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Non Perturbative Renormalization Group Approach to Kraichnan Model:
Calculation of the two point Correlation Function

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Introduction

Statistical Fluid Mechanics and Intermittency

Turbulence is a ubiquitous phenomenon in fluid flows. It is characterized by very peculiar properties generally called intermittency. To describe intermittency flows from Navier Stokes equation is one of the Holy Grail of modern classical theoretical physics. Roughly speaking, intermittency in a stochastic process is the high probability of recurrence of relatively rare events. In experimental measures the fluid velocity increments show sequences of alternate bursts and linear behaviour in the transitional phase from laminar to turbulent flows and also in fully deeloped turbulence, reproducing similar behaviour to the one observed in chaotic motion. This behaviour of a turbulent flow still lacks of a complete and exhaustive theoreitical description.

Let's do a little step back in turbulence history in order to better understand the main topic of the present work. The Navier Stokes equation [3] represents the milestone of fluid mechanics. It is basically a re-expression of the Newton' second law for fluids. One of the first attempt to the solution of this equation is due to G. G. Stokes, which focused on the behaviour of laminar fluids.

It was only thanks to Reynolds in 1883 that turbulence started to be argument of great discussion in theoretical physics. He defined the well known Reynolds number

$$Re = \frac{VL}{\nu} \tag{1}$$

which relates the typical length scale L and velocity V of fluid with its viscosity ν. Physically it expresses the ratio between the strength of the convective force felt by the fluid and the strength of the dissipative one. This value gives also insights on the treshold between what can be considered a turbulent flow and its laminar counterpart. The higher is the Reynolds number, the higher the presence of a flow which is unpredictable, strictly non-diffusive and characterized by interactions involving a wide range of scales. Flow that we then define turbulent.

Let's focus on one very important aspect of the theoretical approach to turbulence. The high complexity and chaotical behaviour of this stochastic process prevents any kind of deterministic study, forcing physicists to takle the problem through a statistical perspective. The first attempt was carried on by Reynolds himself [2] with the definition of the Reynolds averaged Navier Stokes equation. Following his example a new branch of research in this direction was open. This led to the derivation of the stochastic Navier - Stokes equation

$$\partial_t \vec{v} + \vec{v} \cdot \vec{\partial} \vec{v} - \frac{1}{\rho} \Delta \vec{v} - f = 0 \tag{2}$$

which is formally the same expression as the original one but with the forcing term replaced by a stochastic fields. This term was added in order to introduce stocasticity in the system without affecting initial conditions. Furthermore, asking for statistically stationary, homogeneous and isotropic forcing, dismissing external term that could influence the fluid evolution too much, it turned out to be extremely convenient to model f as a memoryless centered gaussian stochastic process. From this model was then possible to extract results about averages of observables.

In order to study the properties of a turbulent stationary state, researchers are forced to continuously inject energy in the system, because of the intrinsic dissipative nature of this process. At very high Reynolds number the range between the scale of energy injection (for example through propellers) and the scale of energy dissipation is very wide. It was observed that between this very different length-scales, there exists an interposed inertial range. In this range, the statistical properties of a turbulent flows are observed to be universal, i.e that is common to many different turbulent flows. Inside this range, the transfer of energy between scales, from larger to lower, happens in a local and self-similar way. In particular it turned out that this energy transfer (when observed far from the boundaries or any obstacles to fluid isotropy and homogeneity) occurs conservatively.

Some physical explanation to this behaviour came when it was assumed that the main responsible for the energy cascade in this inertial range was the evolution of the forcing-generated fluid eddies. This rotating quantities evolves interacting between each other and with the rest of the system, self-destroying in smaller similars from approximately the propellers length-scale to the dissipative one. This process was understood to be conservative and extends until the size of this self-organized structures is comparable to the molecular scale where energy is dissipated.

Starting from those deductions and the existence of universality for the statistical properties of turbulence, A. Kolmogorov in 1941 proposed the first great statistical theory for turbulence [10], [8]. His main results, obtained assuming an homogeneous and stationary turbulent process, are here briefly presented. First of all, he was able to obtain an expression for the Kinetic Energy Spectrum involved in a turbulent flow as

$$E(k) = 4\pi k^2 TF\langle \vec{v}(\vec{x})\vec{v}(0)\rangle = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

where ϵ is the mean energy dissipation rate and C_K is a universal constant.

This power law behaviour with exponent $-\frac{5}{3}$ turned out to be very close to experimental measures, as shown in the following plot

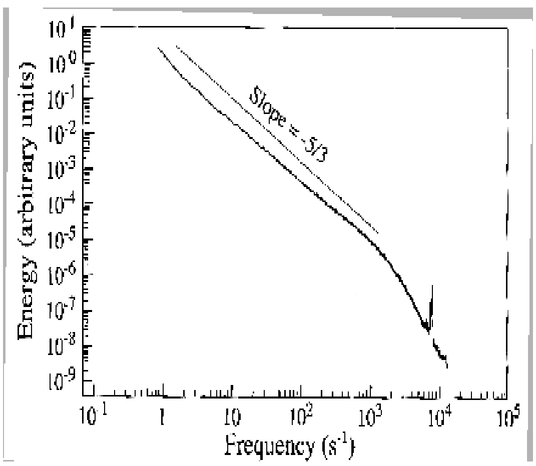

Figure 1: Power law behaviour of E(ω) as function of frequency. *ONERA Wind Tunnel, Anselmet, Gagne, Hopfinger, Antonia. J. Fluid Mech. 140, 1984*

Secondly, A. Kolmogorov analyzed another important observable experimentally measured in turbulent context: the velocity structure function [9]

$$S_n(\ell) = \langle [\delta v_{||}(\ell)]^n \rangle = \langle [\vec{v}(\vec{x}+\vec{\ell},t) - (\vec{v}(\vec{x},t)]\frac{\vec{\ell}}{||\ell||})^n \rangle \sim \ell^{\eta_n}.$$

For $n = 3$ he derived one of the only analytical results for 3d turbulence, which is

$$S_3(\ell) = -\frac{4}{5}\epsilon\ell$$

then assuming the standard scale invariance for the probability distribution function of rescaled velocity differences, i.e. that $r^{-t}\Delta_r v$ could be made r independent with the right choice of t, A. Kolmogorov deduced that for any n $\eta_n = \frac{n}{3}$. Following this insight, the n^{th}-order moment should grow with a power law behaviour linear in n.

However this prediction is not observed in a turbulent flow. The structure functions show an anomalous behaviour and the associated exponents η_n are not linear in n. Hence, standard scale invariance is broken, to give rise to mu

Measuring and then plotting the probability distribution of the velocity increment $\delta v_{||}(\ell)$,one can realize that the probability of rare events is much higher than for a gaussian stochastic process.

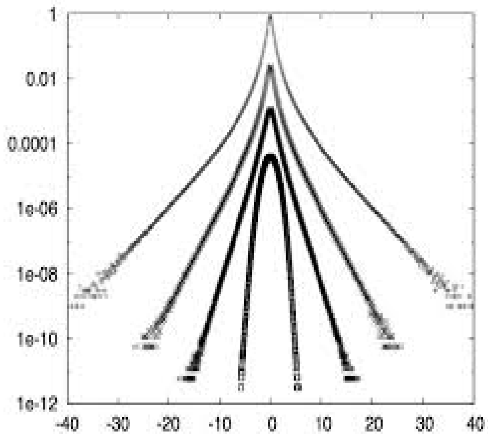

Figure 2: Pdf for $\delta_{||}(\ell)$. *Chevillard et al., Comptes Rendus Physique 2012 [12]*

The distribution of this quantity, lowering in ℓ value, i.e. going towards smaller scales, becomes more and more non-Gaussian, showing high probability for extreme events. These consequential strong fluctuations and calm period which characterizes the intermittency of the turbulent velocity differences.

The wrong assumption from A. Kolmogorov was indeed to consider the usual scale invariance for the turbulent flow and the absence of fluctuations of ϵ. In his theory was substantially assumed that away from boundaries and in isotropic and homogeneous conditions, energy tranfer was associated with a unique variable ϵ, independent on ℓ. What really characterizes the turbulent dynamics is what is referred to as multiscaling behaviour of ϵ. It was later proved that ϵ averaged over a certain length scale ℓ directly depends on the length scale.

The analytical computation of intermittency from the Navier Stokes equation turns out to be mathematically really hard and carries numerous problems. It was suggested that the advection-diffusion equation for a non-reacting scalar field could represent an easier system useful to model intermittent behaviour.

The Kraichan Model

Let's consider the following equation for a passive scalar field Θ which diffuses and is advected by a carrier fluid with velocity $\vec{v}$

$$\partial_t \Theta + \vec{v} \cdot \vec{\partial} \Theta = \nu \Delta \Theta + f \tag{3}$$

This equation is often used to describe the dynamics of a scalar quantity transported by a fluid, such as density or temperature. The attribute passive means the absence of any back-reaction from the scalar field Θ on the velocity flow $\vec{v}$. In principle, $\vec{v}$ should be a solution of a coupled Navier Stokes equation. It was observed that the dynamics of a scalar quantity also shows turbulence and intermittency [5].

Starting from here, A. Kraichnan built a simplified model of passive scalar. His main assumption was to replace the velocity field $\vec{v}$ with a gaussian stochastic random variable. This great insight comes from the observation that even without any intermittency coming from the velocity distribution, intermittent behaviour was found in passive scalar field moments measures. He hence considered a velocity-velocity correlation with an anomalous power-scaling

$$\langle v_a(\vec{x}, t) v_b(\vec{x}', t') \rangle = D_0 \delta(t - t') \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}(\vec{x} - \vec{x}')}}{(\vec{q}^2 + m^2)^{\frac{d+\xi}{2}}} P_{ab}(\vec{q})$$

with ξ anomalous exponent and

$$\Pi^{ab}(\vec{q}) = \delta^{ab} - \vec{q^a} \vec{q^b}$$

the tranverse projector.

In order to introduce stocasticity in the model [1], he modeled f by a gaussian stochastic random field with correlation

$$\langle f(t,\vec{x}) f(t',\vec{x'})\rangle = \delta(t-t') D_f(|\vec{x}-\vec{x'}|)$$

The $\delta(t-t')$ dependence on time points out the memoryless realization of the forcing, and D_f the profile of energy injection. Moreover the Kraichnan model has the advantage of reducing the non-linearity of the advection term compared with Navier Stokes equation.

Predictions on structure functions exponents were analytically computed for this model in a perturbative ($\xi \ll 1$) approximation.

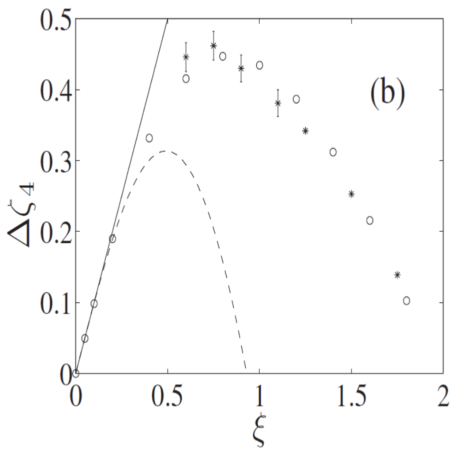

Figure 3: Perturbative computation at first (continue line), and second (dotted line) loop. Comparison with numerical results (dot line).

As illustrated in Fig. perturbative theory breaks down for $\xi \ll 1$ [17]. However, a realistic flow closed to Navier Stokes equation correspond to $\xi = \frac{4}{3}$, which is beyond the reach of perturbative analysis. Hence, our goal is to study behaviours of S_n for higher ξ and the Non Perturbative Renormalization Group Approach turns out to be extremely illuminating in this direction.

Chapter 1

Introduction to non-perturbative renormalization group for an out of equilibrium field theory

In this chapter we will introduce some of the needed theoretical tools in our analysis and the framework of NPRG and we will focus on application to out of equilibrium field theories. There will be a presentation of the MSRJD formalism, useful to map a stochastic equation in a field theory, and a brief treatment on main aspects of this theory. Then, to conclude, we will present the basis of the NPRG formalism and its consequences.

1.1 Notations

In this work, with some exceptions which will be well explained, we will use the usual vector $\vec{v}$ and components v_α notations. Bold symbols will stand for both time and space coordinates $\mathbf{x} = (x, t)$ both in real and Fourier space. Integral notation is abbreviated for clearness:

$$\int_{\mathbf{x}} = \int d^d x dt \tag{1.1}$$

Our convention for the Fourier transform in this work is

$$f(\mathbf{x}) = \int_{\mathbf{p}} f(\mathbf{q}) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{1.2}$$

$$f(\mathbf{p}) = \int_{\mathbf{x}} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{1.3}$$

where we used the shorthand notation:

$$\mathbf{p}\cdot\mathbf{x} = \vec{p}\cdot\vec{x} - \omega t \tag{1.4}$$

In our discussion we will also need to exploit the translational invariance in space and time of some function, in the complex plane. Here below the notation used:

$$\tilde{F}(\mathbf{p_1}, ..., \mathbf{p_n}) = (2\pi)^{d+1} \delta(\sum \omega_i) \delta(\sum \vec{p_i}) \bar{F}(\mathbf{p_1}, ..., \mathbf{p_{n-1}}) \tag{1.5}$$

1.2 Response field formalism for Kraichnan equation

Any problem involving a huge number of degrees of freedom coupled via complex nonlinear interactions, such as turbulence, is subjected to a statistical approach. In general to take account for all the fluctuations involved in a system a field formalism is appropriate. At equilibrium, one defines as Boltzmann weight the probability for any micro-state of the system considered and build the corresponding partition function. Once the related free energy is defined, it is possible to derive all the analytic formulas for the thermodynamic quantities and correlation functions of interest. For instance, in the Ising-spin model, once the partition function is defined, the system can be deeply studied and results obtained.

In an out of equilibrium situation the probability distribution of micro-states is not known. There is not a general theoretical framework to tackle out these systems. However, for stochastic modeling, a field formalism can be constructed through averaging over the noise. We present below a standard procedure to do this.

1.3 Martin Siggia Rose Janssen De Dominicis Formalism

We introduce here the powerful MSRJD formalism [13],[7], [6], useful to exploit the field theoretical formulation for a generic stochastic partial differential equation. Let's resume the equation under analysis

$$\partial_t \Theta + v \cdot \nabla\Theta - \nu\Delta\Theta + f = 0 \tag{1.6}$$

What we want to compute is the average value of observables, such as $A[\Theta]$.

In general we can express the average over the internal noise in the following Path Integral formalism

$$\langle A[\Theta]\rangle_f \propto \int \mathbf{D}[f] A[\Theta(f)] P[f] \tag{1.7}$$

where $P[f]$ represents the probability distribution of the noise.

As first step, we rewrite the constraint expressed by (1.6) under an exponential delta functional expression

$$1 = \int \mathrm{D}\Theta \int \mathrm{D}\tilde{\Theta} e^{-\int_x \int_t \tilde{\Theta}(\partial_t \Theta + \vec{v}\cdot\nabla\Theta - \nu\delta\Theta - f)}, \tag{1.8}$$

$\tilde{\Theta}$ is called the response field, used to exponentiate the functional delta. Of course the integral of a delta is a constant and we can insert (**??**) into (1.7). Remembering our equation possesses two different noises $\vec{v}$ and f, we obtain

$$\begin{aligned} \langle A[\Theta]\rangle_{f,v} \propto \int \mathrm{D}[\Theta, \tilde{\Theta}, \vec{v}] e^{\int_{\vec{x},t} \tilde{\Theta}(\partial_t \Theta + \vec{v}\cdot\nabla\Theta - \nu\Delta\Theta)} A[\Theta] \\ \int \mathbf{D}[f] e^{-\int_{\vec{x}}\int_t \left[\tilde{\Theta} f + \frac{1}{4} f (D_f)^{-1} f\right]} P[\vec{v}] \end{aligned} \tag{1.9}$$

where all the deterministic part of the dynamics is now proportional to the response field and there is a second computable integral over the forcing. We integrate over the gaussian distribution of f to obtain

$$Z = \int \mathrm{D}\Big[\Theta, \tilde{\Theta}, \vec{v}\Big] \, e^{-A[\Theta,\tilde{\Theta},\vec{v}]+\int_{\mathbf{x}} J_i \Theta_i} \tag{1.10}$$

which is the generating functional for the correlation function, and where

$$A[\Theta, \tilde{\Theta}, \vec{v}] = \int_{\vec{x}} \int_t \tilde{\Theta}(\partial_t \Theta + \vec{v} \cdot \nabla\Theta - \nu\Delta\Theta) + \frac{1}{2}\tilde{\Theta} D_f \tilde{\Theta} - \frac{1}{2}\vec{v} D_v^{-1} \vec{v} \tag{1.11}$$

is the action functional related to the field formulation of the Kraichnan model and J_i are the sources. The quantities D_f and D_v are respectively the forcing and velocity correlation.

For completeness, let us gie a precision.Eq. (1.8) involved a change of variables between noise and field $f \to \Theta$. The Jacobian of this change of variable is

$$J = |det \frac{\delta T}{\delta \Theta}| \tag{1.12}$$

where $T[\Theta] = \partial_t \Theta + \vec{v} \cdot \nabla\Theta - \nu\Delta\Theta$.

In this work, we use the Itô convention, which amounts to have a forward discretization scheme of the differential equation. In this case, we can show that J does not depend on the and it can be included, as we did.

One immediately notices that the action (1.11) depends on three fields, while our original equation presents only two fields. This is a general feature of the action obtained through this formalism: it always contains more degrees of freedom than the equation expression of the process. All these new fields, which are here denoted by the *tilda* are called response fields. Each of them expresses one of the constraints coming from the stochastic system that the field theory should fulfill.

Let us emphasize that Z function (1.10) is the generating functional of generating function of the theory. For instatnce, the 2-point correlation function can be obtained through two functional derivatives with respect to the sources

$$G((\mathbf{x}), (\mathbf{x}')) = \frac{\delta^2 Z}{\delta j(\mathbf{x}) \delta \tilde{j}(\mathbf{x}')} = \langle \Theta(\mathbf{x}) \tilde{\Theta}(\mathbf{x}') \rangle \tag{1.13}$$

Let's now discuss some general properties for the Non Perturbative Renormalization Group theory.

1.4 Introduction to non-perturbative renormalization group

As mentioned in the Introduction to this work, perturbative analysis for $\xi \ll 1$ on Kraichnan model was already carried out. We are going now to give some insights on the field theory techniques we will use to study the two point correlation function involved in this system at greater ξ values.

1.4.1 The Wilson Renormalization Group

We will start showing the first formulation of the renormalization approach, done by Wilson in 1976 [16]. Wilson's theory was the functional translation of the Kadanoff's block spin renormalization [4].

The goal of Kadanoff' s idea was to obtain an effective model for "large scale" degrees of freedoms starting from the microscopic degrees of freedom of a real model. There are two main steps involved in this procedure. One starts from a real system with a certain amount of variables (for example spins) and an associated hamiltonian, which contains all the microscopic characteristics of the model. Then, the first step is to collect a sub-set of spins, average over their values, and to promote this new obtained spin to be the variable of a new effective hamiltonian. Finally, the second step is to rescale all the distances by the new effective lattice spacing. The physics of the system is now stated by an effective Hamiltonian, whose variables are average values obtained from the initial degrees of freedom. All the critical properties are then encoded in the behaviour of the flow of the effective theories near critical points.

Wilson's insight was to promote this approximation to act directly on functions and in the Fourier space. So in Fourier space (we omit here the direct expression of the dependency of each function on the momentum variables), we can redefine our field variable ϕ by

$$\phi = \phi_< + \phi_> \tag{1.14}$$

where the $<$ and $>$ signs stands for all the Fourier modes which are larger or lower than a certain renormalization length scale k. The generic system under analysis is described by an associated partition function:

$$Z[j] = \int \mathrm{D}[\phi] e^{j\cdot\phi - S[\phi]} = \int \mathrm{D}[\phi_< + \phi_>] e^{j\cdot(\phi_< + \phi_> - S[\phi_< + \phi_>]} \tag{1.15}$$

The next step is to integrate over all modes which are larger than the value k, so between k and Λ and obtain the effective action. Let's stop to describe the meaning of the Λ value. All the statistical field theories are described over a lattice which, by construction, carries a minimum distance between its vertices. This corresponds to a Λ cut-off in the momentum space. Integrating, for example between Λ and k, is the mathematical expression used to start to take care of a certain amount of fluctuating modes of ϕ. The presence of an upper and lower cut-off in the integration prevents integral divergence. This allows the use of a saddle point approach to compute the corrections to mean field results due to interactions. To build the effective theory one integrates over a little shell the momentum values' set. We obtain

$$Z[j] = \int \mathrm{D}[\phi_<] e^{j\cdot\phi_<} \int \mathrm{D}[\phi_>] e^{-S[\phi_< + \phi_>]} = \int \mathrm{D}[\phi_<] e^{j\cdot\phi_< - S_k^{eff}[\phi_<]} \tag{1.16}$$

We remark here that what is completely new and different from the mean field approximations is the integrating cell. Here the two bounds of integrations $[k, \Lambda]$ prevents any kind of divergences in the saddle-point approximation, which can be applied and the corrections to the coupling constants of our theory can be derived and computed.

The last step consists then in rescaling all the variables of our system. If we recall $k = \Lambda t$ with t smaller but close to 1, we can express this property as

$$\phi(\vec{q}) = t^\alpha \phi'(\vec{q}/t) \tag{1.17}$$

So in general

$$Z[j'] = \int \mathrm{D}[\phi'] e^{j'\cdot\phi' - S'[\phi']} \tag{1.18}$$

Naturally, this process doesn't affect the partition function of the system. It was then demonstrated that [**callan**] one could set up an infinitesimal change of variables and in this way recover the renormalization group differential equation for coupling constants.

1.4.2 The Effective Average action method

The Non Perturbative Renormalization Group is a modern version of the Renormalization Group idea. While the original Kadanoff-Wilson's theory was to perform a coarse-graining and to map hamiltonians into new effective hamiltonians, the idea in NPRG is to compute the Gibbs free energy $\Gamma[\langle\phi\rangle]$ of the rapid modes of our theory $\phi_>$ that have already been integrated out [4]. So the idea is to build a one parameter family of models in a scale k, defined such as

$$\Gamma_{k=\Lambda}[M] = H[\phi = M] \tag{1.19}$$

such that without any fluctuations it describes the original microscopic mean field model. And

$$\Gamma_{k=0}[M] = \Gamma[M] \tag{1.20}$$

which states that, when $k = 0$ all the fluctuations are taken into account, the Gibbs free energy obtained is the real Gibbs Free Energy of the system.

It is worthy to notice here that k acts both as a high-momentum cut-off for the low modes and and a low-momentum cut off for our Gibbs Free Energy Γ. To describe this one parameter family of Gibbs Free Energies we need a way to decouple higher momentum from lower momentum modes.

The way to decouple the two sets was to increase the "mass" of lower momentum degrees of freedom. This idea comes from the theoretical particle physics. Basically a heavy particle cannot interact with the lower energy physics, so supposing the absent or null interaction between the two length scales is intuitively easy. In particle physics, the hamiltonian mass term is the one proportional to ϕ^2. In statistical physics, the quadratic term carries as coupling constant the variable r that the distance of the model from the critical surface. The "increasing" of the mass is achieved through the addition of a supplementary function ΔS_k, which is quadratic in the fields, to the hamiltonian itself. In this way the "weight" of a set of momentum depending variables increases.

To fix the idea, let's write the general shape of the regulator

$$\Delta S_k = \frac{1}{2} \int_{\mathbf{x},\mathbf{y}} [R]_{k,ij}(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y}) \tag{1.21}$$

Because of the regulator, which obviously must be chosen dependent on k, also the partition function of our model becomes scale dependent, in particular

$$Z_k[j] = \int \mathrm{D}[\phi] e^{-S[\phi]-\Delta S_k[\phi]+\int_{\mathbf{x}} j\cdot\phi} \tag{1.22}$$

And so the Free Energy of the model $\Omega_k = \log Z_k$ and all the average values of the fields computed through this partition function.

In general the regulator should satisfy two physical limits. In the $k = \Lambda$ limit all fluctuations are frozen, so it should freeze the propagation of any fluctuations. This can be obtained modeling

$$R_{k=\Lambda}(q) \propto \Lambda^2 \tag{1.23}$$

On the other side, for each momentum over the threshold k, R_k should vanish

$$R_k(q > k) \approx 0 \tag{1.24}$$

for any $0 < k < \Lambda$. In Fig. 1.1 the typical cut-off function in the effective average action approach is shown.

As one can now understand, inserting the regulator inside the partition function exponent, separates in Z_k the lower and higher momentum modes. In particular, the presence of the regulator suppresses the lower momentum modes, ideally deleting them, and keep the dependency on the high momentum shell.

The last step is to define the proper Gibbs Free Energy of the system, in presence of the regulator. Remember our system is completely described by the

$$\Omega_k[j] = \log Z_k[j] \tag{1.25}$$

which can be defined as the Helmoltz free energy of the system. From there the Legendre transform of this function gives

$$\Gamma'_k[\langle\phi\rangle] + \Omega_k[j] = \int \langle\phi\rangle \cdot j \tag{1.26}$$

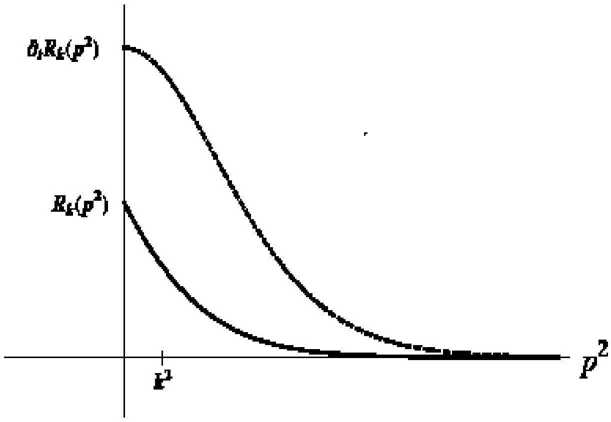

Figure 1.1: Regularizator and derivatives general shape.

called the Effective Average Action, where by definition

$$\langle\phi\rangle(\vec{x}) = \frac{\delta\Omega_k[j]}{\delta j(\vec{x})} \tag{1.27}$$

In general it is easy to verify that the limit $k \to 0$ of this equation restore the $\Gamma'_k \to \Gamma$ limit, defining the original Gibbs Free Energy. In the limit $k \to \Lambda$ instead (1.19) is not respected, so one introduces the modified Legendre transform in order to recover the expected limit.

$$\Gamma'_k[\langle\phi\rangle] + \Omega_k[j] = \int \langle\phi\rangle \cdot j - \frac{1}{2}\Delta S_k \tag{1.28}$$

1.4.3 Flow equations

In presence of the regulator, all thermodynamic functional depend on k. What turns out to be extremely important for this theory are the consequential flow equations. These equations give the evolution of this redefined thermodynamic functional while k is varying. The first one was derived by Polchinski [14] and it is expressed as

$$\partial_k\Omega_k = -\frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \partial_k[R_k]_{ij}(\mathbf{x}-\mathbf{y})\{\frac{\delta^2\Omega_k}{\delta j_i(\mathbf{x})\delta j_j(\mathbf{y})} + \frac{\delta\Omega_k}{\delta j_i(\mathbf{x})}\frac{\delta\Omega_k}{\delta j_j(\mathbf{y})}\} \tag{1.29}$$

Naturally, varying k is exactly the counterpart operation with respect to the shell integration in the Wilson formalism. Within the NPRG equations we still need to rescale the quantities as in the RG formalism in order to reach a fixed point in the couplings space. This turns out to be an extremely important step of the theory and we will come back to it.

Researchers then found extremely worthy to express the same equation in terms of the effective action. Knowing that this last is the Legendre transform of the Ω_k functional, through a mathematical derivation [4] , the equation reads

$$\partial_k \Gamma_k = -\frac{1}{2} \int_{\mathbf{x},\mathbf{y}} \partial_k [R_k]_{ij} (\mathbf{x} - \mathbf{y}) [\Gamma_k^{(2)} + R_k]^{-1}_{ji} (\mathbf{x}, \mathbf{y}) \tag{1.30}$$

which was first derived by Wetterich [15]. Both equations (1.29) and (1.30) are exact.

Initial conditions for these partial differential equations corresponds to the original microscopic model, so basically what was described by the action A 1.11. Deriving each equations w.r.t. the fields involved in the system, one can deduce the flow of the corresponding correlation function (or the 1 particle irreducible correlation function), and all their functional derivatives. As example we show here the equation for $\Gamma^{(2)}$ flow

$$\begin{aligned} \partial_s \bar{\Gamma}^{(2)}_{mn}(\mathbf{p}) = \int_{\mathbf{q}} \left[\partial_s \bar{R}\right]_{ij} (\mathbf{q}) \bar{G}^{(2)}_{jk}(\mathbf{q}) \Bigg[&- \frac{1}{2} \bar{\Gamma}^{(4)}_{klmn}(\mathbf{q}, -\mathbf{q}, \mathbf{p}) \\ &+ \bar{\Gamma}^{(3)}_{kms}(\mathbf{q}, \mathbf{p}) \bar{G}^{(2)}_{st}(\mathbf{q} + \mathbf{p}) \bar{\Gamma}^{(3)}_{tnl}(\mathbf{q} + \mathbf{p}, -\mathbf{p}) \Bigg] \bar{G}^{(2)}_{li}(\mathbf{q}) \end{aligned} \tag{1.31}$$

1.4.4 Rescaling and fixed point solutions

It is now important to explain the relation between scale invariance and fixed point solutions of the NPRG flow equations. In a critical theory, the correlation functions and fields should exhibit scale invariance A scale transformation is mathematically expressed as

$$\phi(\vec{x}, t) \to b^{d_j} \phi_j (b^z t, b\vec{x}) \tag{1.32}$$

where the d_j and z are exponents corresponding to the critical theory studied. This change of variable could be applied to each function involved. Knowing that the partition function Z_k is invariant under the change of dummy variables and taking the infinitesimal limit $b = e^\epsilon = 1 + \epsilon + o(\epsilon^2)$, we can write

$$\langle \delta_\epsilon S \rangle_j + \langle \delta_\epsilon \Delta S_k \rangle_j = j \cdot \delta_\epsilon \phi \tag{1.33}$$

The wonderful idea here is that, choosing an appropriate shape of the regulator we can write

$$\delta_\epsilon \Delta S_k = -\epsilon \partial_s \Delta S_k \tag{1.34}$$

where s is the so called renormalization time $s = \log \frac{k}{\Lambda}$. This mathematical expression state that, with this kind of regulator, the variation due to dilatation of space-time and fields is equal to minus the variation due to a dilatation of the renormalization scale. Now, deriving the partition function by s, we obtain

$$\langle \delta_\epsilon \Delta S_k \rangle = -\epsilon \partial_s \Omega_k[j]|_j \tag{1.35}$$

Resuming, we have

$$\epsilon \partial_s \Omega_k + \langle \delta_\epsilon S_k \rangle_j = \epsilon j \cdot \left[(d_j + \vec{x} \cdot \partial_x + zt\partial_t) \frac{\delta \Omega_k[j]}{\delta j} \right] \tag{1.36}$$

In our treatment we will use the equivalent expression for the Γ_k function, so, taking the Legendre transform

$$\epsilon \frac{\delta \Gamma_k}{\delta \Phi} (d_j + \vec{x} \cdot \partial_x + zt\partial_t) \cdot \Phi = \langle \delta_\epsilon S_k \rangle_j - \epsilon \partial_s \Gamma_k[\Phi] \tag{1.37}$$

This equation is the Ward Identity related to dilatations for the Γ_k function. The lack of scale invariance is then due to the microscopic action's invariance-breaking terms and to the presence of the regulator. In presence of this last, one is forced to introduce dimensionless quantities to recover scale invariance. In particular, a theory is called critical if its microscopic action is scale invariant. So, if we move to adimensional variables

$$\langle \delta_\epsilon S_k \rangle_j = \epsilon \partial_s \hat{\Gamma}_k[\hat{\Phi}] \tag{1.38}$$

we should obtain $\langle \delta_\epsilon S_k \rangle_j = 0$ for $k \to 0$. Mathematically eq. (1.38) shows the equivalence between the presence of a fixed point of the flow equation and the presence of a criticality.

1.4.5 Ward Identities

Let's focus on the definition of a Ward Identity. The study we perform here heavily rely on the Ward Identities, which relate symmetries of Γ_k and of the microscopic action. Thanks to the presence of the regulator ΔS_k we can derive Ward identities for any change of variable $\phi \to \phi'$ which is at most linear in the field. This statement has been already expressed considering the variation of Γ_k under a change of variables. So, if we are dealing with a change linear in fields, the partition function undergoes the following variation

$$\begin{aligned} Z_k[j] = \int \mathbf{D}[\phi'] e^{-S[\phi'] - \Delta S_k[\phi'] + j\cdot[\phi']} = \\ \int \mathbf{D}[\phi] e^{-S[\phi] - \Delta S_k[\phi] + j\cdot[\phi]} e^{\delta(S+\Delta S_k)[\phi] + j\cdot\phi} = \\ Z_k[j] \langle e^{\delta(S+\Delta S_k)[\phi] + j\cdot\phi} \rangle \end{aligned} \tag{1.39}$$

The partition function cannot change under changes of dummy internal variables, so one deduces

$$1 = \langle e^{\delta(S+\Delta S_k)[\phi] + j\cdot\phi} \rangle \tag{1.40}$$

Under the invariance of both the action and the regulator, and the existence of a dual operator which implement the dual field transformation on the current $\bar{A} : j \to j'$, it is easy to prove that

$$\Gamma_k[\bar{A}j] = \Gamma_k[j] \tag{1.41}$$

so, symmetries of the action are also symmetries for Γ_k.

Using the EEA formalism, we can also push forward the transformation consequences. If the transformation relies on an infinitesimal parameter, at linear order in ϵ expansion, we find

$$\langle \delta_\epsilon S \rangle_j + \langle \delta_\epsilon \Delta S_k \rangle_j = j \delta_\epsilon \phi \tag{1.42}$$

Recalling the exact definition of the regulator and of the Γ function, this transformation can be rewritten such as

$$\delta_\epsilon \Gamma_k[\phi] = \langle \delta_\epsilon S \rangle_j + \epsilon \mathrm{Tr}[R_k \cdot T \cdot G^{(2)}[\phi]] \tag{1.43}$$

where T expresses the field variation consequantial to the variables transformation.

Equation (1.43) conserves two important results. In case of exact symmetries for both the action and the regulator, we recover he result already obtained in (1.41). If the change of the action is linear in the field, we obtain an extended version of our Ward Identities, mathematically expressed as

$$\delta S[\langle \Theta \rangle, \langle \tilde{\Theta} \rangle, \langle \vec{v} \rangle] = \langle \delta \mathbf{J} \rangle \tag{1.44}$$

This equality is of fundamental interest for our work.

Chapter 2

Symmetries and Ward Identities

In this chapter we will complete the first and more exhaustive step of our work. We will consider and prove some symmetries for the action associated to the stochastic Kraichnan equation and we will then derive the corresponding Ward Identities. Once those are written, we are ready to set up the flow equations for correlation and vertex functions of our model.

2.1 Symmetries of Kraichnan equation

The first global symmetries considered are very well known. Among these we find

- Space or time translation $\vec{x} \to \vec{x} + \vec{\rho}$, t $\to$ t + τ
- Rotation $\vec{v}(\vec{x}, t) \to A\vec{v}(A^{-1}\vec{x}, t)$
- Field shift $\Theta \to \Theta$ + u
- Field Parity $\Theta \to -\Theta$ and $\tilde{\Theta} \to -\tilde{\Theta}$
- Galilean transformation $\vec{x} \to \vec{x} + \vec{c}$t, t $\to$ t' = t, $\vec{v} \to \vec{v}' = \vec{v} - \vec{c}$

Then, the real step made in this work was to look for extended (or local) symmetries of the equation. Basically we search for invariances under time-gauged transformations, which, as explained in the theoretical section, should generalize the constraints expressed by the Ward Identities. We have studied

- Field time dependent shift $\Theta \to \Theta$ + u(t)
- Response field time dependent shift $\tilde{\Theta} \to \tilde{\Theta} + \epsilon(t)$

- Galilean gauged transformation $\vec{x} \to \vec{x} + c(t)$, t → t' = t, $\vec{v} \to \vec{v}' = \vec{v} - \dot{\vec{c}}$(t)
- Time-gauged rotation

$$\begin{aligned}\delta v_a =& -\epsilon_{abf}x_b\dot{\eta_f}(t) - \epsilon_{abf}v_b\eta_f(t) + \epsilon_{\gamma ln}x_\gamma\partial_l v_a\eta_n(t)\\ \delta\Theta =&\epsilon_{\gamma ln}x_\gamma\partial_l\Theta\eta_n(t)\end{aligned}$$

2.2 Extended Symmetries

Galilean gauged symmetry

It is not worthy to treat here the proof for the space, time translastions and parity invariances. We focus on Galilean gauged symmetry, which will turn out to be extremely useful for our work later on. Physically, this symmetry corresponds to a change from an initial reference frame to one moving at a velocity $\dot{c}(t)$.

Let's write the change of the scalar field under these coordinate transformation

$$\delta\Theta = \Theta'(x,t) - \Theta(x,t) = \dot{\vec{c}}(t)\cdot\nabla\Theta \tag{2.1}$$

We will call from now on *Galilean Scalar densities* all the quantities whose variation "$\delta\Theta$" under coordinate transformation can be rewritten $\dot{\vec{c}}(t)\cdot\nabla\Theta$. Since the action is an integral expression over space-time variables and considering the field physical (so evanescent at infinites), then, this variation vanishes when integrated over space.

Recalling the Kraichnan action

$$A[\Theta,\tilde{\Theta}] = \int_x\int_t \tilde{\Theta}(\partial_{\mathrm{t}}\Theta + v\cdot\nabla\Theta - \nu\Delta\Theta) + \frac{1}{2}\tilde{\Theta}D_f\tilde{\Theta} - \frac{1}{2}\vec{v}D_v^{-1}\vec{v} \tag{2.2}$$

we can express the variation under galilean gauged transform as

$$\delta A[\Theta,\tilde{\Theta}] = -\int_{\vec{x}}\int_t \delta\tilde{\Theta}(\cdot)\Theta + \delta(v_\beta\partial_\beta)\Theta + \tilde{\Theta}(\cdot)\delta\Theta + \delta\tilde{\Theta}D_f\tilde{\Theta} - \delta\vec{v}D_v^{-1}\vec{v} \tag{2.3}$$

and, knowing that the total derivative $D_t = \partial_t + v_\beta\partial_\beta$ is the covariant derivative, we obtain

$$\delta A = 0 \tag{2.4}$$

which states the invariance of the action under this transform. Naturally, this treatment is valid for any $\vec{c}(t) \propto t$ linear in t, such as the usual galilean transformation.

Time Shift Invariance

This transfomation is expressed in the following form

$$\Theta(x,t) \to \Theta(x,t) + u(t) \tag{2.5}$$

so yields the field variation

$$\delta\Theta(x,t) = u(t) \tag{2.6}$$

and, this time, the action variation is not null, but

$$\delta A = \int_{\vec{x},t} \tilde{\Theta}\partial_t u(t) \tag{2.7}$$

is linear in fields. This, as explained in Chapter 1, is an extended symmetry of Γ and from this we can derive another extremely useful Ward Identity.

Response field time Shift Invariance

This symmetry, cannot be worked out for the Kraichnan Equation. This transformation is expressed in the following form

$$\tilde{\Theta}(x,t) \to \tilde{\Theta}(x,t) + \epsilon(t) \tag{2.8}$$

so yields the field variation

$$\delta\tilde{\Theta}(x,t) = \epsilon(t) \tag{2.9}$$

and, the action variation is not null, but

$$\delta A = \int_{\vec{x},t} \epsilon(t)(\partial_t\Theta) + \epsilon(t)D_f\tilde{\Theta} \tag{2.10}$$

is also linear in fields. This is another extended symmetry of Γ.

3D Gauged rotation

Let's show the rotational invariance of Kraichnan action:

$$\int_{x,t} \tilde{\Theta}[\partial_t + v_b\partial_b - \frac{\nu}{2}\Delta]\Theta + \frac{1}{2}\tilde{\Theta}D_{\tilde{\Theta}}\tilde{\Theta} + \frac{1}{2}v_a D_v^{-1} v_a \tag{2.11}$$

The fields vary in the following way:

$$\begin{aligned} \delta v_a =& -\epsilon_{abf}x_b\dot{\eta_f}(t) - \epsilon_{abf}v_b\eta_f(t) + \epsilon_{\gamma ln}x_\gamma\partial_l v_a\eta_n(t) \\ \delta\Theta =& \epsilon_{\gamma ln}x_\gamma\partial_l\Theta\eta_n(t) \end{aligned} \tag{2.12}$$

Plugging this variations in the action, we found, after computations:

$$\delta A = -\frac{1}{2}\int_{x^d,x'^d,t} \epsilon_{abf}x_b\dot{\eta_f}(t)D_v^{-1}v_a + \epsilon_{abf}x'_b\dot{\eta_f}(t)D_v^{-1}v_a \tag{2.13}$$

which is a linear variation in the fields.

2.3 Ward Identities

Each time the action variation is linear in fields or null we can derive, as proved in Chapter 1, a Ward Identity and through its Fourier transform set up a set of constraint useful for the flow equations.

Here we briefly remind the main steps needed to build the WI. Let's remember the fundamental equality

$$\delta A[\langle\Theta\rangle, \langle\tilde{\Theta}\rangle, \langle\vec{v}\rangle] = \langle\delta\mathbf{J}\rangle \tag{2.14}$$

where

$$j_i(\vec{x}) = \frac{\delta\Gamma}{\delta\phi_i(\vec{x})} \tag{2.15}$$

Let's show here below the list of all functional Ward Identities obtained through the previous symmetries

1. $\int dxdt \left[t(\partial_{x^i}\Theta\frac{\delta\Gamma}{\delta\Theta(x)} + \partial_{x^i}\tilde{\Theta}\frac{\delta\Gamma}{\delta\tilde{\Theta}(x)} + \partial_{x^i}v^j\frac{\delta\Gamma}{v_j(x)}) - \frac{\delta\Gamma}{v_i(x)} \right] = 0$

2. $\int dx \left[\partial_{x^i} \Theta \frac{\delta\Gamma}{\delta\Theta(x)} + \partial_{x^i} \tilde{\Theta} \frac{\delta\Gamma}{\delta\tilde{\Theta(x)}} + \partial_{x^i} v^j \frac{\delta\Gamma}{v_j(x)} + \partial_t \frac{\delta\Gamma}{v_i(x)} \right] = 0$

3. $\int_x \partial_t \tilde{\Theta} + \frac{\delta\Gamma}{\delta\Theta(x)} = 0$

4. $\int_x \left[(\partial_t \Theta) + D_f \tilde{\Theta} - \frac{\delta\Gamma}{\delta\tilde{\Theta}(x)} \right] = 0$

5. $-\frac{1}{2} \int_{\vec{x}} \int_t \Big[\epsilon_{abf} x_b \dot{\eta_f}(t) D_v^{-1} v_a + \epsilon_{abf} x_b' \dot{\eta_f}(t) D_v^{-1} v_a -$
$(\epsilon_{\gamma ln} x_\gamma \partial_l \Theta \eta_n(t)) \frac{\delta\Gamma}{\delta\Theta} - (\epsilon_{\gamma ln} x_\gamma \partial_l \tilde{\Theta} \eta_n(t)) \frac{\delta\Gamma}{\delta\tilde{\Theta}} - (\epsilon_{abf} x_b \dot{\eta_f}(t) -$
$\epsilon_{abf} v_b \eta_f(t) + \epsilon_{\gamma ln} x_\gamma \partial_l v_a \eta_n(t)) \frac{\delta\Gamma}{\delta v_a} \Big] = 0$

which are currently written in real space. In order, they come from

- *Galilean tranformation*
- *Galilean gauged transformation*
- *field time-gauged shift*
- *Response field time-gauged shift*
- *3D time-gauged rotation*

. From here it is then possible to derive an infinite set of ward identities functionally deriving this expression n times by Θ, m times by $\tilde{\Theta}$ and ℓ times by v_a

$$\Gamma^{(n,m,\ell)}_{\alpha_1,...,\alpha_\ell}[\vec{x}_1, ..., \vec{x}_{n+m+\ell}] = \frac{\delta^{n+m+\ell}}{\delta\Theta_1...\delta\Theta_n \delta\tilde{\Theta}_1...\delta\tilde{\Theta}_m \delta v_{\alpha_1}...\delta v_{\alpha_\ell}} \Gamma \tag{2.16}$$

and evaluating at zero fields. Taking the Fourier transform of those expressions, we obtain

1. $\sum_{j=1}^{m+n+\ell} [(k_j^i)(\frac{\partial}{\partial\omega_j})] \tilde{\Gamma}^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k, \omega_k, .., k_j, \omega_j, .., p_k, \omega_k) =$
$\tilde{\Gamma}^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k, \omega_k, .., p = 0, \omega = 0, .., p_k, \omega_k)$
2. $-\sum_{j=1}^{m+n+\ell} [(k_j^i)] \tilde{\Gamma}^{(n;m;\ell)}_{\alpha_1,...,\alpha_\ell}(p_k, \omega_k, .., k_j, \omega + \omega_j, .., p_k, \omega_k) =$
$\omega \tilde{\Gamma}^{(n;m;\ell+1)}_{i,\alpha_1,...,\alpha_\ell}(p_k, \omega_k, .., p = 0, \omega, .., p_k, \omega_k)$
3. $\tilde{\Gamma}^{(m,n,\ell)}(\vec{p_k}) = i\omega \delta_{m1} \delta_{n1} \delta_{\ell 0}$
4. $\tilde{\Gamma}^{(m,n,\ell)}(\vec{p_k}) = D_f \delta_{m0} \delta_{n2} \delta_{\ell 0}$

5. $-\epsilon_{ab}\omega\frac{\partial}{\partial q^b}\tilde{\Gamma}^{(n,m,\ell+1)}_{a,\alpha_1,...,\alpha_{\ell+1}}(\vec{p}=0,\omega,\{p_k\}_i^{\ell}) =$
$\sum_{j=1}^{m+n+\ell}(\delta_{a\alpha_j}+\delta_{a\alpha_j}\epsilon_{\gamma\ell}p^{\gamma}\frac{\partial}{\partial q^{\ell}})\tilde{\Gamma}^{(n,m,\ell)}_{\alpha_1,..,\alpha_{\ell+1}}(\{p_k\}_1^{j-1},q_j,\omega+\omega_j,\{p_k\}_{j+1}^{m+n+\ell})$

All these identities valid for the vertex functions, can be written in term of $\bar{\Gamma}$ by extracting the delta function for momentum and frequency conservation. All steps are shown in Appendices. So, the identities read, for galilean time-gauged transformation

1. $\bar{\Gamma}^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p=0,\omega,p_1,\omega_1,...,p_{n+m+l-1},\omega_{n+m+l-1}) =$
$\sum_{j=1}^{m+n+l-1}[\frac{(k^i_j)}{\omega}]\{\bar{\Gamma}^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k)-\bar{\Gamma}^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_i,\omega_i)\}$

Chapter 3

NPRG Equation and the two point correlation function

This chapter is devoted to explain step by step the derivation of the flow equations for the two point two times correlation function of the Kraichnan Model and their solution. We will give in the end some considerations on the result found and possible insights on future developments.

3.1 Brief theoretical introduction

3.2 Functional derivative

Let's start with the exact RG flow equation for Γ in an equilibrium theory for a scalar field Θ.

$$\partial_k \Gamma_k = \frac{1}{2} \int_{\vec{q}} \partial_k R_k(\vec{q})(\Gamma^{(2)}[\Theta] + R_k)^{-1}(\vec{q}, -\vec{q}) \tag{3.1}$$

already written in the Fourier space. We need to take derivatives of this flow equations, so let's first see how to derive the operator over the fields. We proceed in a very standard way

$$\int_{\vec{k}} G^{(2)}_{ij}(\vec{p} - \vec{k}) G^{(2)-1}_{jm}(\vec{k} - \vec{\ell}) = \delta(\vec{p} - \vec{\ell})\delta_{im} \tag{3.2}$$

where

$$G^{(2)}(\vec{q},-\vec{q}) = \left[\Gamma^{(2)}[\Theta] + R_k\right]^{-1}(\vec{q},-\vec{q}) \tag{3.3}$$

So deriving functionally with respect to Θ:

$$\frac{\delta}{\delta(\Theta(\vec{x}))}\int_{\vec{z}} G^{(2)}_{ij}(\vec{y}-\vec{z})G^{(2)-1}_{jm}(\vec{z}-\vec{\ell}) = \int_{\vec{z}}\left(\frac{\delta G^{(2)}_{ij}}{\delta(\Theta(\vec{x}))}G^{(2)-1}_{jm} + G^{(2)}_{ij}\frac{\delta G^{(2)-1}_{jm}}{\delta(\Theta(\vec{x}))}\right) \tag{3.4}$$

and then we obtain:

$$\frac{\delta G^{(2)-1}_{ij}}{\delta(\Theta(\vec{x}))}(\vec{z}-\vec{y}) = \int_{\vec{f}}\int_{\vec{e}} G^{(2)}_{im}(\vec{z}-\vec{f})\frac{\delta\Gamma^{(2)}_{mu}(\vec{f}-\vec{e})}{\delta(\Theta(\vec{x}))}G^{(2)}_{uj}(\vec{e}-\vec{y}) \tag{3.5}$$

Plugging this to various order we obtain what we want, with all delta function where needed. For example:

$$\partial_k\frac{\delta\Gamma_{k,ij}(\vec{\ell}-\vec{y})}{\delta(\Theta(\vec{x}))} = \frac{1}{2}\int_{\vec{f}}\partial_k R_{k,il}(\vec{\ell}-\vec{f})\int_{\vec{e}}\int_{\vec{g}} G^{(2)}_{lm}(\vec{f}-\vec{e})\frac{\delta\Gamma^{(2)}_{mu}(\vec{e}-\vec{g})}{\delta(\Theta(\vec{x}))}G^{(2)}_{uj}(\vec{g}-\vec{y}) \tag{3.6}$$

This can be extended to higher order derivatives.

3.2.1 Ingredients

The Kraichnan model field formalism is expressed through three different fields. This lead us to express our functionals in matrix form and so we will insert in our equations the following expressions. The matrix of $\Gamma^{(2)}$:

$$\mathbf{\Gamma^{(2)}_k} = \begin{bmatrix} 0 & i\omega & 0 \\ -i\omega & D_f(q) & 0 \\ 0 & 0 & D_v^{-1} \end{bmatrix}$$

Each matrix elements express a double functional derivatives in the fields. All those definitions come from the action (1.11).

Then we choose our Regulator to separate fluctuations modes as

$$\mathbf{R_k} = \begin{bmatrix} 0 & R_{k,\Theta\tilde{\Theta}} & 0 \\ R_{k,\tilde{\Theta}\Theta} & R_{k,\tilde{\Theta},\tilde{\Theta}} & 0 \\ 0 & 0 & R_{k,vv} \end{bmatrix}$$

Remembering here the definition of the Propagator for our theory $G_k^{(2)} = \left[\Gamma_k^{(2)} + R_k\right]^{-1}$, it follows

$$\mathbf{G^{(2)}} = \begin{bmatrix} G_{\Theta\Theta} & G_{\Theta\tilde{\Theta}} & 0 \\ G_{\tilde{\Theta}\Theta} & 0 & 0 \\ 0 & 0 & G_{v_\alpha v_\beta} \end{bmatrix}$$

3.2.2 Equation for $\Gamma^{(2)}$

Starting from equation (3.6) we can write the equivalent equations for $\Gamma^{(2),(1,1,0)}$ and $\Gamma^{(2),(0,2,0)}$. Where with this notation $(\cdot)^{(\alpha),(i,j,k)}$ we refer to the $(\Theta, \tilde{\Theta}, \vec{v})$ (i, j, k)-times derivation of the α-order vertex functions. So deriving two times (3.6) we obtain:

$$\begin{aligned} \partial_k \Gamma_k^{(2),(0,2,0)}(\mathbf{p}) = -\frac{1}{2}\int_{\mathbf{q}} \partial_k R_k(\mathbf{q})_{ij} G_{jl}^{(2)}(\mathbf{q}) \Gamma_{lk}^{(2),(0,2,0)}(\mathbf{q}, -\mathbf{q}, \mathbf{p}) G_{ni}^{(2)}(\mathbf{q}) + \\ \partial_k R_k(\mathbf{q})_{ij} G_{jl}^{(2)}(\mathbf{q}) \Gamma_{ln}^{(2),(0,1,0)}(\mathbf{q}, \mathbf{p}) G_{np}^{(2)}(\mathbf{p}+\mathbf{q}) \Gamma_{pq}^{(2),(0,1,0)}(\mathbf{q}+\mathbf{p}, -\mathbf{p}) G_{qi}^{(2)}(\mathbf{q}) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \partial_k \Gamma_k^{(2),(1,1,0)}(\mathbf{p}) = -\frac{1}{2}\int_{\mathbf{q}} \partial_k R_k(\mathbf{q})_{ij} G_{jl}^{(2)}(\mathbf{q}) \Gamma_{lk}^{(2),(1,1,0)}(\mathbf{q}, -\mathbf{q}, \mathbf{p}) G_{ni}^{(2)}(\mathbf{q}) + \\ \partial_k R_k(\mathbf{q})_{ij} G_{jl}^{(2)}(\mathbf{q}) \Gamma_{ln}^{(2),(1,0,0)}(\mathbf{q}, \mathbf{p}) G_{np}^{(2)}(\mathbf{p}+\mathbf{q}) \Gamma_{pq}^{(2),(0,1,0)}(\mathbf{q}+\mathbf{p}, -\mathbf{p}) G_{qi}^{(2)}(\mathbf{q}) \end{aligned} \tag{3.8}$$

which sets to, after taking the trace of this matricial operation:

$$\begin{aligned} \partial_k \Gamma_k^{(0,2,0)} = -\frac{1}{2}\int_{\mathbf{q}} \partial_k R_k(\mathbf{q})_{ij} G_{ji}^{(2)}(\mathbf{q}) \Gamma_{ij}^{(2),(0,2,0)}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) G_{ji}^{(2)}(\mathbf{q}) + \\ \partial_k R_k(\mathbf{q})_{ij} G_{ji}^{(2)}(\mathbf{q}) \Gamma_{ij}^{(2),(0,1,0)}(\mathbf{q}, \mathbf{p}) G_{ji}^{(2)}(\mathbf{p}+\mathbf{q}) \Gamma_{ij}^{(2),(0,1,0)}(\mathbf{q}+\mathbf{p}, -\mathbf{p}) G_{ji}^{(2)}(\mathbf{q}) \end{aligned} \tag{3.9}$$

with (i,j)=(1,2),(2,1),(3,3)

This is the entire expression we should deal with. Clearly this is a non-closed equation which involves 3 and 4 point functions. Of course in this way this is not solvable. We need to resort some approximations and to make use of the previously derived Ward Identities to close them.

3.3 Expansion at large wavenumber

We are going to explain here a strategy already used by [11] to approach turbulence analysis.

The idea comes from the regulator properties. On one side, it limits the wave-number $\vec{q}$ to be of order k or lower in the integral expression. In particular, it follows that at large $\vec{p}$ one has $\vec{q}/\vec{p} \ll 1$. On the other side, the regulator force the Γ function to be smooth, so it can be expanded in a Taylor series in the $\vec{q}/\vec{p}$ variable. The vertex functions are function of the internal wave-number only through scale invariant ratios, such as $\vec{q}/\vec{p}$. So it is mathematically equal to evaluate expansion in $\vec{q} \sim 0$ or in $\vec{p} \to \infty$.

In this limit as in [11] results are more conveniently expressed in terms of the flow of the connected correlation function

$$\partial_k \Omega_k = -\frac{1}{2} \mathrm{Tr} \left[\partial_k R_k \cdot (G^{(2)} + \phi \otimes \phi) \right] \tag{3.10}$$

It is fundamental to remember that the Taylor expansion is valid only for Γ functions, since they only depend on internal wave number, while generic $G^{(n)}$ functions not. In this case we have firstly to express all our $G^{(n)}$ as functions of Γ and then to make use of the approximation on those lasts.

So now, let's focus on the real results for our example. Basically any Γ can be rewritten such as

$$\Gamma^{(n)}(\vec{q}, \omega, ...) = \Gamma^{(n)}(\vec{q} = 0, \omega, ...) + o(\vec{q}) \tag{3.11}$$

So we can approximate any $\Gamma^{(n)}(\vec{q}, \omega, ...)$ of equations like (3.9) by the corresponding $\Gamma^{(n)}(\vec{q} = 0, \omega, ...)$. We will make use of this result in the next section.

Closing flow equation for $\Gamma^{(0,2,0)}$

Now we are ready to close the flow equations. We are not going to write each time the dependence of any function on k.

First of all remember expression for the shift in time Ward Identity:

$$\Gamma^{(m,n,\ell)}(\vec{p_k}) = i\omega\delta_{m1}\delta_{n1}\delta_{\ell 0} \tag{3.12}$$

which allows us to delete all terms independent on derivatives in Θ and $\tilde{\Theta}$ of (3.9). So we consider from now on the expression

$$\partial_k\Gamma_k^{(0,2,0)} = -\frac{1}{2}\int_{\mathbf{q}} \partial_k R_k(\mathbf{q})_{v_\alpha v_\gamma} G^{(2)}_{v_\gamma v_\delta}(\mathbf{q})\Gamma^{(2),(0,2,0)}_{v_\delta v_\epsilon}(\mathbf{p},-\mathbf{p},\mathbf{q})G^{(2)}_{v_\epsilon v_\alpha}(\mathbf{q})+$$
$$\partial_k R_k(\mathbf{q})_{v_\alpha v_\gamma} G^{(2)}_{v_\gamma v_\delta}(\mathbf{q})\Gamma^{(2),(0,1,0)}_{v_\delta v_\epsilon}(\mathbf{q},\mathbf{p})G^{(2)}_{v_\epsilon v_\mu}(\mathbf{p}+\mathbf{q})\Gamma^{(2),(0,1,0)}_{v_\mu v_\nu}(\mathbf{q}+\mathbf{p},-\mathbf{p})G^{(2)}_{v_\nu v_\alpha}(\mathbf{q}) \tag{3.13}$$

The overwritten equation is not closed. We implement here the large wave number expansion and the constraints coming from to the Ward Identities previously found to close them. In particular we will plenty use the Galilean Gauged Ward Identity.

Higher order Γ function can be rewritten as

$$\begin{aligned}
&\Gamma^{(0,2,2)}_{v_\alpha v_\beta}(\mathbf{q},-\mathbf{q},\mathbf{p}) = \Gamma^{(0,2,2)}_{v_\alpha v_\beta}(\mathbf{q_1}=\mathbf{0},\omega_\mathbf{1},\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu) = \\
&-\frac{q_{2\alpha}}{\omega}\Gamma^{(0,2,1)}_{v_\beta}(\mathbf{q_2},\omega_\mathbf{2}+\omega,\mathbf{p},\nu) - \frac{p_\alpha}{\omega}\Gamma^{(0,2,1)}_{v_\beta}(\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu+\omega) \\
&+\frac{p_\alpha+q_{2\alpha}}{\omega}\Gamma^{(0,2,1)}_{v_\beta}(\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu)
\end{aligned} \tag{3.14}$$

and lowering once more the order, putting then $q_1 = -q_2 = 0$, we obtain:

$$\begin{aligned}
&\Gamma^{(0,2,2)}_{v_\alpha v_\beta}(\mathbf{q}=\mathbf{0},\omega,-\mathbf{q}=\mathbf{0},-\omega,\mathbf{p},\nu) = \\
&-\frac{p_\alpha p_\beta}{\omega^2}(-\Gamma^{(0,2,0)}(\omega+\nu,\mathbf{p})+2\Gamma^{(0,2,0)}(\nu,\mathbf{p})-\Gamma^{(0,2,0)}(-\omega+\nu,\mathbf{p}))
\end{aligned} \tag{3.15}$$

We apply the same deduction on the two 3rd order Γ term of the second line of equation (3.21). So:

$$\begin{aligned}
&\Gamma^{(0,1,2)}_{v_\alpha v_\beta}(\mathbf{q}=\mathbf{0},\omega,\mathbf{p},\nu) = \\
&-\frac{p_\alpha}{\omega}(\Gamma^{(0,1,1)}_{v_\beta}(\omega+\nu,\mathbf{p})-\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p}))
\end{aligned} \tag{3.16}$$

and

$$\Gamma^{(0,1,2)}_{v_\alpha v_\beta}(\mathbf{q}=\mathbf{0}+\mathbf{p},\omega+\nu,-\mathbf{q}=\mathbf{0},-\omega)= \\ \frac{p_\alpha}{\omega}(\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p})-\Gamma^{(0,1,1)}_{v_\beta}(\nu+\omega,\mathbf{p})) \tag{3.17}$$

So, then we can write the complete expression:

$$\partial_k\Gamma^{(0,2,0)}_k=-\frac{1}{2}\int_{\mathbf{q}}\partial_k R_k(\mathbf{q})_{v_\delta v_\gamma}G^{(2)}_{v_\gamma v_\alpha}(\mathbf{q})(-\frac{p_\alpha p_\beta}{\omega^2}(-\Gamma^{(0,2,0)}(\omega+\nu,\mathbf{p})+2\Gamma^{(0,2,0)}(\nu,\mathbf{p})- \\ \Gamma^{(0,2,0)}(-\omega+\nu,\mathbf{p})))G^{(2)}_{v_\beta v_\delta}(\mathbf{q})+\partial_k R_k(\mathbf{q})_{v_\delta v_\gamma}G^{(2)}_{v_\gamma v_\alpha}(\mathbf{q})(\frac{p_\alpha}{\omega}(\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p})- \\ \Gamma^{(0,1,1)}_{v_\beta}(\nu+\omega,\mathbf{p}))-\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p}))G^{(2)}_{v_\beta v_\delta}(\mathbf{p}+\mathbf{q})(\frac{p_\delta}{\omega}(\Gamma^{(0,1,1)}_{v_\mu}(\nu,\mathbf{p}) \\ -\Gamma^{(0,1,1)}_{v_\mu}(\nu+\omega,\mathbf{p}))G^{(2)}_{v_\mu v_\delta}(\mathbf{q}) \tag{3.18}$$

Or, in a concise way

$$\partial_k\Gamma^{(0,2,0)}_k=-\frac{1}{2}\int_{\mathbf{q}}\tilde{\partial}G^{(2)}_{v_\beta v_\alpha}(\mathbf{q})(-\frac{p_\alpha p_\beta}{\omega^2}(-\Gamma^{(0,2,0)}(\omega+\nu,\mathbf{p})+2\Gamma^{(0,2,0)}(\nu,\mathbf{p})- \\ \Gamma^{(0,2,0)}(-\omega+\nu,\mathbf{p})))+\tilde{\partial}G^{(2)}_{v_\mu v_\alpha}(\mathbf{q})(\frac{p_\alpha}{\omega}(\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p})- \\ \Gamma^{(0,1,1)}_{v_\beta}(\nu+\omega,\mathbf{p}))-\Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p}))G^{(2)}_{v_\beta v_\delta}(\mathbf{p}+\mathbf{q})(\frac{p_\delta}{\omega}(\Gamma^{(0,1,1)}_{v_\mu}(\nu,\mathbf{p}) \\ -\Gamma^{(0,1,1)}_{v_\mu}(\nu+\omega,\mathbf{p})) \tag{3.19}$$

3.3.1 Close flow equation for $\Gamma^{(1,1,0)}$

The same reasoning can be done for the second equation of interest

$$\partial_k\Gamma^{(1,1,0)}_k=-\frac{1}{2}\int_{\mathbf{q}}\partial_k R_k(\mathbf{q})_{v_\alpha v_\gamma}G^{(2)}_{v_\gamma v_\delta}(\mathbf{q})\Gamma^{(2),(1,1,0)}_{v_\delta v_\epsilon}(\mathbf{p},-\mathbf{p},\mathbf{q})G^{(2)}_{v_\epsilon v_\alpha}(\mathbf{q})+ \\ \partial_k R_k(\mathbf{q})_{v_\alpha v_\gamma}G^{(2)}_{v_\gamma v_\delta}(\mathbf{q})\Gamma^{(2),(1,0,0)}_{v_\delta v_\epsilon}(\mathbf{q},\mathbf{p})G^{(2)}_{v_\epsilon v_\mu}(\mathbf{p}+\mathbf{q})\Gamma^{(2),(0,1,0)}_{v_\mu v_\nu}(\mathbf{q}+\mathbf{p},-\mathbf{p})G^{(2)}_{v_\nu v_\alpha}(\mathbf{q}) \tag{3.20}$$

Following entirely the same procedure we are able to obtain

$$\partial_k \Gamma_k^{(1,1,0)} = -\frac{1}{2}\int_{\mathbf{q}} \tilde{\partial} G^{(2)}_{v_\beta v_\alpha}(\mathbf{q})(-\frac{p_\alpha p_\beta}{\omega^2}(-\Gamma^{(1,1,0)}(\omega+\nu,\mathbf{p}) + 2\Gamma^{(1,1,0)}(\nu,\mathbf{p}) -$$
$$\Gamma^{(1,1,0)}(-\omega+\nu,\mathbf{p}))) + \tilde{\partial} G^{(2)}_{v_\mu v_\alpha}(\mathbf{q})(\frac{p_\alpha}{\omega}(\Gamma^{(1,0,1)}_{v_\beta}(\nu,\mathbf{p}) - \tag{3.21}$$
$$\Gamma^{(1,0,1)}_{v_\beta}(\nu+\omega,\mathbf{p})) - \Gamma^{(0,1,1)}_{v_\beta}(\nu,\mathbf{p}))G^{(2)}_{v_\beta v_\delta}(\mathbf{p}+\mathbf{q})(\frac{p_\delta}{\omega}(\Gamma^{(0,1,1)}_{v_\mu}(\nu,\mathbf{p})$$
$$-\Gamma^{(0,1,1)}_{v_\mu}(\nu+\omega,\mathbf{p}))$$

We state here only the results of this second computation which details are present in appendices.

So, now the flow equations are closed. Let us stress here an important detail about the equations found. We started this analysis with the goal of understanding the intermittency problem. In particular it would been nice to capture the behaviour of *equal-time* effects due to intermittency, in oder to compare results with the Kolmogorov analysis. This first approach prevents us in doing this. What we obtained are equation for evolution of two point two time correlations functions. Our equations vanish at equal time, so the analysis of intermittency, related to equal time quantities cannot be performed here. This effect can be captured by computing the next term in the large wave expansion.

3.3.2 Closed flow equation for $G^{(2)}_{\Theta,\Theta}$

Let's focus on the main character of our work, the two point two time correlation function. We express here the physical meaning of this function in the Kraichnan model

$$G^{(2)}_{\Theta\Theta}(\vec{x},\vec{x}',t,t') = \langle\Theta(\vec{x}',t')\Theta(\vec{x},t)\rangle - \langle\Theta(\vec{x}',t')\rangle\langle\Theta(\vec{x},t)\rangle \tag{3.22}$$

which is the correlator between two values of Θ at different space points and different times. We know the expression for $G^{(2)}_{\Theta,\Theta}$ in terms of Γ function

$$G^{(2)}_{\Theta\Theta} = \left[\Gamma^{(1,1,0)}_{\Theta\tilde{\Theta}}\right]^{-1}\left[\Gamma^{(0,2,0)}_{\tilde{\Theta}\tilde{\Theta}}\right]\left[\Gamma^{(1,1,0)}_{\tilde{\Theta}\Theta}\right]^{-1} \tag{3.23}$$

and the expression of G correlation function as function of $\Gamma_k + R_k$. So exploiting each term as fuction of G we obtain

$$\partial_k \Gamma_k^{(2)}(\mathbf{p}) = \partial_k [G^{(2)}]^{-1}(\mathbf{p}) \tag{3.24}$$

where we can omit de dependance on R_k.

Recalling the expression for the derivative of the inverse function, explicitely written in the beginning of this chapter, we express

$$\partial_k G^{(2)}_{k,i\ell}(\mathbf{q}-\mathbf{z}) = -\int_{kp} G^{(2)}_{k,iu}(\mathbf{q}-\mathbf{k})\partial_k [G^{(2)}]^{-1}_{k,un}(\mathbf{k}-\mathbf{p}) G^{(2)}_{k,n\ell}(\mathbf{p}-\mathbf{z}) \tag{3.25}$$

and so, substituting what we found previously and summing over proper indices (do not forget this is a matricial operation, pay attention to lower indices)

$$\begin{aligned}\partial_k G^{(2)}_{k,\Theta\Theta}(\mathbf{p}) = &-\int_{\mathbf{q}} G^{(2)}_{k,\Theta\tilde{\Theta}}(\mathbf{q})[\partial_k \Gamma^{(2)}]_{k,\tilde{\Theta}\Theta}(\mathbf{p}) G^{(2)}_{k,\Theta\Theta}(\mathbf{q}) + \\ &G^{(2)}_{k,\Theta\Theta}(\mathbf{q})[\partial_k \Gamma^{(2)}]_{k,\Theta\tilde{\Theta}}(\mathbf{p}) G^{(2)}_{k,\tilde{\Theta}\Theta}(\mathbf{q}) + G^{(2)}_{k,\Theta\tilde{\Theta}}(\mathbf{q})[\partial_k \Gamma^{(2)}]_{k,\tilde{\Theta}\tilde{\Theta}}(\mathbf{p}) G^{(2)}_{k,\tilde{\Theta}\Theta}(\mathbf{q})\end{aligned} \tag{3.26}$$

and, plugging in all the expressions found found for the equations for Γ, we reach

$$\begin{aligned}\partial_k G^{(2)}_{k,\Theta\Theta}(\mathbf{p}) = -\int_{\mathbf{q}} G^{(2)}_{k,\Theta\tilde{\Theta}}(\mathbf{q})\{-\frac{1}{2}\int_{\mathbf{k}} \partial_k R_k G^{(2)}_{k,v_\alpha v_\gamma} G^{(2)}_{k,v_\gamma v_\beta}(\frac{p^\alpha p^\beta}{\omega^2} G^{-1}(\omega+\nu,\mathbf{p}) - 2G^{-1} \\ (\nu,\mathbf{p}) + G^{-1}(-\omega+\nu,\mathbf{p}))_{k,\tilde{\Theta}\Theta}(\mathbf{p})\} G^{(2)}_{k,\Theta\Theta}(\mathbf{q}) +\end{aligned} \tag{3.27}$$

then, exploiting the product between inverse functions $\int_{\mathbf{p}} G^{(2)-1}_{ij}(\omega+\nu,\mathbf{p}) G^{(2)}_{jk}(\mathbf{q},\mu) = \delta(\mathbf{p}+\mathbf{q})\delta(\omega+\nu+\mu)\delta_{ik}$, the final result is

$$\begin{aligned}\partial_k G^{(2)}_{k,\Theta\Theta}(\mathbf{p},\nu) = \frac{3}{2}\int_{\mathbf{k},\omega} \partial_k R_{k,v_\alpha v_\gamma}(\mathbf{k}) G^{(2)}_{k,v_\gamma v_\epsilon}(\mathbf{k},\omega) G^{(2)}_{k,v_\epsilon v_\beta}(\mathbf{k},\omega) \\ \left[\frac{p^\alpha p^\beta}{\omega^2}(G_{k,\Theta\Theta}(-\omega-\nu,-\mathbf{p}) - 2G_{k,\Theta\Theta}(-\nu,-\mathbf{p})) + G_{k,\Theta\Theta}(\omega-\nu,-\mathbf{p})\right]\end{aligned} \tag{3.28}$$

and with the shorter notation

$$\partial_k G^{(2)}_{k,\Theta\Theta}(\mathbf{p},\nu) = \frac{3}{2}\int_{\mathbf{k},\omega} \tilde{\partial}_k G^{(2)}_{k,v_\alpha v_\beta}(\mathbf{k},\omega) \\ \left[\frac{p^\alpha p^\beta}{\omega^2}(G_{k,\Theta\Theta}(-\omega-\nu,-\mathbf{p}) - 2G_{k,\Theta\Theta}(-\nu,-\mathbf{p})) + G_{k,\Theta\Theta}(\omega-\nu,-\mathbf{p})\right] \tag{3.29}$$

It has been shown it is much easier to compute a solution of this equation when rewritten in a mixed representation in time and momentum coordinates. Exploiting the Fourier time transform and taking the kernel of our equation, the closed equation for the $G^{(2)}_{\Theta,\Theta}$ is written as

$$\partial_k G^{(2)}_{\Theta\Theta,k}(\vec{p},t) = -G^{(2)}_{\Theta\Theta,k}(\vec{p},t)\frac{3}{2}p^\alpha p^\beta \int_{\vec{k},\omega} \tilde{\partial}_s G^{(2)}_{v_\alpha v_\beta,k}(\vec{k},\omega)(\frac{2cos(\omega t)-2}{\omega^2}) \tag{3.30}$$

This equation represent the second central result of my work.

3.4 Fixed point solution

In order to compute the solution of our equation at criticality, we assume to be near a fixed point of the theory. So then we first adimensionalize all involved quantities.

3.4.1 Adimensionalization

Let us recall here the action (1.11)

$$A[\Theta,\tilde{\Theta},\vec{v}] = \int_{\vec{x}}\int_t \tilde{\Theta}(\partial_{\mathrm{t}}\Theta + \vec{v}\cdot\nabla\Theta - \nu\Delta\Theta) + \frac{1}{2}\tilde{\Theta}D_f\tilde{\Theta} - \frac{1}{2}\vec{v}D_v^{-1}\vec{v} \tag{3.31}$$

The adimensionalization constraints takes care of any components of this action. Since the action is dimensionless, all our deduction must leave the action still adimensional. So let's rescale the momentum variables $\vec{q} = k\hat{\vec{q}}$. Asking for the kinetic term of the action to be invariant under rescaling, we obtain also $t = \hat{t}k^{-2}\nu_k^{-1}$, where ν_k is the diffusivity. Then, asking for

$$\begin{aligned} \Theta(\vec{x},t) &= K^{-d_\Theta}\hat{\Theta}(\hat{x},\hat{t}) \\ \tilde{\Theta}(\vec{x},t) &= K^{-d_{\tilde{\Theta}}}\hat{\tilde{\Theta}}(\hat{x},\hat{t}) \end{aligned} \tag{3.32}$$

it's straight forward to show that $d_\Theta + d_{\tilde{\Theta}} = d$.
Moreover, from the term proportional to $D_f = k^{-d_{D_f}}\hat{D}_f$, we obtain the complete adimensionalization of the fields as

$$\begin{aligned} \Theta(\vec{x},t) &= \left(\frac{D_{fk}k^d}{k^2\nu_k}\right)^{\frac{1}{2}}\hat{\Theta}(\hat{x},\hat{t}) \\ \tilde{\Theta}(\vec{x},t) &= \left(\frac{k^2\nu_k k^d}{D_{fk}}\right)^{\frac{1}{2}}\hat{\tilde{\Theta}}(\hat{x},\hat{t}) \end{aligned} \tag{3.33}$$

Last and most important we look for the adimensionalization of the velocity correlator D_v, which will be extremely useful for our equation. Let's recall here the shape

$$D_v(\vec{x},\vec{x}',t,t') = \langle v_a(\vec{x},t)v_b(\vec{x}',t')\rangle = D_0\delta(t-t')\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{i\vec{q}(\vec{x}-\vec{x}')}}{(\vec{q}^2+m^2)^{\frac{d+\xi}{2}}}P_{ab}(\vec{q}) \tag{3.34}$$

Taking the momentum Fourier transform on the left side of this equation we obtain

$$D_v(\vec{q},t,t') = \frac{1}{(2\pi)^d}\frac{P_{ab}(\vec{q})}{(\vec{q}^2+m^2)^{\frac{d+\xi}{2}}}\delta(t-t') \tag{3.35}$$

So adimensionalizing we obtain

$$D_v(\vec{q},t,t') = k^{-d-\xi+2}\nu_k k^{-D_0}\hat{D}_v(\vec{\hat{q}},\hat{t},\hat{t}') \tag{3.36}$$

3.4.2 Solution

Now, firstly integrating, with a proper choice of the regulator

$$R(\vec{q})_{vv} = \frac{q^{\frac{d+\xi}{2}}}{e^{\frac{q^2}{k^2}}-1} \tag{3.37}$$

we obtain

$$\int_{\vec{k},\omega} \tilde{\partial}_s G^{(2)}_{v_\alpha v_\beta,k}(\vec{k},\omega)(\frac{2cos(\omega t)-2}{\omega^2}) = (d-(d+\xi)2^{-2-\frac{d}{4}+\frac{\xi}{4}})\Gamma\left[\frac{d-\xi}{4}\right](-\pi|t|) \quad (3.38)$$

Let's call this value αt from now on. Proceeding in adimensionalization of the flow equation near the fixed point we find

$$(\partial_s - d_G - \hat{p}\cdot\partial_{\hat{p}} + (2-\eta_{\nu_k})\hat{t}\partial_{\hat{t}} - \frac{3}{2}\alpha p^\alpha p^\beta |t|)\hat{G} = 0 \quad (3.39)$$

where η_{ν_k} is the anomalous exponent for diffusivity.

Since this equation is linear, the solution can be derived defining defining the new variable $y = \rho^{2-\eta_{\nu_k}} t$ and knowing that the correlation function does not depend anymore on k at the fixed point, we rewrite

$$(-d_G - \hat{p}\cdot\partial_{\hat{p}} + (2-\eta_{\nu_k})\hat{t}\partial_{\hat{t}} - \frac{3}{2}\alpha|\rho|^{(2-\eta_{\nu_k})}|y|)\hat{G}(y,\rho) = 0 \quad (3.40)$$

We then define the variable $u = \log\rho$, so

$$log_u \hat{G}^{(2)}_{\Theta\Theta}(y,u) = -d_G u - 3\frac{e^{(2\eta_\nu - 2)u}}{2\eta_\nu - 2} y_k \hat{I}_k + \hat{F}^{(2)}_{\Theta\Theta}(y) \quad (3.41)$$

The solution, leaving all subleading terms in p, is

$$G^{(2)}_{\Theta\Theta}(\vec{p},t) \propto e^{-\alpha p^2|t|} \quad (3.42)$$

And the behaviour of α in $d=3$ is shown in the following plot

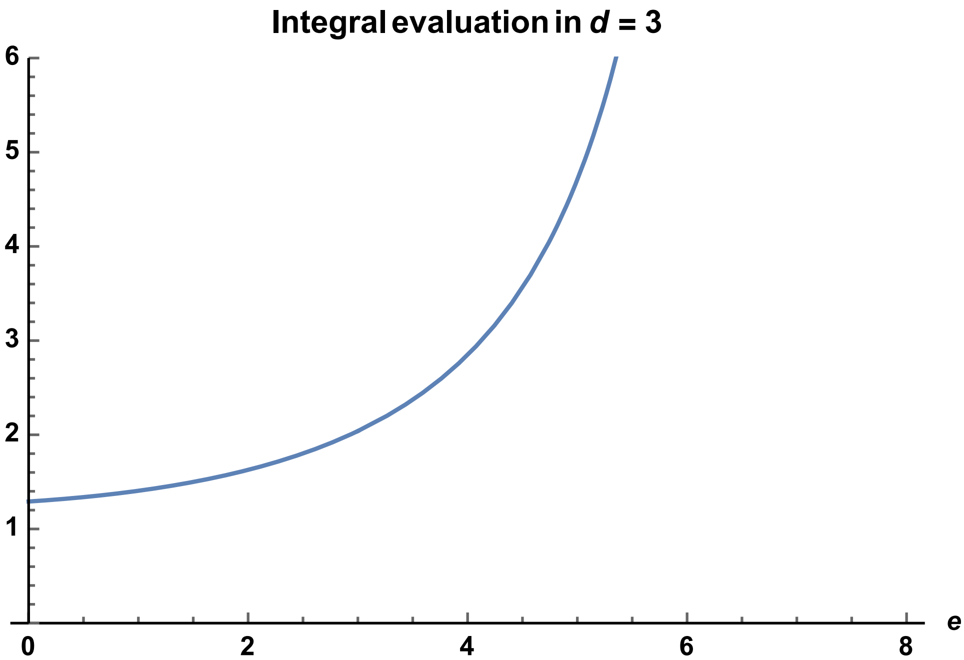

Figure 3.1: Plot of α in d=3.

Conclusion

In the present work we studied extended symmetries for the Kraichnan action. After stating the field formalism associated to this model we carried out some analyses on the symmetries useful to express Ward Identity for the Effective Average Action. In a second step we apply the tools of Nonperturbative Renormalization Group to study the two point two time correlation function of the model. Thanks to the large wave-number expansion and the Ward Identities written we were able to close this equation and to find the solution at the fixed point. Newly discovered symmetries, such as the 3D time gauged rotation invariance of the action could lead to further results in future works.

Appendices

Appendix A

In detail computation for WI

Here we will go through the main steps needed to compute the Fourier transform of the Ward Identities. We will show passages for two of them. The same computation can be carried on over all other identities found.

A.1 Full calculation of n-order Ward identities (Global Galilean)

Let's start with variation in the functional Ward Identity :

$$\int dxdt \left[t(\partial_{x^i}\Theta \frac{\delta\Gamma}{\delta\Theta(x)} + \partial_{x^i}\tilde{\Theta}\frac{\delta\Gamma}{\delta\tilde{\Theta(x)}} + \partial_{x^i}v^j \frac{\delta\Gamma}{v_j(x)}) - \frac{\delta\Gamma}{v_i(x)} \right] = 0 \tag{A.1}$$

Taking n derivatives in Θ, m in $\tilde{\Theta}$, l in v_i, we obtain:

$$\begin{aligned}\int dxdt \Bigg[\sum_{j=1}^{m+n+l} t[\partial_{x^i}\delta(x-x_j)]\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(x_k,t_k,..,x,t,..,x_k,t_k)+ \\ -\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(x_k,t_k,..,x,t,..,x_k,t_k)\Bigg] = 0\end{aligned} \tag{A.2}$$

Fourier transform of all:

$$\int dxdt \sum_{j=1}^{m+n+l} t[\partial_{x^i} \int dk_j d\omega_j e^{-ik_j(x-x_j)+i\omega_j(t-t_j)}]$$
$$\int_{p_1,..,p_{n+m+l-1},p} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t}$$
$$-\int_{p_1,..,p_{n+m+l},p} \Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t} = 0 \tag{A.3}$$

where all integrals are both on momentum and frequencies, then explicitate the derivatives:

$$\int dxdt \sum_{j=1}^{m+n+l} \int dk_j d\omega_j e^{-ik_j(x-x_j)+i\omega_j(t-t_j)} (-ik^i_j)(i\frac{\partial}{\partial\omega})$$
$$\int_{p_1,..,p_{n+m+l},p} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t} -$$
$$\int_{p_1,..,p_{n+m+l+1},p} \Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t} = 0 \tag{A.4}$$

integrate now over x and over t:

$$\sum_{j=1}^{m+n+l} \int dk_j d\omega_j [(-ik^i_j)(i\frac{\partial}{\partial\omega}) e^{+ik_j x_j+i\omega_j(t-t_j)}]$$
$$\int_{p_1,..,p_{n+m+l-1},p} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t} \delta(p-k_j)\delta(\omega-\omega_j) -$$
$$\int_{p_1,..,p_{n+m+l},p} \Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)-i\omega t} \delta(p)\delta(\omega) = 0 \tag{A.5}$$

sum over delta variables:

$$\sum_{j=1}^{m+n+l} \int_{k_j,\omega_j,p_1,..,p_{n+m+l}} (k_j^i)(\frac{\partial}{\partial\omega_j})\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega_j,..,p_k,\omega_k)e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a-\omega_a t_a)+ik_j x_j-i\omega_j(t-t_j)}$$
$$-\int_{p_1,..,p_{n+m+l+1}} \Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p=0,\omega=0,..,p_k,\omega_k)e^{i\sum_{a=1}^{n+m+l}(k_a x_a-\omega_a t_a)} = 0 \tag{A.6}$$

now we obtain, taing the kernel:

$$\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p=0,\omega=0,..,p_k,\omega_k) = \sum_{j=1}^{m+n+l}[(k_j^i)(\frac{\partial}{\partial\omega_j})]\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega_j,..,p_k,\omega_k) \tag{A.7}$$

A.2 Full calculation of n-order Ward identities (Galilean gauged)

Let's start with variation in the functional Ward Identity:

$$\int dx\left[\partial_{x^i}\Theta\frac{\delta\Gamma}{\delta\Theta(x)}+\partial_{x^i}\tilde{\Theta}\frac{\delta\Gamma}{\delta\tilde{\Theta(x)}}+\partial_{x^i}v^j\frac{\delta\Gamma}{v_j(x)}+\partial_t\frac{\delta\Gamma}{v_i(x)}\right]=0 \tag{A.8}$$

Taking n derivatives in Θ, m in $\tilde{\Theta}$, l in v_i, we obtain:

$$\int dx\left[\sum_{j=1}^{m+n+l}[\partial_{x^i}\delta(x-x_j)]\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(x_k,t_k,..,x,t,..,x_k,t_k)+ \partial_t\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(x_k,t_k,..,x,t,..,x_k,t_k)\right]=0 \tag{A.9}$$

Fourier transform of all:

$$\int dx \Bigg[\sum_{j=1}^{m+n+l} [\partial_{x^i} \int dk_j d\omega_j e^{-ik_j(x-x_j)+i\omega_j(t-t_j)}]$$
$$\int_{\mathbf{p_1,..,p_{n+m+l-1},p}} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t}$$
$$\partial_t \int_{\mathbf{p_1,..,p_{n+m+l},p}} \Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t}\Bigg] = 0 \tag{A.10}$$

where all integrals are both on momentum and frequencies, then explicitate the derivatives:

$$\int dx \Bigg[\sum_{j=1}^{m+n+l} \int dk_j d\omega_j e^{-ik_j(x-x_j)+i\omega_j(t-t_j)}(-ik^i_j)$$
$$\int_{p_1,..,p_{n+m+l},p} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t}$$
$$\int_{p_1,..,p_{n+m+l+1},p} (-i\omega)\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)+ipx-i\omega t}\Bigg] = 0 \tag{A.11}$$

integrate now over x:

$$\sum_{j=1}^{m+n+l} \int dk_j d\omega_j [(-ik^i_j) e^{+ik_j x_j + i\omega_j(t-t_j)}]$$
$$\int_{p_1,..,p_{n+m+l-1},p} \left[\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l}(k_a x_a - \omega_a t_a)-i\omega t}\delta(p-k_j)\right]$$
$$\int_{p_1,..,p_{n+m+l},p} \left[(-i\omega)\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a)-i\omega t}\delta(p)\right] = 0 \tag{A.12}$$

then sum over delta variables:

$$\sum_{j=1}^{m+n+l} \int dk_j d\omega_j [(-ik_j^i) e^{+ik_j x_j + i\omega_j (t-t_j)}]$$
$$\int_{p_1,..,p_{n+m+l}} \left[\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l-1}(k_a x_a - \omega_a t_a) - i\omega t} \right]$$
$$\int_{p_1,..,p_{n+m+l+1}} \left[(-i\omega)\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p=0,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a) - i\omega t} \right] = 0 \tag{A.13}$$

so now, shift the frequencies to have equal complex exponents:

$$\sum_{j=1}^{m+n+l} \int_{k_j,\omega_j,p_1,..,p_{n+m+l}} [(-ik_j^i)]$$
$$\left[\Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega,..,p_k,\omega_k) e^{i\sum_{a=1\neq j}^{n+m+l-1}(k_a x_a - \omega_a t_a) - i\omega t + ik_j x_j + i\omega_j (t-t_j)} \right]$$
$$\int_{p_1,..,p_{n+m+l+1}} \left[(-i\omega)\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p=0,\omega,..,p_k,\omega_k) e^{i\sum_{a=1}^{n+m+l}(k_a x_a - \omega_a t_a) - i\omega t} \right] = 0 \tag{A.14}$$

now, renaming in the first intgral $\omega = \omega + \omega_j$, we obtain, taing the kernel:

$$\Gamma^{(n;m;l+1)}_{i,\alpha_1,...,\alpha_l}(p_k,\omega_k,..,p=0,\omega,..,p_k,\omega_k) = \\ -\sum_{j=1}^{m+n+l} \frac{[(k_j^i)]}{\omega} \Gamma^{(n;m;l)}_{\alpha_1,...,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k) \tag{A.15}$$

A.3 Fourier reduction

Using the translational invariance properties of Γ functions is possible to reduce the number of variables in our Γ equations.

Let's consider:

$$\begin{aligned}&\sum_{j=1}^{m+n+l}[(k_j^i)]\Gamma^{(n;m;l)}_{\alpha_1,\ldots,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k=\\&-\omega\Gamma^{(n;m;l+1)}_{i,\alpha_1,\ldots,\alpha_l}(p=0,\omega,p_1,\omega_1,\ldots,p_{n+m+l},\omega_{n+m+l})\end{aligned} \tag{A.16}$$

and write the translational invariance properties. Remember this is a kernel of an operation, omitting here integrals to help notations:

$$\begin{aligned}&\forall j\in 1;\ldots;n+m+l-1\\&-[(k_j^i)]\Gamma^{(n;m;l)}_{\alpha_1,\ldots,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k\delta(\sum_{k=1}^{m+n+l}p_k=0)\delta(\sum_{k=1}^{m+n+l}\omega_k=0)-\ldots\\&-[(k_j^i)]\Gamma^{(n;m;l)}_{\alpha_1,\ldots,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k)\delta(\sum_{k=1}^{m+n+l}p_k=0)\delta(\sum_{k=1}^{m+n+l}\omega_k=0)=\\&\omega\Gamma^{(n;m;l+1)}_{i,\alpha_1,\ldots,\alpha_l}(p_k,\omega_k,..,p=0,\omega,..,p_k,\omega_k)\delta(\sum_{k=1}^{m+n+l}p_k=0)\delta(\sum_{k=1}^{m+n+l}\omega_k=0)\end{aligned} \tag{A.17}$$

In the first equation we get, when we contract on the j element, the sum of all moments multiplied by the general Γ, which j element has been lost by contraction. Resuming:

$$\begin{aligned}&\Gamma^{(n;m;l+1)}_{i,\alpha_1,\ldots,\alpha_l}(p=0,\omega,p_1,\omega_1,\ldots,p_{n+m+l-1},\omega_{n+m+l-1})=\\&\sum_{j=1}^{m+n+l-1}\frac{[(k_j^i)]}{\omega}\left[\Gamma^{(n;m;l)}_{\alpha_1,\ldots,\alpha_l}(p_k,\omega_k,..,k_j,\omega+\omega_j,..,p_k,\omega_k)-\Gamma^{(n;m;l)}_{\alpha_1,\ldots,\alpha_l}(p_i,\omega_i)\right]\end{aligned} \tag{A.18}$$

Appendix B

Closed equation

In this appendix we give detail to derive the closed equation for $\Gamma^{(1,1,0)}$.

B.1 Detailed compuation of $\Gamma^{(1,1,0)}$ closed equation

With the same method of the previous section we derive, rewriting higher order Γ:

$$\begin{aligned}\Gamma^{(1,1,2)}_{\alpha,\beta}(\mathbf{q},-\mathbf{q},\mathbf{p}) = \Gamma^{(1,1,2)}_{\alpha,\beta}(\mathbf{q_1}=\mathbf{0},\omega_\mathbf{1},\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu) = \\ -\frac{q_2^\alpha}{\omega}\Gamma^{(1,1,1)}_{\beta}(\mathbf{q_2},\omega_\mathbf{2}+\omega,\mathbf{p},\nu) - \frac{p}{\omega}\Gamma^{(1,1,1)}_{\beta}(\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu+\omega) \\ +\frac{p^\alpha+q_2^\alpha}{\omega}\Gamma^{(1,1,1)}_{\beta}(\mathbf{q_2},\omega_\mathbf{2},\mathbf{p},\nu)\end{aligned} \tag{B.1}$$

and lowering once more the order, putting then $q_1 = -q_2 = 0$, we obtain:

$$\begin{aligned}\Gamma^{(1,1,2)}_{\alpha,\beta}(\mathbf{q}=\mathbf{0},\omega,-\mathbf{q}=\mathbf{0},-\omega,\mathbf{p},\nu) = \\ -\frac{p^\alpha p^\beta}{\omega^2}(-\Gamma^{(1,1,0)}(\omega+\nu,\mathbf{p}) + 2\Gamma^{(1,1,0)}(\nu,\mathbf{p}) - \Gamma^{(1,1,0)}(-\omega+\nu,\mathbf{p}))\end{aligned} \tag{B.2}$$

We apply the same deduction on the two $3-order$ Γ term of the second line of equation 3.21. So:

$$\begin{aligned}\Gamma^{(1,0,2)}_{\alpha,\beta}(\mathbf{q}=\mathbf{0},\omega,\mathbf{p},\nu) = \\ -\frac{p^\alpha}{\omega}(\Gamma^{(1,0,1)}_{\beta}(\omega+\nu,\mathbf{p}) - \Gamma^{(1,0,1)}_{\beta}(\nu,\mathbf{p})\end{aligned} \tag{B.3}$$

$$\Gamma^{(0,1,2)}_{\alpha,\beta}(\mathbf{q}=\mathbf{0}+\mathbf{p},\omega+\nu,-\mathbf{q}=\mathbf{0},\nu)=$$
$$\frac{p^\alpha}{\omega}(\Gamma^{(0,1,1)}_{\beta}(\nu,\mathbf{p})-\Gamma^{(0,1,1)}_{\beta}(\nu+\omega,\mathbf{p}) \tag{B.4}$$

Or rewriting the complete equation:

$$\begin{aligned}\partial_k\Gamma^{(1,1,0)}_k(\mathbf{p}) = &-\frac{1}{2}\int_{\mathbf{q}}\partial_k R_k(\mathbf{q})_{v_\epsilon v_\gamma}G^{(2)}_{v_\gamma v_\alpha}(\mathbf{q})(-\frac{p^\alpha p^\beta}{\omega^2}(-\Gamma^{(1,1,0)}(\omega+\nu,\mathbf{p})+2\Gamma^{(1,1,0)}(\nu,\mathbf{p})\\ &-\Gamma^{(1,1,0)}(-\omega+\nu,\mathbf{p})))G^{(2)}_{v_\beta v_\epsilon}(\mathbf{q})+\partial_k R_k(\mathbf{q})_{v_\epsilon v_\gamma}G^{(2)}_{v_\gamma v_\alpha}(\mathbf{q})(-\frac{p^\alpha}{\omega}(\Gamma^{(1,0,1)}_{\beta}(\omega+\nu,\mathbf{p})\\ &-\Gamma^{(1,0,1)}_{\beta}(\nu,\mathbf{p}))G^{(2)}_{v_\beta v_\mu}(\mathbf{p},\omega+\nu)(\frac{p^\mu}{\omega}(\Gamma^{(0,1,1)}_{\theta}(\nu,\mathbf{p})-\Gamma^{(0,1,1)}_{\theta}(\nu+\omega,\mathbf{p}))G^{(2)}_{v_\theta v_\epsilon}(\mathbf{q})\end{aligned} \tag{B.5}$$

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