategy for a node with at least one in-link is to point to an arbitrarily chosen node within its in-links. Moreover, we are capable of giving a complete characterization of Nash equilibria (strict or not) through Theorem 5 and Corollary 6. Furthermore, in Theorem 7, we prove that

the game is order potential, in this way insuring that the best response dynamics always converges to the set of Nash equilibria. For the case $m = 2$, Theorem 9 proves that the nodes towards which a generic node s should direct its out links have to be searched among the ones with distance smaller or equal to 2 from s. This result expresses the fact that the best strategy for every node s to maximize its global PageRank centrality, is to act locally, in an in-neighborhood of radius 2 of s. In addition to Theorem 9, Theorem 10 shows that strict Nash equilibria for such a game are all and only the undirected graphs.

In the following chapters results are presented by increasing the complexity of the model studied. The first chapter provides the reader with the theoretical background and notation needed to address the following dissertation. It specially stresses the various interpretations that can be attributed to certain measures of centrality, to better understand the applications in which it is used.

Chapter 2 gives some Game Theory background and the theoretical formulation of the Centrality Maximization Game.

Chapter 3 presents the first results on the Centrality Maximization Game under the constraint of 1 out-link for every node. It firstly focuses on the best response actions, it then describes the structure of strict and non-strict Nash equilibria and it counts them depending on the number of nodes n. It then identifies an ordinal potential function and, hence, it proves that the best response dynamic converges.

Chapter 4 can be intended as a first attempt to generalize some results. Under the constraint that every node owns exactly two out-links both the structure of strict and non-strict Nash is analyzed. Furthermore, considerations on the best response dynamics are drown out of some Matlab routines.

We may emphasize that the results obtained in Chapter 4 are often easily generalisable in the case in which every node has exactly m out-links. In the conclusions we therefore present some ideas on how to extend the findings to generic Centrality Maximization Games.

Chapter 1

Background

In our daily life, even without realizing it, we are constantly immersed in networks and we interact through them: the social network consisting of the people we meet everyday, the traffic network we use to get to work, the World Wide Web (WWW) we get information from. Even though all these systems may seem very different from each other, they can all be effectively modeled by the concept of graph, that we are shortly going to introduce. Prior to that, let us stress that, once a graph is defined as a mathematical object, the mathematical properties we will describe can all be brought back to each of the context listed above with different meaning. For instance, the notion of centrality can be interpreted as a measure of visibility of a web page when referring to the internet network, or a measure of influence of the opinion of a given agent when brought in a social network context.

1.1 Graph Theory Background

In order to describe the structure of a network we can rely on the notion of *graph*. A graph $\mathcal{G}$ is defined by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where

- $\mathcal{V}$ is the set of nodes representing the units that constitute the network (e.g. web pages, social media accounts, people).
- $\mathcal{E}$ is a set of links described as ordered pairs (i, j), where $i, j \in \mathcal{V}$. A link (i, j) represents the existence of a relation between i and j. Many interpretations can be attributed to the existence of a link between two nodes. For the seek of our work we will mainly focus on two on them: when referring to the WWW, a link (i, j) can be interpreted as page i directing an hyperlink to page j while, in the context of social networks, the link (i, j) stands for the possibility of agent i to access the opinion of agent j and to get influenced by it.
- W is a matrix that associate to each link (i, j) a weight W_{ij}. The link's weight can be interpreted as a measure of the strength of the connection between two

nodes. In the specific case of social network, the weight of a link can be an index of the impact of one agent's opinion over another one.

We now introduce notions that will be frequently used in the following chapters:

- *unweighted graphs* are graphs where $W_{ij} \in \{0,1\}$ for all nodes $i, j \in \mathcal{V}$. In this case the structure of the graph can be univocally deduced from the set $\mathcal{E}$. When the graph is unweighted we call matrix W the *adjacency matrix* of $\mathcal{G}$.
- *undirected graphs* are graphs where $W = W'$. This implies that, for every link (i, j), there also exists a link with reversed direction (j, i), and they both have the same weight $W_{ij} = W_{ji}$.
- *simple graphs* are undirected, unweighted graphs where the weight matrix W has zero diagonal, i.e. $\mathcal{G}$ contains no self-loops.
- *complete graphs* are simple graphs in which $\forall i, j \in \mathcal{V}, i \neq j$, $(i, j) \in \mathcal{E}$.

We also introduce some definitions and notations that will be used in the next chapters:

- The *out-* and *in-neighborhood* of a node $i \in \mathcal{V}$ are defined, respectively, as

$$N_i = \{j \in \mathcal{V} \,|(i, j) \in \mathcal{E}\}, \qquad N_i^- = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}. \tag{1.1}$$

  We will refer to elements in N_i and N_i^- as *out-neighbors* and *in-neighbors*.

- The *out-degree* and *in-degree* of a node i are defined, respectively, as

$$w_i = \sum_{j \in \mathcal{V}} W_{ij}, \qquad w_i^- = \sum_{j \in \mathcal{V}} W_{ji}.$$

  Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, we use the compact notation

$$w = W\mathbb{1},$$

  where $\mathbb{1} \in \mathbb{R}^n$ is a vector with all entries equal to one.
  Note that, when the graph is unweighted, then

$$w_i = |N_i| \quad w_i^- = |N_i^-|.$$

- A node i is *balanced* if $w_i = w_i^-$. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is balanced if all its node are so.
- A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is m-*regular* if all its nodes have the same degree, i.e. if $w = w^- = m\mathbb{1}$. Observe that a regular graph is always balanced but not necessarily undirected. On the other hand, undirected graphs are always balanced but not necessarily regular.

- A *walk* from node i to node j in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a finite sequence of nodes $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ such that $\gamma_0 = i$, $\gamma_l = j$, and $(\gamma_{h-1}, \gamma_h) \in \mathcal{E}$ for all $h = 1, \ldots, l$. We call l the *length* of the walk.

- A node j is said to be *reachable* from node i if there exists a walk from i to j.

- We call *path* a walk $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ such that $\gamma_h \neq \gamma_k$ for all $0 \leq h < k \leq l$, except for possibly $\gamma_0 = \gamma_l$. In other words, a path is a walk that does not pass through a previously visited node except possibly for ending in its start node.

- A path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ of length $l \geq 3$ that starts and ends in the same node $\gamma_0 = \gamma_l$ is called a *cycle*.

- $\mathcal{G}$ is *strongly connected* if, given any two nodes i and j, we have that i is reachable from j.

In the following chapters we will mainly focus on unweighted graphs. Moreover, we will often ask the out-weights w_i to be equal for any $i \in \mathcal{V}$. It then proves convenient to introduce a *normalized weight matrix* P defined as follows:

$$P = D^{-1}W, \qquad D = \mathrm{diag}(w).$$

Let us stress the main features of matrix P:

- P is *non-negative*, i.e. all its entries are non-negative;

- $P\mathbb{1} = \mathbb{1}$, i.e., all rows of P sum up to 1. Non-negative square matrices satisfying this property are referred to as *stochastic* matrices.

Notice that, given a stochastic matrix P, we can always think of it as the normalised weight matrix of a graph. It is sufficient to consider $\mathcal{G}_P = (\mathcal{V}, \mathcal{E}, P)$ where $\mathcal{E} = \{(i, j) : P_{ij} > 0\}$. We shall refer to such $\mathcal{G}_P$ as the *graph associated to the stochastic matrix P*. Sometimes we use the following terminology: a stochastic matrix P is referred to as *irreducible* if $\mathcal{G}_P$ is strongly connected.

We now recall some general results obtained by *Perron* and *Frobenius* on non-negative matrices whose proof can be found in [**?**], Ch. 2, Th. 1.1. This result will be very useful to define the concept of invariant distribution on a graph.

Proposition 1. Let W be a non-negative matrix. Then there exists $\lambda_W \geq 0$ and nonnegative vectors $x \neq 0$, $y \neq 0$ such that

- $Wx = \lambda_W x$, $W'y = \lambda_W y$;

- every eigenvalue μ of W is such that $|\mu| \leq \lambda_W$.

We call λ_W the *dominant eigenvalue* of W. Let us now consider a stochastic matrix P and the graph $\mathcal{G}_P$ associated to it. Recalling that P is non-negative, proposition 1 can be extended as follows:

Proposition 2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P)$ be a graph, and let $P = D^{-1}W$ be its normalized weight matrix. Then,

1. $\lambda_P = 1$;
2. there exists a nonnegative vector π such that $\mathbb{1}'\pi = 1$ and $P'\pi = \pi$;
3. if $\mathcal{G}$ is strongly connected, then $\lambda_P = 1$ is geometrically and algebraically simple and there exists $y > 0$ such that $P'y = y$.

A Proof of this result can be found in [9]. We call π the *invariant distribution* of the graph $\mathcal{G}$. The following section will exploit two of the most common interpretations associated to such a vector.

1.2 Interpretations of the centrality π

The invariant distribution of a graph can be seen as a measure of the importance of each node's position in a graph, a measure of each node's centrality. In fact we would like a measure π_i representing the centrality of a node in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ to

- grow as the centrality of the in-neighbors $\pi_j \in N_i$ grows, i.e., to satisfy

$$\pi_i \propto \sum_{j \in N_i} W_{ij}\pi_j = \sum_{j \in \mathcal{V}} W_{ij}\pi_j,$$

- weight the contribution of any in-neighbors $j \in \mathcal{V}$ by their out-degree, hence

$$\pi_i \propto \sum_{j \in N_i} \frac{W_{ij}}{w_j}\pi_j = \sum_{j \in \mathcal{V}} \frac{W_{ij}}{w_j}\pi_j = \sum_{j \in \mathcal{V}} P_{ij}\pi_j. \tag{1.2}$$

Writing equation (1.2) in matrix form, we ask a measure of centrality to satisfy

$$\pi = cP'\pi. \tag{1.3}$$

By choosing $c = 1$, we obtain

$$\pi = P'\pi. \tag{1.4}$$

Again, if $\mathcal{G}$ is strongly connected and $\pi'\mathbb{1} = 1$, then we know from Proposition 2 that π is unique. We call π the Bonacich centrality [1] of $\mathcal{G}$ and note that it corresponds to the leading left eigenvector of the normalized weight matrix P.[1] The Bonacich centrality therefore corresponds to the invariant distribution over graph $\mathcal{G}$. Let us have a look in detail at the interpretation of the Boancich centrality in two specific contexts: social networks and the WWW.

[1]Sometimes the Bonacich centrality is defined as the leading left eigenvector of the weight matrix W. To avoid any confusion we will refer to it as the vector that satisfies (1.4)

1.2.1 The vector π as a measure of influence in Average Opinion Dynamics

If we interpret graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P)$ as a social system, every node $j \in \mathcal{V}$ represents an agent of the population $\mathcal{V}$, a link (i, j) indicates that agent i is influenced by node j and P_{ij} weights the influence of node j over node i. Assume that every agent i has an opinion $x_i(t) \in \mathbb{R}$ at a discrete time-step $t = 0, 1, \dots$ that is updated at time-step $t + 1$ as follows:

$$x_i(t+1) = \sum_{j \in \mathcal{V}} P_{ij} x_j(t), \qquad i \in \mathcal{V}.$$

The above equation means that every node i updates its state at time $t+1$ with an average of the current opinions $x_j(t)$ of its in-neighbors $j \in N_i$. We can rewrite the update of the dynamics as

$$x(t+1) = Px(t), \quad t = 0, 1, \dots \tag{1.5}$$

where $x(t) = (x_i(t))_{i \in \mathcal{V}}$ is the vector of agents' opinions at time t. The dynamics described by (1.5) is called *linear averaging dynamic* on $\mathcal{G}_P$ and was firstly introduced by DeGroot in [10]. If $\mathcal{G}_P$ is a strongly connected graph, we know from Proposition 2 that $\lambda_P = 1$ has algebraic and geometric multiplicity equal to one. As the equilibria vectors of the dynamics should satisfy

$$x = Px,$$

then, if $\mathcal{G}_P$ is strongly connected, the only equilibria are constant vectors $\alpha\mathbb{1}$. This vectors are called *consensus vectors* as all the agents in the population end up in the same state α. Let us now investigate how α depends on an initial opinion vector $x(0)$. We observe that $\pi' x(t+1) = \pi' P x(t) = \pi' x(t)$. Therefore, starting with an initial opinion vector $x(0)$ we obtain that

$$\pi' x(t) = \pi' x(0) \quad t = 0, 1, \dots$$

Taking the limit as t grows large, in case of a strongly connected graph, we obtain that, when the dynamic converges, the only equilibria are consensus vectors $\alpha\mathbb{1}$ where

$$\alpha = \pi' x(0).$$

Without focusing on the conditions that grant the dynamics to converge, we remark that in a linear averaging dynamics, if we reach an equilibrium state, such a state will be a weighted average of the initial opinions. The weight attributed to the initial opinion of agent i will be π_i, i.e., its centrality in the network. In other words, the centrality π_i is a measure of the **influence** that the initial opinion of node i will have in the final consensus.

1.2.2 The vector π as a measure of visibility over the Internet

Centrality measures, as we said repeatedly, can be used to order web-pages by importance in an online search. Therefore, in this case centrality becomes a synonym of **visibility** over the network. Bonacich centrality, unfortunately, in such a context suffer from the limitation that nodes can increase the centrality of a given node arbitrarily, by adding self-loop on this node of very large weight. In the limit, as the weight of a self-loop grows large, the ratio between the centrality of this node and the total centrality and the total centrality of all other nodes grows unbounded, even without loosing connectivity. Even if self-loops are not allowed, one can easily take two nodes and add an undirected link between them of larger and larger weight: in the limit as the weight of this undirected link grows large, the ratio between the sum of the two nodes' centralities and the centralities of all other nodes grows to infinity. This drawback is overcome by modifying the notion of centrality by allowing nodes to get some centrality, independently of their in-neighbors. Consider a graph $G = (\mathcal{V}, \mathcal{E}, R)$. If we choose a parameter $\beta \in (0,1)$ and a nonnegative vector μ to be thought of some intrinsic centrality, we can define the *PageRank* centrality as

$$\pi = \beta R' \pi + (1-\beta)\mu. \tag{1.6}$$

Note that, for every $0 < \beta < 1$ the dominant eigenvalue of $\beta R'$ is smaller than 1, so that the matrix $(I - (1-\beta)R')$ is invertible. This implies the PageRank centrality vector is well defined, unique, and can be represented as

$$\pi = (I - (1-\beta)R')^{-1}(1-\beta)\mu.$$

Such a measure of centrality was first introduced by Brin and Page in brinpage to measure the relative importance of webpages in the WWW. Vector μ is a measure of intrinsic centrality: it can both be a uniform vector $\mu = \frac{1}{n}\mathbb{1}$ or a vector that expresses the pertinence of the web-pages to context-sensitive searches. Let us now consider the case of $\mu = \frac{1}{n}\mathbb{1}$, i.e. the case in which every web-page has the same intrinsic centrality, for example a set of pages with the same pertinence to a certain topic. We can rewrite

$$\pi = \beta R' \pi + (1-\beta)\frac{1}{n}\mathbb{1} = \left(\beta R' + (1-\beta)Q'\right)\pi,$$

where $Q \in \mathbb{R}^{n \times n}$, $n = |\mathcal{V}|$, and

$$Q = \frac{1}{n}\mathbb{1}\mathbb{1}'.$$

Such a matrix Q can be thought of as a matrix associated with a complete unweighted graph with self loops. Notice that the PageRank centrality vector over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, R)$ corresponds to the Bonacich centrality vector over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, R)$, where

$$P = (1-\beta)Q + \beta R$$

1.3 Expected Hitting Times

In this section we will present another way to evaluate the invariant distribution π of a matrix P by introducing a new quantity: the *expected hitting times* over a Markov chain associated to P. Let us interpret a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P)$ as a network over which a particle moves. We denote the state of the particle by $X(t)$ for $t = 1, 2, \ldots$. At every time-step the state of the particle will be a node in the graph $\mathcal{G}$, i.e., $X(t)$ belongs to $\mathcal{V}$. At any time-step the particle will change its position choosing between all the nodes of the graph with probability equal to P_{ij} if $X(t) = i$. Notice that, by moving to a node j

$$\begin{aligned} &P(X(t+1) = j|\, X(0) = j, X(1) = j, \ldots, X(t) = j) \\ &= P(X(t+1)) = j|\, X(t)) = j) = P_{ij}, \end{aligned}$$

that is the probability of assuming a certain position at time $t+1$, depends only on the current position and not on the way the particle got there. Such a stochastic process is called a *Markov chain* X(t). One may ask how long it takes for $X(t)$ to move from a state i to another state s, or how long it takes to go back at i. As $X(t)$ is a random variable, both this quantities will be random variables. We define the *hitting time* and the *return time* on a given node $s \in \mathcal{V}$ as:

$$T_s := \inf\{t \geq 0 :\ X(t) = s\} \quad \text{and} \quad T_s^+ := \inf\{t \geq 1 :\ X(t) = s\}$$

with the convention that the infimum of an empty set is $+\infty$.
When considering a deterministic initial condition $X(0) = i$ for some $i \in \mathcal{V}$ we can use the following notation for the probability induced by the process $X(t)$.

$$\mathbb{P}_i(\,\cdot\,) = \mathbb{P}_i(\,\cdot\,|X(0) = i).$$

Similarly, we use the notation

$$\mathbb{E}_i[\,\cdot\,] = \mathbb{E}_i[\,\cdot\,|X(0) = i]$$

for the corresponding conditional expected value. It can be proved that

Proposition 3. If X(t) is a Markov chain with finite state space $\mathcal{V}$, transition probability matrix P and an initial state $i \in \mathcal{V}$, then the hitting time $T_s,\ s \in \mathcal{V}$ satisfies

$$\mathbb{P}_i(T_s < +\infty) = 1 \qquad \mathbb{E}_i[T_s] < +\infty$$

if and only if s is reachable from i.

For a proof of Proposition 3 see [11].

Remark 1. Let $X(t)$ be a Markov chain. In the following we will refer to the expected hitting times $\mathbb{E}_i[T_s]$ from a node i to s with the notation:

$$\tau_i^s = \mathbb{E}_i[T_s]$$

We can now state the following Proposition:

Proposition 4. Let $X(t)$ be a Markov chain with finite state space $\mathcal{V}$, transition probability matrix P and initial state $i \in \mathcal{V}$. Let $s \in \mathcal{V}$ be a node reachable from every state $i \in \mathcal{V}$. Then, the expected hitting times τ_i^s, $i \in \mathcal{V}$ are the only family of finite values satisfying the relations:

$$\begin{cases} \tau_i^s = 0 & \text{if } i = s \\ \tau_i^s = 1 + \sum_{j \in \mathcal{V}} P_{ij} \tau_j^s & \text{if } i \neq s, \end{cases} \tag{1.7}$$

From the previous result, the following Corollary can be drawn.

Corollary 1. *Let $X(t)$ be a Markov chain with finite state space $\mathcal{V}$, transition probability matrix P. Then, for any state $i \in \mathcal{V}$, the expected return times satisfy*

$$\mathbb{E}_i[T_i^+] = 1 + \sum_{j \in \mathcal{V}} P_{ij} \tau_j^s. \tag{1.8}$$

As said in the beginning of this section, our aim is to find a relation between the centrality measure π_s and the expected hitting times τ_i^s, $i \in \mathcal{V}$. In order to do so, the following result, presented by M.Kac in [12] proves to be particularly useful.

Proposition 5. (Kac's Formula)
Let $X(t)$ be a Markov chain over a finite state space $\mathcal{V}$ with irreducible transition probability matrix P, associated invariant distribution π and initial state $s \in \mathcal{V}$. Then,

$$\mathbb{E}_s[T_s^+] = \frac{1}{\pi_s}. \tag{1.9}$$

For a proof of Proposition 1.7 and 5 see [11]. By using equation (1.8) in equation (1.9) we obtain the following.

Corollary 2. *Let $X(t)$ be a Markov chain over a finite state space $\mathcal{V}$ with irreducible transition probability matrix P, associated invariant distribution π and initial state $s \in \mathcal{V}$. Then,*

$$\pi_s = \frac{1}{1 + \sum_{j \in \mathcal{V}} P_{ij} \tau_j^s}. \tag{1.10}$$

Corollary 1.10 plays a fundamental role in this dissertation as it shows that, if a node s wants to maximize its own centrality, it has to direct its out-links to those nodes with smallest hitting times τ_j^s to s.

Chapter 2

Centrality Maximization Game

This chapter aims at providing the reader with some Game Theory background and it introduces the main topic of this thesis. In many networks that can be modeled by a graph one may rise the question of how an agent can maximize its own centrality. For example, for various reasons there could be a desire to affect the opinion distribution of a network, for instance to favor one opinion or to reduce polarization in a network. Alternatively a web page may want to maximize its visibility in the internet to get more viewers and maybe increase its profitability. This could potentially be achieved by modifying the network in many different ways, still the most interesting from a node's point of view would be the one only depending on its own choices. How can a web-page place its hyperlinks to maximize its visibility? How should an agent in a population manage its relations to maximize its influence on the total opinion distribution? Moreover, what kind of structures and shapes does the network assume when each agent (i.e. each node) tries to maximize its centrality? To answer these questions, Section 2.1 firstly provides the notion of strategic form game and of best response dynamics. Section 2.2 expresses the theoretical formulation of the Centrality Maximization Game and then highlights a characterization of the best response function.

2.1 Game Theory Background

To be able to formulate the problem this work will mainly focus on, it is necessary to first introduce the notion of *strategic game*. A strategic form game Γ is univocally defined by the triple $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$, where

- $\mathcal{V}$ constitutes the set of *players*, i.e., the set of agents taking part of the game.
- $\mathcal{A}_i$ is the set of *actions* of player i, i.e., the set of his possible choices. The assignment of an action to each player is described by a vector $x \in \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$ which is called *configuration* or *action profile*. The *configuration space* will be

$$\mathcal{X} = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n.$$

- $u_i : \mathcal{X} \to \mathbb{R}$ is the *utility function* of node i. It attributes to every action x_i of player i a certain value, given the actions $x_j \in \mathcal{A}_j$ of other players. By calling

$$x_{-i} = x|_{\mathcal{V}\setminus\{i\}}, \tag{2.1}$$

we can then refer to the utility function of player i with the following notation

$$u_i(x_i, x_{-i}) = u_i(x).$$

Notice that, in a strategic form game every player i is assumed to be rational, i.e. his aim consists of maximizing his utility function. Moreover we assume that every player knows the state of all other players $j \in \mathcal{V}$. As the best action every player can choose to maximize his utility depends on other's players state, we can define a *best response function*

$$\mathcal{B}_i(x_{-i}) = \arg\max_{x_i \in \mathcal{A}} u_i(x_i, x_{-i})$$

as the function that associates to every action configuration $x \in \mathcal{X}$ the best rational action, or actions, for player i. We call the elements of the set $\mathcal{B}_i(x_{-i})$ *best response actions*. We can now define

Definition 1. (Pure strategy Nash equilibrium). A *(pure strategy) Nash equilibrium* for a game $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$ is an action configuration $x^* \in \mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \qquad \forall i \in \mathcal{V}. \tag{2.2}$$

A Nash equilibrium is therefore a configuration x^* where every player's action belongs to the set of best responses, i.e., there is *no incentive* to *unilaterally* deviate from his current action, as the utility it is getting with the current action is the best possible, given the actions of other players. Note that, even if there is no incentive to change action for any player, it doesn't mean that for every player there is a unique action x_i^* with utility $u_i(x_i^*, x_{-i}^*)$, given x_{-i}^*. Moreover, being in a Nash equilibrium doesn't mean that every player has the greatest possible utility in general, but that he has the greatest possible utility given the actions of the other nodes. A subset of Nash equilibria is the set of

Definition 2. (Pure strategy strict Nash equilibrium). A *(pure strategy) strict Nash equilibrium* for a game $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$ is an action configuration $x^* \in \mathcal{X}$ such that

- x^* is a Nash equilibrium;
- $u_i(x_i^*, x_{-i}^*) > u_i(x_i^*, x_{-i}^*) \quad \forall i \in \mathcal{V}$.

In other words, a strict Nash equilibrium is a Nash equilibrium in which every node i, given x_{-i}^*, has got a unique best response action x_i^*. In addition to the analysis of best response functions and Nash equilibria, in the following chapters we will also focus on the question of whether a discrete dynamics in which at every time-step t a node i, randomly chosen, plays one of his best response actions converges to specific configurations. In particular, we will study the convergence of a *best response dynamics*, which is defined as follows:

Definition 3. Consider a strategic-form game $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$. The continuous-time asynchronous *best response* dynamics is a Markov chain $X(t)$ with state space $\mathcal{X}$, where every player $i \in \mathcal{V}$ is given an independent rate-1 Poisson clock. When his clock ticks at time t, player i updates his action to some y_i chosen from the action set $\mathcal{A}_i$ with conditional probability distribution that is uniform over the best response set

$$\mathcal{B}_i\left(X_{-i}(t)\right) = \underset{x_i\in\mathcal{A}_\rangle}{\arg\max}\{u_i(x_i, X_{-i}(t))\}.$$

In particular, when the best response is unique, player i updates his action according to such best response action. To better understand the conditions under which such a dynamic converges to the set of Nash equilibria we can introduce the definition of *ordinal potential* and *generalized ordinal potential.*

Definition 4. Ordinal potential Let $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$ be a strategic game. A function $C : \mathcal{X} \to \mathbb{R}$ is called ordinal potential function if, for all $i \in \mathcal{V}$, for all $a_{-i} \in A_{-i}$, and all $a_i, a_i' \in A_i$, it holds that

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \Longleftrightarrow C(a_i, a_{-i}) > C(a_i', a_{-i}).$$

Definition 5. Generalized ordinal potential Let $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$ be a strategic game. A function $C : \mathcal{X} \to \mathbb{R}$ is called generalized ordinal potential function if, for all $i \in \mathcal{V}$, for all $a_{-i} \in A_{-i}$, and all $a_i, a_i' \in A_i$,

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \Longrightarrow C(a_i, a_{-i}) > C(a_i', a_{-i}).$$

Notice that, if C is an ordinal potential, then it is also a generalized ordinal potential.

Theorem 3. *The best response dynamics of a finite game with a generalized order potential converges to the set of Nash equilibria, which is not empty.*

For a proof of this theorem see [13].

2.2 Theoretical Formulation of the Centrality Maximization Game

Let us now assume to have a complete graph in which every node has got a link to all the nodes belonging to the graph, including itself, with a weight $\frac{1-\beta}{n}$. Let us also assume that every node can decide how to uniformly spread a remaining weight

β within m different nodes aiming at maximizing its own centrality. We refer to such a game as to the *Centrality Maximization Game* and we define it through the following theoretical formulation.

We name *Centrality Maximization Game* the triple $CMG(\mathcal{V}, \beta, m) = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$ where

- the set of actions $\mathcal{A}_i$ is the set of cardinality m sub-sets of $\mathcal{V}\backslash\{i\}$, where $1 \leq m \leq n$ and $n = |\mathcal{V}|$.
  Notice that every action configuration $x \in \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$ defines a matrix $R(x) \in \mathbb{R}^{n\times n}$ with entries

$$R_{ij}(x) = \begin{cases} \frac{1}{m} & \text{if } j \in x_i, \\ 0 & \text{if } j \notin x_i. \end{cases} \tag{2.3}$$

  We name the graph associated to matrix $R(x)$

$$\mathcal{G}(x) = \mathcal{G}_{R(X)}.$$

  Interpreting the set of players $\mathcal{V}$ as the set of nodes of graph $\mathcal{G}(x)$, action x_i of node i can therefore be thought of as the set of m nodes towards which node i is directing its m out-links in $\mathcal{G}(x)$.

- the utility function u_i, given an action configuration x is

$$u_i(x_i, x_{-i}) = (I - (1-\beta)Q - \beta R(x))^{-1}\mathbb{1}. \tag{2.4}$$

  where

$$Q = \frac{1}{n}\mathbb{1}'\mathbb{1}.$$

  and $\beta \in (0, 1)$.
  Such a utility function corresponds to the Bonacich centrality over a graph $\mathcal{G}_{P(x)}$ associated to matrix

$$P(x) = (1-\beta)Q + \beta R(x). \tag{2.5}$$

  Hence, it is the Bonacich centrality over a graph obtained by overlapping graph $\mathcal{G}(x)$ with a complete graph, associated to matrix Q, that represents an underlying unalterable structure for the network.

Remark 2. In the following Chapters, we may refer to an action configuration x in $CMG(\mathcal{V}, \beta, m)$ through matrix $R(x)$ or graph $G(x)$. We will, for instance, say that $\mathcal{G}(x)$ represents, under certain conditions, a Nash equilibrium for the game. Moreover, using the notation introduced in (2.1) we can refer to $R(x)$ and $G(x)$ through the following

$$R(x_i, x_{-i}) = R(x)$$

$$G(x_i, x_{-i}) = G(x)$$

Remark 3. Notice that $\mathcal{G}(x)$ is a graph related only to the nodes' choices and that it does not take account of the underlying complete structure set out by matrix Q.

By recalling Corollary 2, the best response function in a CMG($\mathcal{V}, \beta, m$) satisfies the following Proposition.

Proposition 6. Consider a CMG($\mathcal{V}, \beta, m$) and an action configuration x. The set of best response actions $\mathcal{B}_i(x_{-i})$ of a node i satisfies

$$\mathcal{B}_i(x_{-i}) = \arg\min_{x_i \in \mathcal{A}_i} \sum_j R_{ij}(x_i, x_{-i}) \tau_j^s(x_i, x_{-i}) \tag{2.6}$$

where $R(x_i, x_{-i})$ is the matrix defined in (2.3) and $\tau_j^s(x_i, x_{-i})$ are the expected hitting times over a Markov chain governed by $P(x) = (1-\beta)Q + \beta R(x_i, x_{-i})$.

In other terms, the problem of finding a best response action of node s in a CMG($\mathcal{V}, \beta, m$) boils down to detecting which are the m nodes that minimize the hitting time on s.

Remark 4. In the following, given an action configuration x, we will refer to the *expected hitting times associated to* x as to the expected hitting times $\tau_i^s(x)$ defined over a Markov chain governed by $P(x) = (1-\beta)Q + \beta R(x)$. We may stress that such expected hitting times are defined through matrix $P(x)$ and not through matrix $R(x)$. In fact, every player wants to maximize its own Bonacich centrality over $G_{P(x)}$ and not over $\mathcal{G}(x)$.

Recalling the different interpretations we attributed to Bonacich centrality in Section 1.2, two different reading keys can be used to parse the Centrality Maximization Game described above. In can be interpreted as:

- *a maximization of visibility on the Internet game.*
  Given a $CMG(\mathcal{V}, \beta, m)$, let us interpret the set $\mathcal{V}$ as a collection of webpages with the same bearing on a specific subject matter. Action x_i of webpage i can be interpret as the set of webpages page i has a direct hyperlink to. Given an action configuration x, let us interpret $\mathcal{G}(x)$ as the internet network restricted to the set of webpages $\mathcal{V}$. From the definition of $CMG(\mathcal{V}, \beta, m)$ we know that the aim of every webpage consist of maximizing its own Bonacich centrality over $\mathcal{G}_{P(x)}$ where $P(x)$ is defined as in (2.5). As stated in Section 1.2, when webpages have the same relevance to a context sensitive search, maximizing Bonacich centrality over $\mathcal{G}_{P(x)}$ is equivalent to maximize the PageRank centrality over $\mathcal{G}(x)$. Therefore we can interpret $CMG(\mathcal{V}, \beta, m)$ as a game in which webpages belonging to $\mathcal{V}$ seek to maximize their PageRank centrality in $\mathcal{G}(x)$.

- *a maximization of influence in a community game*
  Given a $CMG(\mathcal{V}, \beta, m)$, let us interpret the set of nodes $\mathcal{V}$ as a community

in which every member can access and be influenced by the opinion of all the other agents (for instance a community in which everybody knows each other) but gets more influenced by certain members of the community (for instance the people he hangs around most or those he gives most credit to). Again, as the aim of a player in $CMG(\mathcal{V}, \beta, m)$, given an action configuration x, is to maximize his Bonacich centrality over $\mathcal{G}(x)$, $CMG(\mathcal{V}, \beta, m)$ can be interpreted as a game in which people belonging to a community $\mathcal{V}$ seek to maximize their influence in a final consensus of a DeGroot model (1.5) by choosing, for example, which people to spend more time with.

Chapter 3

One-link Game

The current section aims at studying the Centrality Maximization Game CMG($\mathcal{V}$, β, m), presented in Chapter 2, in the particular case of $m = 1$, i.e. when every node can choose towards where to direct only one out-link. We study the structure of Nash equilibria, both strict and not, and we make some considerations about the convergence of the best response dynamics.

3.1 Nash equilibria

In order to study Nash equilibria, the best response of a node s has to be analyzed. Let us consider a node s and describe its best response depending on the number of links pointing at it.

Proposition 7. Let us consider a CMG($\mathcal{V}, \beta, 1$) as defined in Section 2.2. Consider an action configuration x and let action x_s belong to the set of best responses $\mathcal{B}_s(x_{-i})$. If node j belong to x_s, it holds that:

- $j \in N_s^-(x)$ if $N_s^-(x) \neq \emptyset$,
- j is any node in $\mathcal{V}$, if $N_s^-(x) = \emptyset$,

where, as defined in (1.1), $N_s^-(x)$ is the set of in-neighbor of s.

Proof. Let us analyze the best response action in the three possible scenarios:

1. *j is the only node pointing at s in $\mathcal{G}(x)$* , i.e. $N_s^-(x) = \{j\}$. Let us call k any other node of the graph without any direct link to s in $\mathcal{G}(x)$ and $k+1$ the node towards which k is directing its out-link in $\mathcal{G}(x)$. Consider the expected hitting times $\tau_i^s(x)$ associated to x and, for the seek of simplicity, let us call

refer to them as $\tau_i^s = \tau_i^s(x)$. Subtracting the return times

$$\tau_j^s = 1 + \frac{1-\beta}{n} \sum_{i \in \mathcal{V}} \tau_i^s \tag{3.1}$$

$$\tau_k^s = 1 + \frac{1-\beta}{n} \sum_{i \in \mathcal{V}} \tau_i^s + \beta \tau_{k+1}^s \tag{3.2}$$

we get

$$\tau_j^s - \tau_k^s = -\beta \tau_{k+1}^s$$

As τ_{k+1}^s is a strictly positive quantity, it holds that

$$\tau_j^s < \tau_k^s.$$

In other terms, we proved that a node pointing at s has always got a smaller return time than a node which does not have a direct link to s. Hence, according to Proposition 6, node j is the best choice for node s.

2. *two or more nodes pointing at s in $\mathcal{G}(x)$*, i.e. $|N_s^-(x)| \geq 2$. As seen in point 1, a node pointing at s has always got a smaller hitting time to s than a node with no direct link to s. Let us now consider two nodes, j and k, both pointing at s. Which one should s direct its link towards?

$$\tau_j^s = 1 + \frac{1-\beta}{n} \sum_{i \in \mathcal{V}} \tau_i^s \tag{3.3}$$

$$\tau_k^s = 1 + \frac{1-\beta}{n} \sum_{i \in \mathcal{V}} \tau_i^s \tag{3.4}$$

hence

$$\tau_j^s = \tau_k^s.$$

A node s can therefore randomly choose a node within the ones that point at itself to maximize its centrality.

3. *no node pointing at s in $\mathcal{G}(x)$*, i.e., $N_s^-(x) = \emptyset$. Let us consider the expected hitting time from a node i to node s. At every discrete time t the probability to hit node s is $\left(1 - \frac{1-\beta}{n}\right)^{t-1} \frac{1-\beta}{n}$. Therefore τ_k^s can be written as:

$$\tau_k^s = \frac{1-\beta}{n} \sum_{t=0}^{\infty} \left(1 - \frac{1-\beta}{n}\right)^{t-1} t$$

and we can see that it does not depend on k, i.e., the expected hitting time to s is the same from every node. Hence, player s can choose any node $j \in \mathcal{V}$.

□

Notice that:

Corollary 4. *The best response action of node s is unique if and only if $|N_s^-(x)| = 1$.*

Now that the best responses of a certain node s are known in every possible scenario, Nash equilibria can be investigated. In order to do so let us first formulate the following definition.

Definition 6. Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and name N_i and N_i^- respectively the sets of out-neighbors and in-neighbors of node i as defined in (1.1). We say $\mathcal{G}$ is a graph of *type* C_k^1 when

- $|N_i| = 1, \ \ \forall i \in \mathcal{V}$,
- there are k cycles of length 2,
- if node i does not belong to a cycle of length 2, then $|N_i^-| = 0$.

Using the previous definition we can characterize the Nash equilibria as follows:

Theorem 5. *Consider a CMG($\mathcal{V}, \beta, 1$) and let $n = |\mathcal{V}|$. When $n \geq 2$*

1. *the set of Nash equilibria is non empty.*
2. *x^* is a Nash equilibrium if and only if $\mathcal{G}(x^*)$ is of type C_k^1 for some $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$.*

Proof. Let us start by considering that, if x^* is a Nash equilibrium for CMG($\mathcal{V}, \beta, 1$), in $\mathcal{G}(x)$ there is at least one node i hit by an arrow. Therefore, in a Nash equilibrium there will be at least one cycle of length 2, because node i will point back at one of its in-links. All nodes i with at least one in link $j \in N_i^-$ have to point back at it. Hence there will be $k \geq 1$ couples. Nodes not in a couple will be nodes j such that $N_j^- = \emptyset$. They can choose to direct their out-link to any node in a couple. A generic non-strict Nash looks like Figure 3.2. □

A direct consequence of Theorem 5 is the following

Corollary 6. *Consider a CMG($\mathcal{V}, \beta, 1$) and let $n = |\mathcal{V}|$. If n is even it holds that*

1. *the set of strict Nash equilibria is non empty.*
2. *x^* is a strict Nash equilibria if and only if $\mathcal{G}(x^*)$ is of type $C_{n/2}^1$.*

If n is odd, there are no strict Nash equilibria.

Proof. According to the definition of strict Nash equilibrium and by Corollary 4, in order to have a unique best response, every node s needs to have exactly one in-link j. Node s best response is then $x_s = j$, i.e. it needs to point right back at j. In other words we need $\mathcal{G}(x^*)$ to be a collection of couples as shown in Figure 3.1. Note that this is possible only when $n = |\mathcal{V}|$ is even. □

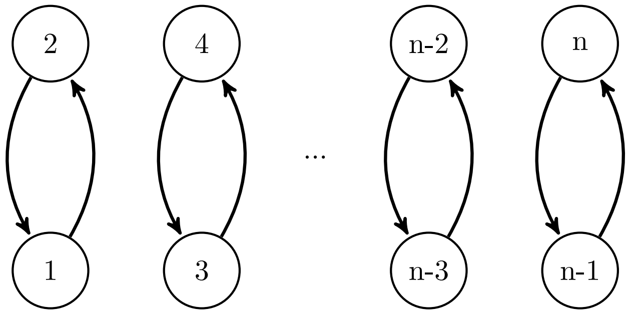


Figure 3.1: Generic strict Nash

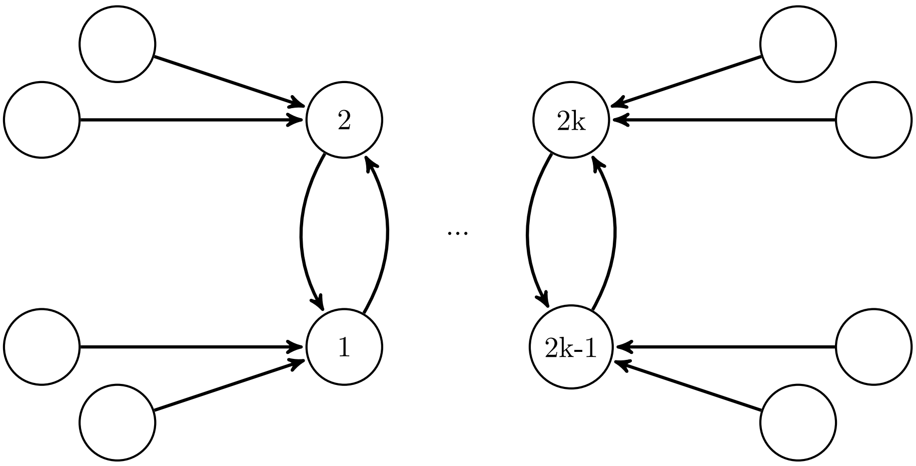


Figure 3.2: Generic non-strict Nash

Let us now analyze more in detail the possible structures of Nash equilibria and their number by varying n. Naming E_n the number of Nash equilibria and S_n the number of strict Nash equilibria, the following relations can be obtained.

- $\boldsymbol{n = 2}$. The solution is trivial as there is just one possible configuration.

$$E_2 = 1 \quad S_2 = 1.$$

- $\boldsymbol{n = 3}$. According to the previous reasoning, in a Nash equilibrium there will be at least one direct link. The third node can therefore arbitrarily chose where to place its out-link. Figure 3.3 shows the structure of such configurations. For every couple with a direct link, i.e., $\binom{3}{2}$ possibilities, we have 2 Nash equilibria. As the third node has no unique choice, Nash equilibria are not strict.

$$E_3 = 2\binom{3}{2} \quad S_3 = 0.$$

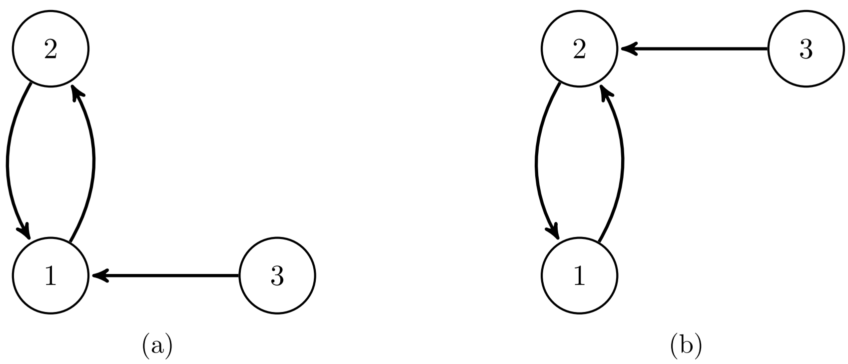


Figure 3.3: Example of non strict Nash equilibria when $n = 3$

- $\boldsymbol{n = 4}$. We know that we have at least one couple. If we have exactly one couple the other two nodes can either choose both the same node, or direct them self to two different nodes, as Figure 3.4 shows. This gives $4\binom{4}{2}$ non strict Nash, as the non-coupled links can arbitrarily choose within the set of nodes. We can also have the cases with two couples, i.e. other $\frac{1}{2}\binom{4}{2}$ equilibria, which this time are strict as no node can change its action. Therefore:

$$E_4 = 4\binom{4}{2} + \frac{1}{2}\binom{4}{2} \quad S_4 = \frac{1}{2}\binom{4}{2}.$$

- ...

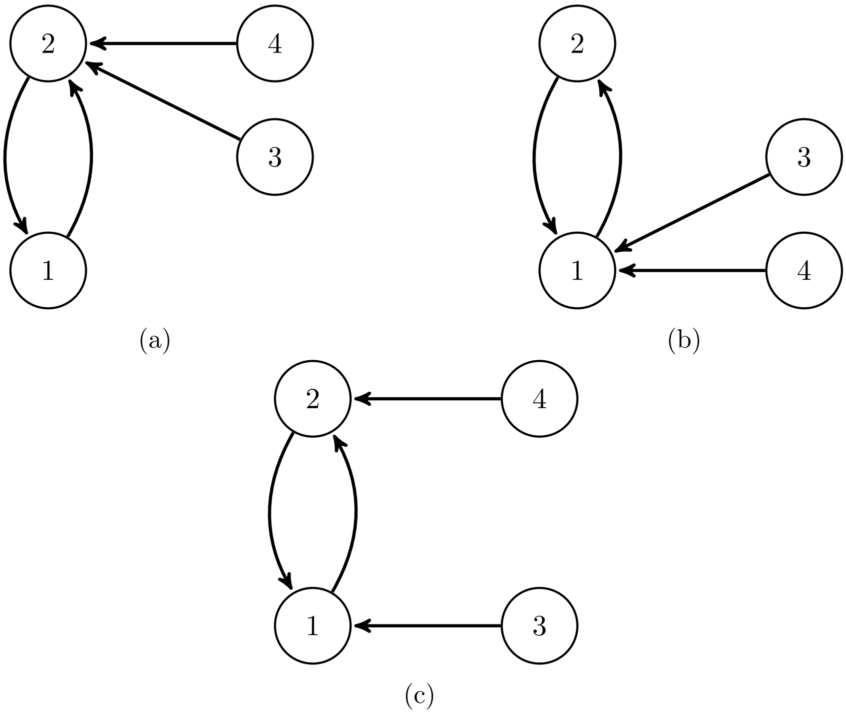


Figure 3.4: Example of non strict Nash equilibria when $n = 4$

- $\boldsymbol{n = 2r, \quad r} \in \mathbb{Z}$. When we have an even number of nodes the number of strict Nash equilibria becomes the multinomial coefficient, without repetitions:

$$S_{2r} = \frac{1}{r!}\binom{n}{2,2,\ldots,2} = \frac{1}{r!}\frac{n!}{\prod_{i=1}^{r} 2!} = \frac{1}{r!}\frac{n!}{2^r} \quad r \in \mathbb{Z}.$$

  To investigate the non-strict Nash equilibria we proceed by considering separately the number of combinations depending on the number of couples k. If

  - $\mathbf{k = 1}$, we have

  $$\binom{n}{2}\left(2^{(n-2)}\right).$$

    Nash equilibria. In fact, there are $\binom{n}{2}$ possible way to extract a couple in n nodes. Any node which is not in the couple can direct its out-link to any node in the couple. Hence, for each couple, we have $\left(2^{(n-2)}\right)$ Nash.

  - $\mathbf{k = 2}$ the number of Nash corresponds to

  $$\frac{1}{2}\binom{n}{2,2,n-4}\left(4^{(n-4)}\right).$$

as $\frac{1}{2}\binom{n}{2,2,n-4}$ represents the number of distinct ways in which a set of n object can be divided in 3 subsets of which one has dimension $n-4$ and the others have dimension 2. Again, every node not in a couple, i.e., $n-4$ nodes, can choose within 4 nodes (the one in couples). We end up with $4^{(n-4)}$ Nash per couple.

– **k**. For any other k, the number of Nash becomes:

$$\frac{1}{k}\binom{n}{2,2,\ldots,n-2k}\left((2k)^{(n-2k)}\right).$$

As k in this case goes up to $n/2$, i.e. r, the total number of Nash is:

$$E_{2r}=\sum_{k=1}^{r}\frac{1}{k}\binom{n}{v_k}\left((2k)^{(n-2k)}\right)+\frac{1}{r!}\frac{n!}{2^r},$$

where v_k is a vector of length $k+1$ with k-first components equal to 2 and the $(k+1)$ component equal to $n-2k$, i.e.,

$$v_k=(\underbrace{2,2,\ldots,2}_{k\text{ times}},n-2k).$$

- $\boldsymbol{n=2r+1,}\quad \boldsymbol{r}\in\mathbb{Z}$. As previously said, when n is odd, there are no strict equilibria.

$$S_{2r+1}=0\quad r\in\mathbb{Z}.$$

The number of non strict equilibria can be evaluated analogously to the previous case, when n was even. Therefore we get that

$$E_{2r+1}=\sum_{k=1}^{r}\frac{1}{r}\binom{n}{v_k}\left((2k)^{(n-2k)}\right),$$

where

$$v_k=(\underbrace{2,2,\ldots,2}_{k\text{ times}},n-2k).$$

To better summarize the results obtained up here, the following proposition can be formulated:

Proposition 8. Let us consider a CMG$(\mathcal{V},\beta,1)$. By calling $n=|\mathcal{V}|$ and E_n the number of Nash equilibria of such a game and S_n the number of strict Nash equilibria, when $n\geq 2$ it holds that:

- if n is even, i.e., $\exists r\in\mathbb{Z}$ such that $n=2r$:

$$E_{2r}=\sum_{k=1}^{r}\frac{1}{k}\binom{n}{v_k}\left((2k)^{(n-2c)}\right)+\frac{1}{r!}\frac{n!}{2^r}\qquad S_{2r}=\frac{1}{r!}\frac{n!}{2^r}$$

- if n is odd, i.e. $r \in \mathbb{Z}$ such that $n = 2r + 1$:

$$E_{2r+1} = \sum_{k=1}^{r} \frac{1}{k} \binom{n}{v_k} \left((2k)^{(n-2k)} \right) \qquad S_{2r} = 0$$

where $v \in \mathbb{R}^{k+1}$ and

$$v_k = (\underbrace{2, 2, \ldots, 2}_{k \text{ times}}, n - 2k).$$

Notice that, by Theorem 5, it can be drawn that:

Proposition 9. No Nash equilibria for CMG$(\mathcal{V}, \beta, 1)$ depends on β.

3.2 Best response dynamics

Let now focus on a best response dynamics over a CMG$(\mathcal{V}, \beta, 1)$. In order to study the convergence of the dynamics a MATLAB routine has been implemented. Results are shown in Figure 3.5. Let us call x^t the action configuration at a certain time t. Each curve represents a different initial configuration of the graph, i.e. a different x^0. In all the simulations $\beta = 0.3$ and $n = 20$. To make the convergence visible, a time block of 20 time steps has been defined, and the number of changes between x^t and x^{t+1} has been aggregated in every time block. As shown by the

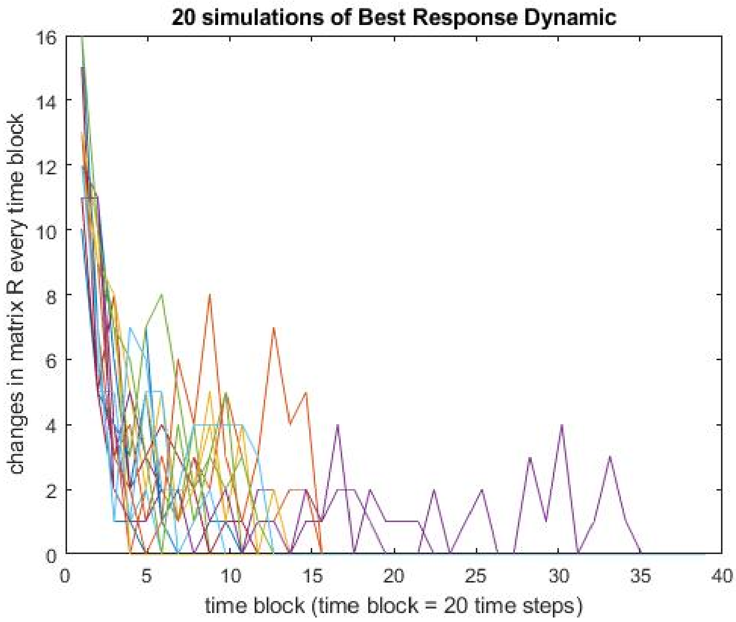


Figure 3.5

picture, in all the 20 simulations the number of changes every 20 time steps tends to decrease along time and the dynamics always converges. Figure 3.6 compares two structures of $\mathcal{G}(x^0)$, i.e. two initial configurations, with the respective structures of $\mathcal{G}(x^{800})$, i.e. the the action configurations after 800 time steps. It points out that both the simulations converge to strict Nash equilibria. Hence the game seems to

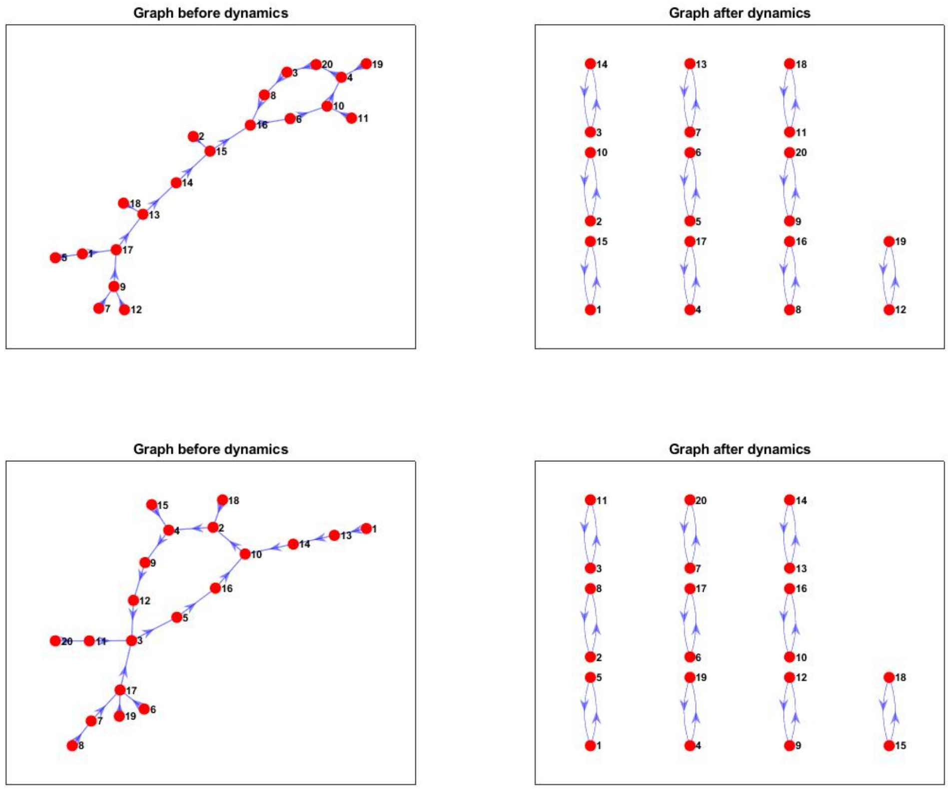


Figure 3.6: Example of convergence in CMG($\mathcal{V}, \beta, 1$)

behave as a potential game as its best response dynamics seems to always converge to a Nash equilibria. It can be proved that there exist an ordinal potential function for CMG($\mathcal{V}, \beta, 1$), which grants the dynamics to converge.

Theorem 7. *The best response dynamics of CMG($\mathcal{V}, \beta, 1$) always converges to a Nash equilibrium.*

Proof. Let us call $C : x \to \mathbb{N}$ the function counting the number of cycles of length 2 in graph $\mathcal{G}(x)$. Recalling Proposition 7, we observe that function C satisfies Definition 4 of ordinal potential as

$$u_i(x_i, x_{-i}) > u_i(x_i', x_{-i}) \Leftrightarrow C(x_i, x_{-i}) > C(x_i', x_{-i}).$$

In fact, $u_i(x_i, x_{-i})$ is strictly grater than $u_i(x_i', x_{-i})$ if and only if node j such that $x_i = \{j\}$, belongs to N_i^- in $\mathcal{G}(x)$ while node j' such that $x_i' = \{j'\}$, does not belong to N_i^-. As $j \in N_i^-$, action x_i creates a new couple, while x_i' does not. Hence, function $C(x)$ is an ordinal potential for the game.
As a consequence from Theorem 3, the best response dynamics over CMG$(\mathcal{V}, \beta, 1)$ always converges to a Nash equilibrium. □

As a final consideration, which will be useful when trying to generalise the conclusions drawn in the $m = 1$ case, we can notice that:

Proposition 10. In CMG$(\mathcal{V}, \beta, 1)$, both best response actions and Nash equilibria do not depend on β.

Chapter 4

Two-link Game

Let us now consider the CMG($\mathcal{V}, \beta, 2$). As in the previous chapter, at first, we will try to understand what is the structure of Nash equilibria and whether they are strict or not and then focus on the convergence of the best response dynamics.

4.1 Characterization of strict Nash equilibria

To be able to study the set of best response actions for a node s in CMG($\mathcal{V}, \beta, 2$), given a configuration x, an upper bound for the hitting times $\tau_i^s(x)$ associated to x proves to be of practical use.

Proposition 11. Consider a CMG($\mathcal{V}, \beta, m$). Let $\mathcal{X}$ be the space of action configurations for the game and $x \in \mathcal{X}$. By calling $\tau_i^s(x)$ the expected hitting times associated to action x, it holds that

$$\tau_i^s(x) \leq \frac{n}{1-\beta} \quad \forall i, s \in \mathcal{V} \;\; i \neq s, \; \forall x \in \mathcal{X},$$

where $n = |\mathcal{V}|$.

Proof. Let us consider a matrix $A \in \mathbb{R}^{n \times n}$ with diagonal entries equal to $\beta + \frac{1-\beta}{n}$ and all the other entries equal to $\frac{1-\beta}{n}$, i.e.

$$A_{ij} = \begin{cases} \beta + \frac{1-\beta}{n}, & \text{if } i = j, \\ \frac{1-\beta}{n}, & \text{if } i \neq j. \end{cases} \tag{4.1}$$

By solving system (1.7), the expected hitting times $\hat{\tau}_i^s$ over a Markov chain governed by A are

$$\hat{\tau}_i^s = \frac{n}{1-\beta} \quad \forall i, s \in \mathcal{V}, \quad i \neq s.$$

We now need to show that the expected hitting times in $\mathcal{G}(x)$ are smaller or equal to the expected hitting times in $\mathcal{G}_A$. Consider the Markov chain described by matrix A. At every time step t, from any initial node, we have a probability to get to node s

exactly equal to $\frac{1-\beta}{n}$. On the other hand, if we consider the Markov chain described by matrix P, at every time step, from any initial node, we have a probability to get to node s greater or equal to $\frac{1-\beta}{n}$ as the link to s has a weight equal to $\frac{1-\beta}{n}$ or $\frac{1-\beta}{n} + \frac{\beta}{m}$. Hence

$$\tau_i^s(x) \leq \hat{\tau}_i^s = \frac{n}{1-\beta}.$$

□

From Proposition 11, two trivial but useful results, collected in the following Corollary, can be derived.

Corollary 8. *Consider a CMG($\mathcal{V}, \beta, 2$) and let x be an action configuration for the game. Let $\tau_i^s(x)$ be the expected hitting times from i to s associated to x and $R(x)$ the transition matrix associated to x. It holds that:*

1. $R_{is}(x) = 0, \forall i \in \mathcal{V} \Longleftrightarrow \tau_i^s = \frac{n}{1-\beta}, \ \ \forall i \in \mathcal{V}, i \neq s$;
2. *if* $\exists\, i \in \mathcal{V}$ *such that* $\tau_i^s = \frac{n}{1-\beta}$, *then* $\tau_j^s = \frac{n}{1-\beta}, \ \ \forall j \in \mathcal{V}, j \neq s$.

Proof.

1. ($\Leftarrow$)
   Let us assume, by contradiction, that there exists a node j such that $R_{js}(x) \neq 0$. Let us call k the node different form s in the out-neighborhood $N_j(x)$ of j in $\mathcal{G}(x)$. Then the expected hitting time form j to s can be written as

$$\tau_j^s(x) = 1 + \frac{1-\beta}{n} \sum_{i \in \mathcal{V}} \tau_i^s(x) + \frac{\beta}{2} \tau_k^s(x).$$

As $\tau_i^s(x) = \frac{n}{1-\beta}, \ \ \forall i \neq s, i \in \mathcal{V}$, the above equation can be rewritten as:

$$\frac{n}{1-\beta} = 1 + n - 1 + \frac{\beta}{2} \frac{n}{1-\beta} \quad \Rightarrow \left(1 - \frac{\beta}{2}\right) = 1 - \beta,$$

which is impossible. Hence we proved the left implication of Corollary 8, item 1.

($\Rightarrow$)
Let us consider again matrix A, as in (4.1). Notice that the expected hitting times $\hat{\tau}_i^s$ from i to s in a Markov chain governed by A is

$$\hat{\tau}_i^s = \frac{1-\beta}{n} \sum_{t=1}^{\infty} \left(1 - \frac{1-\beta}{n}\right)^{t-1} t = \frac{n}{1-\beta}.$$

If $R_{is}(x) = 0\ \ \forall i \in \mathcal{V}$, it means that at every time step t we have a probability $\frac{1-\beta}{n}$ to get to node s while a probability of $\left(1 - \frac{1-\beta}{n}\right)$ not to. Therefore, also τ_i^s can be expressed as

$$\tau_i^s(x) = \frac{1-\beta}{n} \sum_{t=1}^{\infty} \left(1 - \frac{1-\beta}{n}\right)^{t-1} t = \frac{n}{1-\beta} \quad \forall i \in \mathcal{V},\ i \neq s.$$

2. Assume $\exists\, i \in \mathcal{V}$ such that

$$\tau_i^s = \frac{n}{1-\beta}. \tag{4.2}$$

Calling h, k the nodes towards which i points to, we can rewrite

$$\tau_i^s(x) = 1 + \frac{1-\beta}{n} \sum_{j \in \mathcal{V}} \tau_j^s(x) + \frac{\beta}{2}(\tau_k^s(x) + \tau_h^s(x)).$$

By contradiction, let us assume there exist a node j such that $\tau_j^s(x) < \frac{n}{1-\beta}$ then we obtain

$$\tau_i^s(x) < 1 + n - 1 + \beta \frac{n}{1-\beta}$$

$$\tau_i^s(x) < (1+\beta)\frac{n}{1-\beta} \Rightarrow \tau_i^s(x) < \frac{n}{1-\beta}$$

which contradicts 4.2. Hence, $\tau_j^s = \frac{n}{1-\beta},\ \ \forall j \in \mathcal{V}, j \neq s$.

□

We now present two results which allows to restrict the search for a best response action for a node s into a subset of $\mathcal{V}$. Given a CMG$(\mathcal{V}, \beta, 2)$ and a state x, we name $\text{dist}_x(i,j)$ the distance within i and j in $\mathcal{G}(x)$ and

$$N_j^{-d}(x) = \{i \in \mathcal{V} \mid \text{dist}_x(i,j) \leq d\}.$$

the set of nodes with distance from j smaller or equal to d. In the following we will refer to $N_j^{-d}(x)$ as to the *in-neighborhood of radius d of s.*

Proposition 12. Consider a CMG$(\mathcal{V}, \beta, 2)$ and an action configuration x. Assume $x_s \in \mathcal{B}_s(x_{-i})$, then it holds that:

1. if $|N_s^{-2}(x)| = 0, \quad x_s = \Big\{\{i,j\} \mid i,j \in \mathcal{V}\backslash\{s\}\Big\}$
2. if $|N_s^{-2}(x)| = 1, \quad x_s = \Big\{\{i,j\} \mid i \in N_s^{-2}(x), j \in \mathcal{V}\backslash\{s,i\}\Big\}$

Proposition 12 it's an obvious consequence of Corollary 8.

Theorem 9. *Consider a CMG($\mathcal{V}, \beta, 2$) and an action configuration x. Consider a node s and assume $|N_s^{-2}(x)| \geq 2$. If $x_s \in \mathcal{B}_s(x_{-i})$, then $\forall j \in x_s$ it holds that*

$$dist_x(j,s) \leq 2.$$

Moreover, if $x_s \in \mathcal{B}_s(x_{-i})$, and there exist a node $j \in x_s$ such that $dist_x(j,s) = 2$, by calling $\gamma = (j,i,s)$ the shortest path from j to s in $\mathcal{G}(x)$, it holds that $i \in x_s$.

Proof. For the seek of simplicity in notation, given the action configuration x, let us refer to the expected hitting times $\tau_i^s(x)$ as τ_i^s. Let us call nodes $s, 1, 2, \ldots, n-1 \in \mathcal{V}$ in such a way that

$$\tau_s^s = 0 < \tau_1^s \leq \tau_2^s \leq \ldots \leq \tau_{n-1}^s.$$

Let us assume that node 1 has no direct link in $\mathcal{G}(x)$ to s (i.e. its out-neighbors $x, y \neq s$). Then

$$\tau_1^s = 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_x^s + \tau_y^s) \tag{4.3}$$

$$\geq 1 + \frac{1-\beta}{n}(n-1)\tau_1^s + \beta\tau_1^s \tag{4.4}$$

, which implies that

$$\frac{1-\beta}{n}\tau_1^s \geq 1 \Rightarrow \tau_1^s \geq \frac{n}{1-\beta}. \tag{4.5}$$

But by Proposition 11 we know that $\tau_1^s \leq \frac{n}{1-\beta}$, therefore $\tau_1^s = \frac{n}{1-\beta}$. From Corollary 8 we get that, $\forall j \neq s, j \in \mathcal{V}$, $R_{js}(x) = 0$, which contradicts the hypothesis, as it means that $\forall j \neq s, j \in \mathcal{V}$, $\text{dist}_x(j,s) = +\infty$. Then node 1 has to have a direct link to s.
Let us assume that 2 has no direct link in $\mathcal{G}(x)$ to s or to 1, i.e. by calling a and b the out-neighbors of 2:

$$\tau_2^s = 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_a^s + \tau_b^s)$$

$$\geq 1 + \frac{1-\beta}{n}(n-2)\tau_2^s + \frac{1-\beta}{n}\tau_1^s + \beta\tau_2^s,$$

which implies:

$$2\tau_2^s \geq \frac{n}{1-\beta} + \tau_1^s. \tag{4.6}$$

From the hypothesis, there exists another node k, different from 1, with $\text{dist}_x(k,s) \leq 2$. Let us consider the two different cases:

- if $\text{dist}_x(k,s) = 1$, s is an out-neighbor of k. By calling h the out-neighbor of k different from s we can write the system:

$$\begin{cases} \tau_k^s &= 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_h^s), \\ \tau_2^s &= 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_a^s + \tau_b^s). \end{cases} \tag{4.7}$$

By subtracting the two equations we get

$$\frac{2}{\beta}(\tau_k^s - \tau_2^s) = \tau_h^s - (\tau_a^s + \tau_b^s).$$

As $\tau_k^s \geq \tau_2^s$, necessarily

$$\tau_h^s \geq \tau_a^s + \tau_b^s \geq 2\tau_2^s. \tag{4.8}$$

Substituting (4.6) in (4.10) we obtain

$$\tau_h^s \geq \frac{n}{1-\beta} + \tau_1^s.$$

which, by Proposition 11, is impossible as τ_1^s is a positive quantity.

- if $\mathrm{dist}_x(k,s) = 2$, 1 is an out-neighbor of k as in the previous item we proved that there is no other node with distance from s equal to 1 but node 1. By calling h the out-neighbor of k different from 1 we can write the system:

$$\begin{cases} \tau_k^s &= 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_h^s + \tau_1^s), \\ \tau_2^s &= 1 + \frac{1-\beta}{n}\sum_{i\in\mathcal{V}}\tau_i^s + \frac{\beta}{2}(\tau_a^s + \tau_b^s). \end{cases} \tag{4.9}$$

By subtracting the two equations we get

$$\frac{2}{\beta}(\tau_k^s - \tau_2^s) = \tau_h^s + \tau_1^s - (\tau_a^s + \tau_b^s).$$

As $\tau_k^s \geq \tau_2^s$, necessarily

$$\tau_h^s + \tau_1^s \geq \tau_a^s + \tau_b^s \geq 2\tau_2^s. \tag{4.10}$$

Substituting (4.6) in (4.10) we obtain

$$\tau_h^s + \tau_1^s \geq \frac{n}{1-\beta} + \tau_1^s,$$

$$\tau_h^s \geq \frac{n}{1-\beta}.$$

which, by Proposition 11 and Corollary 8, when $|N_s^- 2(x)| \geq 2$, is impossible.

Therefore we proved that, if node 2 has no link to 1 or to s, then no other node has a distance from s smaller or equal to 2. This contradicts the hypothesis $|N_s^{-2}(x)| \geq 2$. Hence, node 2 has to satisfy

$$\mathrm{dist}_x(2,s) \leq 2.$$

Moreover, if $\mathrm{dist}_x(2,s) = 2$, then $1 \in N_2(x)$.

□

A first result on Nash equilibria is the following:

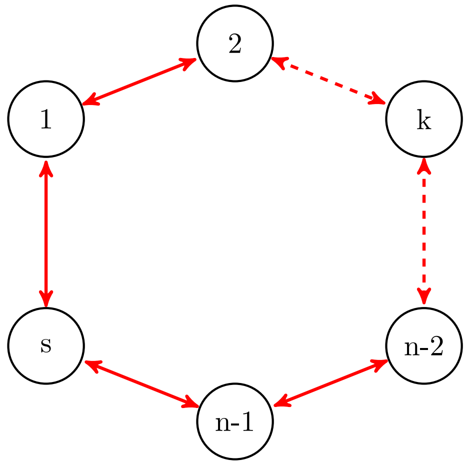


Figure 4.1: Cycle graph with n nodes

Proposition 13. If $\mathcal{G}(x)$ is a cycle graph, then it is a strict Nash equilibrium for CMG$(\mathcal{V}, \beta, 2)$.

Proof. Let us consider a generic cycle graph, as the one presented by Figure 4.1 and study the best response of node s. As a consequence from Proposition 9, it is sufficient to show that $\tau_2^s(x) > \tau_1^s(x)$ and $\tau_{n-1}^s(x) > \tau_{n-2}^s(x)$. By subtracting

$$\tau_2^s(x) = 1 + \frac{1-\beta}{n}\sum_{j\in\mathcal{V}}\tau_j^s(x) + \frac{\beta}{2}(\tau_1^s(x) + \tau_3^s(x))$$
$$\tau_1^s(x) = 1 + \frac{1-\beta}{n}\sum_{j\in\mathcal{V}}\tau_j^s(x) + \frac{\beta}{2}\tau_2^s(x)$$

we obtain that

$$\left(1 - \frac{\beta}{2}\right)(\tau_2^s(x) - \tau_1^s(x)) = \frac{\beta}{2}\tau_3^s(x) \quad \Rightarrow \tau_2^s(x) > \tau_1^s(x)$$

The same consideration can be made for $\tau_{n-1}^s(x) > \tau_{n-2}^s(x)$. The Proposition is hence proved. □

To prove that $\mathcal{G}(x)$ is a strict Nash if and only if it is an undirected graph, we may firstly stress the following

Remark 5. When considering CMG$(\mathcal{V}, \beta, 2)$, it follows from Proposition 12, that the best response of a node s with $N_s^- = \emptyset$ in $\mathcal{G}(x)$ is not unique.

As a result of the above considerations we can now give a characterization of strict Nash equilibria.

Theorem 10. *A graph $\mathcal{G}(x)$ is a strict Nash equilibrium for CMG$(\mathcal{V}, \beta, 2)$ if and only if it is undirected.*

Proof. Let us consider the two implications separately:

- $\mathcal{G}(x)$ undirected $\Rightarrow \mathcal{G}(x)$ strict Nash equilibrium.
  If $\mathcal{G}(x)$ is undirected, i.e. we have only direct links, it has to be a set of disjoint cycle graphs. As any circle graph is a Nash and there is no incentive for a node to link to another circle, $\mathcal{G}(x)$ undirected $\Rightarrow \mathcal{G}(x)$ strict Nash.

- $\mathcal{G}(x)$ strict Nash $\Rightarrow \mathcal{G}(x)$ undirected.
  By contradiction, let us assume that there exist two nodes $s, j \in \mathcal{V}$ such that $R_{js}(x) \neq 0$ while $R_{sj}(x) = 0$. Therefore node s points at two nodes $s+1, s-1 \neq j$. As we want $\mathcal{G}(x)$ to be a strict Nash, according to Remark 5, j needs to have at least one in-neighbor i, otherwise its best response won't be unique. The structure described until now is presented by Figure 4.2.

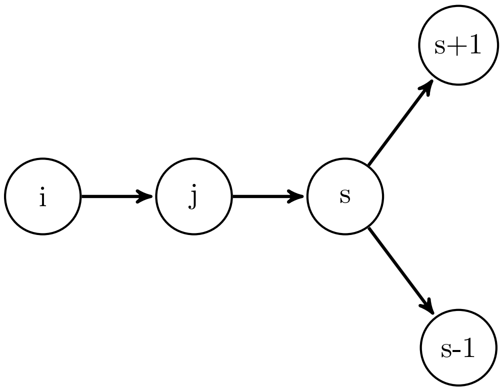
  

  Figure 4.2: Structure implied by having a directed link (j, s)

  According to Proposition 9, if j has one in-neighbour i, it has to link back to it and to also have a link to one of the in-neighbours of i. Hence, if s is the best response of j, s has to be a in-neighbour of i. The structure implied by having a directed link from j to s becomes the one presented by Figure 4.3. If $i \in x_s$,

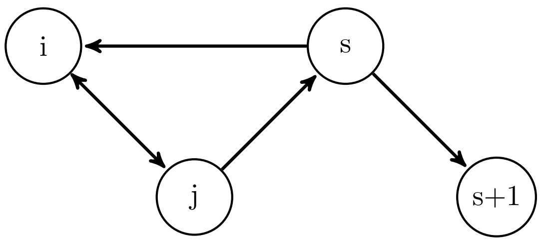
  

  Figure 4.3: Structure implied by having a directed link (j, s)

  Proposition 9 implies that also $j \in x_s$, which contradicts the hypothesis.

□

4.2 Non-strict Nash equilibria

When dealing with a $CMG(\mathcal{V}, \beta, 2)$, the complexity of Nash equilibria structures significantly increases compared to the one of CMG$(\mathcal{V}, \beta, 1)$. Therefore, in the following, instead of characterizing all the Nash equilibria, we will analyze some specific examples that result to of particular interest when studying the best response dynamics.

Example 1. (The hourglass) When $n = 5$, Figure 4.4 shows an example of non-strict Nash

Notice that, from Proposition 12, nodes k, h, j, i are all playing their best response

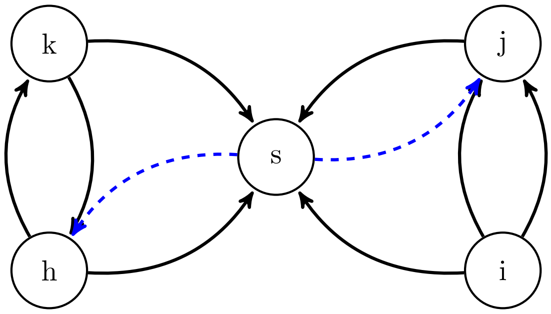


Figure 4.4: Example 1: the hourglass

action. Moreover, they all have the same expected hitting time on s, given the symmetry of the graph. Therefore, s can change the direction of its out-links as it likes.

One may rise the question of whether this object can evolve to a strict Nash equilibria or not. The shape of this object can change only when s is directing its out-links on the same side, for instance towards k and h. Afterwards, nodes i, j can choose equivalently any two nodes to direct their out-links to. Hence the only other shape that can be reached from the hourglass is shown in Figure 4.5. Is the structure

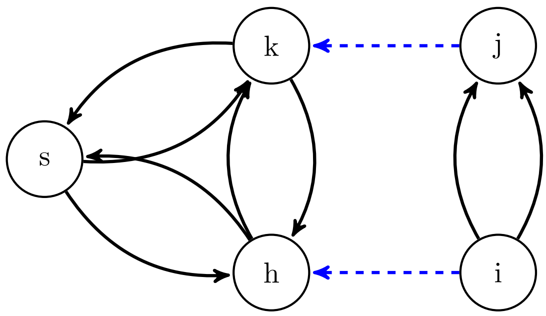


Figure 4.5: Possible evolution of the hourglass

shown by Figure 4.5 a Nash equilibrium? It's know from Proposition 12, that both

i and j are in their best responses and from Theorem 8 that node s is in its best response. Let x be the action configuration such that $\mathcal{G}(x)$ is the graph shown in Figure 4.5. Let us now focus on $\tau_j^k(x)$ and $\tau_s^k(x)$. We know that

$$\tau_j^k(x) = 1 + \frac{1-\beta}{n}\sum_{l\in\mathcal{V}}\tau_l^k(x) + \frac{\beta}{2}\tau_i^k(x)$$

$$\tau_s^k(x) = 1 + \frac{1-\beta}{n}\sum_{l\in\mathcal{V}}\tau_l^k(x) + \frac{\beta}{2}\tau_h^k(x)$$

Therefore, by subtracting the two previous equations we obtain

$$\tau_s^k(x) < \tau_j^k(x) \;\Leftrightarrow\; \tau_h^k(x) < \tau_i^k(x).$$

But, following the same reasoning, we obtain

$$\tau_h^k(x) < \tau_i^k(x) \;\Leftrightarrow\; \tau_s^k(x) < \tau_j^k(x) + \tau_h^s(x).$$

As $\tau_h^s(x) = \tau_s^k(x)$, the last equation it is always true. Hence

$$\tau_s^k(x) < \tau_j^k(x).$$

Therefore the structure showed by Figure 4.5 it is another non-strict Nash where the only links that can change are the blue ones. Hence, from structures as the ones in Figure 4.5 and 4.4 it is not possible to evolve to a strict Nash equilibrium.

Example 2. (Node linking two cycles)
Another non-strict Nash equilibrium can be identified in the structure presented

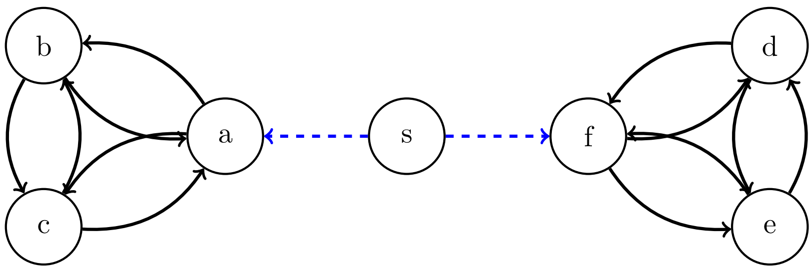


Figure 4.6: Example 2: Node linking two cycles

by Figure 4.6. From Proposition 9, nodes c, b, d, e, s are playing their unique best response action. By solving system (1.7) it is possible to check that also nodes a and f are playing a best response action. Therefore, in Figure 4.6, the links colored in blue correspond to not unique best responses actions. The black ones, i.e., the links inside cycles are unique best responses actions.

Example 3. (Couple linking two cycles) Another non-strict Nash equilibrium can be identified in the structure presented by Figure 4.7. Again the links colored in blue correspond to not unique best responses actions. The black ones, i.e., the links inside cycles are unique best responses actions.

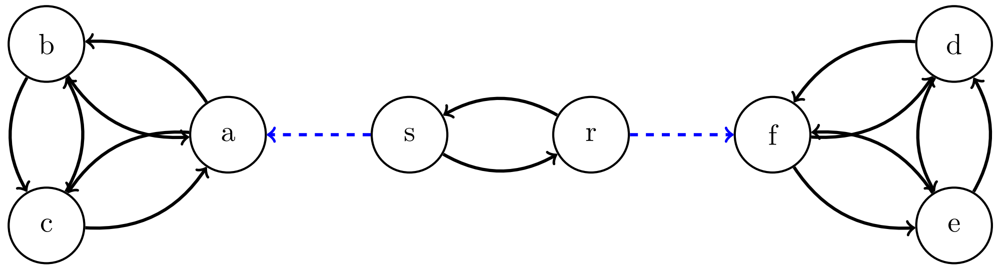


Figure 4.7: Example 2: Couple linking two cycles

4.3 Best response dynamics

Let now focus on the best response dynamics related to CMG($\mathcal{V}, \beta, 2$). As in Chapter 3, in order to study the convergence of the dynamics a MATLAB routine has been implemented. Results are shown by Figure 4.8. Let us call x^t the action configuration at a certain time t. Each curve represents a different initial configuration of the graph, i.e. a different x^0. In all the simulations $\beta = 0.3$ and $n = 20$. To make the convergence visible, a time block of 20 time steps has been defined, and the number of changes between x^t and x^{t+1} has been aggregated in every time block. Each curve represents a different initial configuration of the graph.

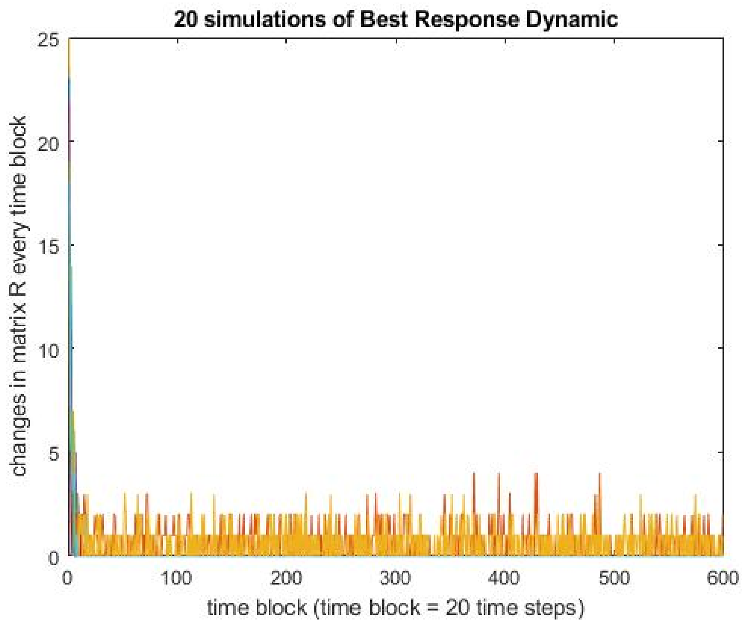
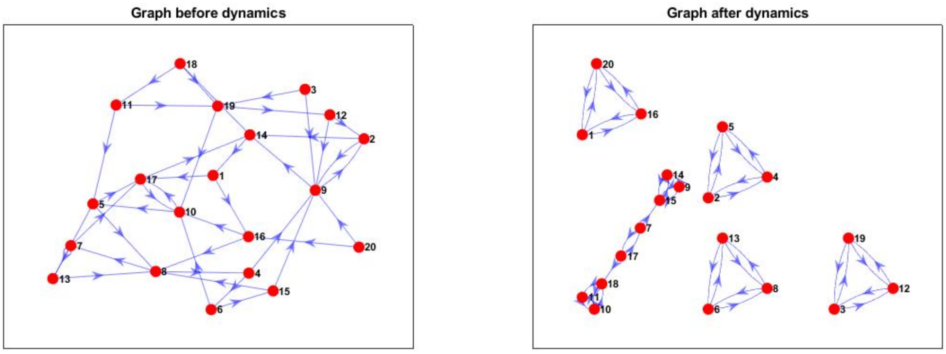
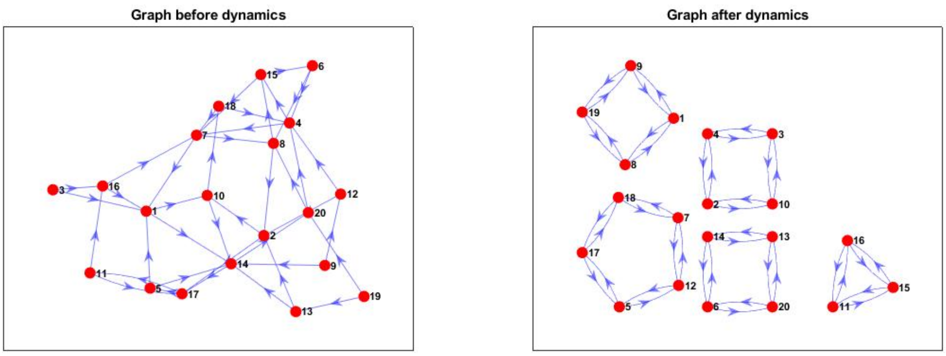


Figure 4.8

As shown by the picture, in all the 20 simulation the number of changes every 20 time steps tends to decrease along time. Nevertheless, while some simulations converge pretty fast, some others keep oscillating within different configurations. That begs the question of what the structures the dynamics is converging to look like in this two cases. Figure 4.9 shows the structure of $\mathcal{G}(x^{800})$ before and after the dynamics in two of the previously considered simulations.



(a) Convergence to non strict Nash equilibrium



(b) Convergence to a strict Nash equilibrium

Figure 4.9: Example of convergences in CMG$(\mathcal{V}, \beta, 2)$

Figure 4.9 could suggest that the dynamics converges to the set of Nash equilibria. Nevertheless, when the Nash is strict then, of course, the configuration does not change anymore, while when the Nash equilibrium is non strict, (for instance in the case of Example 1 or Example 3) the system oscillates within some non-strict Nash equilibria.

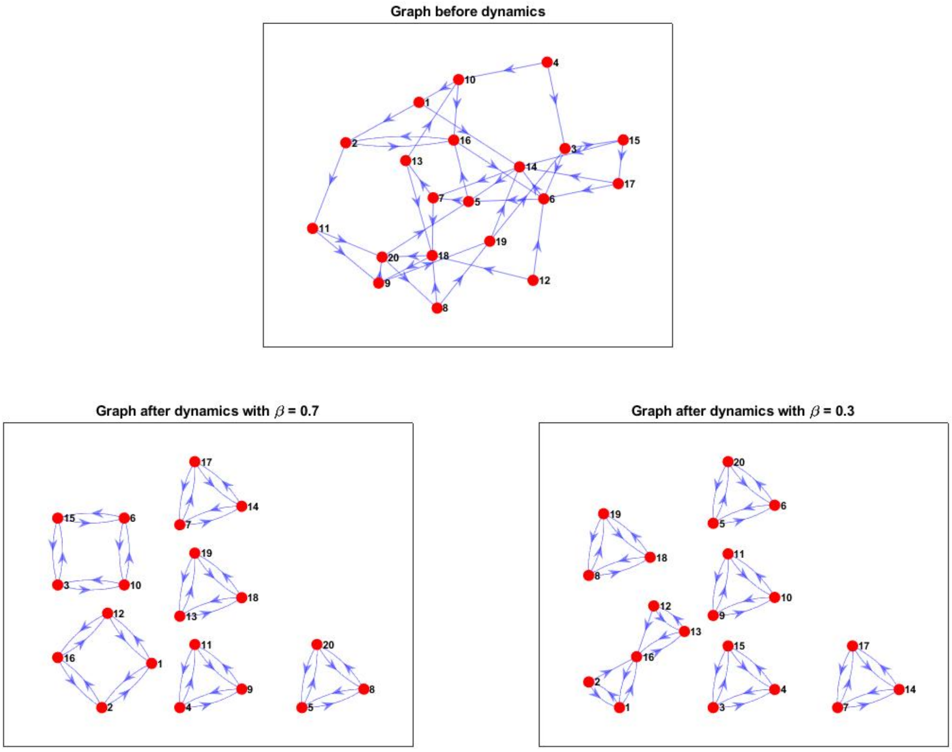


Figure 4.10: Dependence of convergence on beta

4.4 The role of β

To better understand the role of parameter β in the convergence of the dynamics some simulations have been implemented. As we know that strict Nash do not depend on the value of β we could ask our-selves whether the best response functions depend on such a parameter. In Figure 4.10 the same dynamics has been implemented with two different $\beta = 0.7$ and $\beta = 0.3$ and the Nash equilibrium reached after 600 iterations have been plotted. Note that for both value of β, at every time step t, the same node was turning on and choosing his best action.

Figure 4.10 shows that different Nash equilibria are reached with different β. This implies that the best response functions depend on β.
Another interesting question related to β is whether it influences the speed of convergence of the dynamics. By using two values of β pretty far from each other ($\beta = 0.1$ and $\beta = 0.9$) and simulating the dynamics for 40 times each, the mean number of changes every 20 time steps has been calculated with Matlab. The result is shown in Figure 4.11.

The speed of convergence does not seem to be related to β.

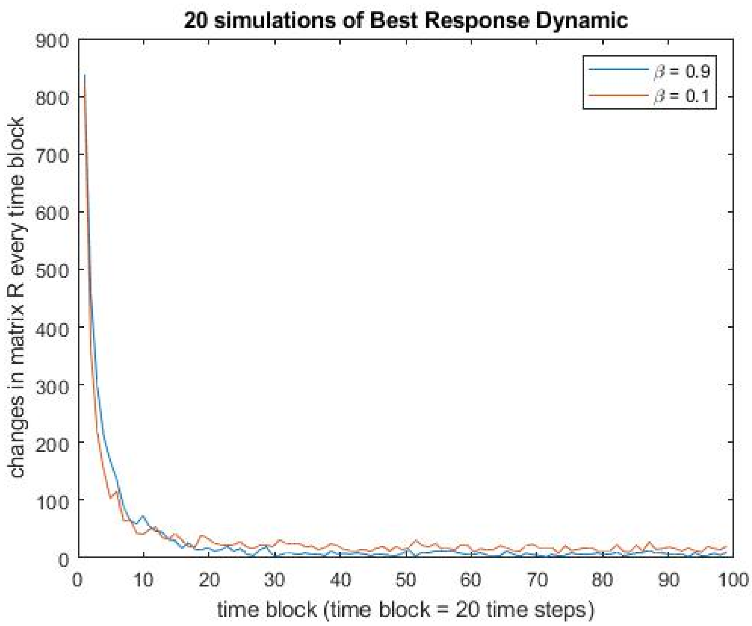

20 simulations of Best Response Dynamic
β = 0.9
β = 0.1
changes in matrix R every time block
time block (time block = 20 time steps)
0
100
200
300
400
500
600
700
800
900
10
20
30
40
50
60
70
80
90
100

Figure 4.11

Chapter 5

Conclusions

Centrality measures, as broadly illustrated by this dissertation, play an important role in many systems modeled by graphs. With a centrality measure we can, for instance, investigate how visible a webpage is on the internet or how influential a person's opinion is in a social network. Strategies to improve a node's centrality are thus of particular interest as, in general, they allow a node to play a major role in a given framework. This thesis aims at better describing these strategies on networks presenting an underlying immutable structure over which a network built by nodes' decisions is overlapped. More specifically, the work focused on what we defined as *Centrality Maximization Game* $CMG(\mathcal{V}, \beta, m)$, a game in which every node is given the chance to chose where to direct m unweighted out-links to achieve the goal of maximizing its centrality. Every nodes' action configuration x therefore defines a graph that we call $\mathcal{G}(x)$.

In the case of $m = 1$, i.e. the situation in which every node is allowed to place only a single link, we exhibit that the best strategy for a node with at least one in-link is to point to an arbitrarily chosen node within its in-links. Furthermore, Theorem 5 proves that Nash equilibria are all and only those graphs $\mathcal{G}(x)$ with $c \geq 1$ couples of nodes connected by an undirected link and all the nodes not forming a couple pointing at coupled nodes. Moreover, the investigation pointed out that strict Nash equilibria, when $m = 1$, are all and only those graphs $\mathcal{G}(x)$ in which every node is in a couple. In other terms they are all and only the undirected graphs with 1 out-links from every node. Another result related to $CMG(\mathcal{V}, \beta, 1)$ is provided by Theorem 7 which states that the best response dynamics of such a game always converges and proves it by finding an ordinal potential function.

In a first generalization attempt, the same properties have been studied for $CMG(\mathcal{V}, \beta, 2)$. The prominent result in this case is that the search of best response actions for a node s can be limited to a subset of nodes with maximum distance from s equal to 2. This result proves to be particularly helpful when analyzing Nash equilibria and brings to the formulation of Theorem 10 which states that strict Nash equilibria are all and only the undirected graphs with 2 out-links from every node.

It should also be noted that results obtained when $m = 1$ and $m = 2$ present some recurrences and hence suggest a possible generalization. In fact, in both cases, even if nodes can potentially choose any nodes to link to, they still find their optimal choice in a set of nodes within a certain (small) distance from themselves. Proposition 7 and Theorem 9, which state this result respectively when $m = 1$ and $m = 2$, could therefore be generalized by the following

Conjecture 1. Consider a CMG$(\mathcal{V}, \beta, m)$ and a state x. Assume there exist at least m nodes with distance in $\mathcal{G}(x)$ from s smaller or equal to m. Then if $x_s \in \mathcal{B}_s(x_{-i})$, then $\forall j \in x_s$ it holds

$$\text{dist}_x(j, s) \le m.$$

Moreover, if $x_s \in \mathcal{B}_s(x_{-i})$, and there exist a node $j \in x_s$ such that $\text{dist}_x(j, s) = k$, with $k \le m$, by calling $\gamma = (j, i_2, i_3, \dots, i_k, s)$ the shortest path from j to s in $\mathcal{G}(x)$, it holds that $i_2, i_3, \dots, i_k \in x_s$.

Moreover, in both cases $m = 1$ and $m = 2$, strict Nash equilibria are all and only undirected graphs with m out-links from every node. Therefore, this result may be generalized as well. We formulate it through Conjecture 2 for further developments.

Conjecture 2. $\mathcal{G}(x)$ is a strict Nash equilibrium for CMG$(\mathcal{V}, \beta, m)$ if and only if it is undirected.

As a final remark, we may stress that, when dealing with the best response dynamics over a CMG$(\mathcal{V}, \beta, m)$, a general convergence to Nash equilibria has been observed through Matlab simulations. Even though such a convergence has been proved to be true in the case of $m = 1$, it still has to be formalized for a generic $m \in \mathbb{Z}$. Therefore we can formulate the last Conjecture.

Conjecture 3. Consider a CMG$(\mathcal{V}, \beta, m)$. The best response dynamics over such a game always converges to the set of Nash equilibria.

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