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MASTER THESIS

Ising model subject to resetting dynamics

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All'insaziabile desiderio della scoperta,
all'incontenibile bellezza dell'Universo,
al mio inesauribile amore per la fisica.

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Introduction

In everyday life it is quite common to observe unexpected interruptions of what is happening around us. These abrupt stops are often followed by a restart of the dynamics from a fixed initial configuration. This mechanism can be modeled in a general way by considering a system that, starting from a fixed initial state and evolving in time according to a prescribed dynamics, stochastically resets to its fixed initial state at a constant rate r and then restarts the dynamics [1, 2]. At long times, the system reaches a non trivial nonequilibrium stationary state (NESS), due to the coexistence of the prescribed dynamics and the resetting dynamics.

This phenomenon is found in many different situations. For example, in biology, the process of RNA polymerization, which is responsible for the synthesis of RNA from a DNA template, is stochastically interrupted by backtracking [3, 4]. Similar notions are also found in the ecological context: for instance, animals, while searching for food, often perform stochastic resetting to come back to their nest and restart the search, thus operating a particular kind of intermittent search strategies [5]. In computer science, stochastic restarts can be used to reduce the running time of randomized search algorithms [6]. The effects induced by the stochastic resetting mechanism turn out to be remarkable in several stochastic processes, like simple diffusion processes [1, 2, 7, 8], Lévy flights [9], continuous-time random walks [10], coagulation-diffusion processes [11] and fluctuating interfaces [12]. Very recently, stochastic resetting has been also studied in the context of active Brownian motion [13] and quantum dynamics [14, 15].

While most of the works involve only single-particle systems, very little is known about the effects induced by the stochastic resetting mechanism in extended many-body systems (see however [11, 12]). Nevertheless, it would be interesting to understand the influence of the resetting process on a system that exhibits a phase transition and fully characterize its behaviour in the different phases. One of the paradigmatic non equilibrium systems with these properties is certainly the Ising model evolving from a fixed initial configuration with the Glauber dynamics at temperature T. A well studied peculiarity of this model is the growth of order through the coarsening dynamics when the system is quenched from the disordered phase to the ferromagnetic phase [16]. In particular, considering the non trivial time dependence of the typical length scale of the ordered domains, the Ising model, when subject to the stochastic resetting mechanism, constitutes an interesting object of study.

In this thesis, we characterize the stationary state of the Ising model when, while performing the Glauber dynamics, it is stochastically reset to its fixed initial configuration with magnetization m_0 at a constant rate r. We present an exact solution of the problem in the one dimensional case, where we show the existence of a transition in the behaviour of the stationary PDF of the magnetization density at a particular value of

the resetting rate $r^*(T)$ that depends on the temperature T. In two dimensions, we exploit the correspondence between resetting processes and renewal theory to derive the stationary PDF of m in the paramagnetic and ferromagnetic phases and at the critical temperature. The most striking feature of our results is that the behaviour of the PDF strongly depends on the temperature T at which the Glauber dynamics is performed, showing three different forms at high T (paramagnetic phase), low T (ferromagnetic phase) and at the critical temperature T_c. In particular, only in the paramagnetic phase, it is possible to find a sufficiently small resetting rate r such that the equilibrium magnetization still remains the most probable value of the magnetization assumed by the system. Remarkably, this effect does not happen in the ferromagnetic phase and at the critical temperature, i.e. an arbitrarily small value of the resetting rate r destroys the divergence of the stationary PDF of m at the equilibrium magnetization.

Chapter 1

Equilibrium and dynamical properties of the Ising model

In this chapter we present the main equilibrium and dynamical properties of the Ising model, focussing mostly on the two dimensional case. We also present the concept of dynamical criticality and domain growth through scaling relations. We conclude the chapter with an application of stochastic resetting on simple one particle diffusion process. Even in this simple setting, the introduction of a resetting rate r provokes rather dramatic changes in the long time properties of the system.

1.1 Glauber dynamics

Before dealing with the effects of stochastic resetting in the Ising model, it is necessary to completely characterize the model in its simplest form. Considering the two dimensional case, the hamiltonian that describes the system is given by

$$H = -h \sum_{i=1}^{N} s_i - J \sum_{<i,j>} s_i s_j, \tag{1.1}$$

where h and N denote the field and the total number of spins respectively, $s_i \in \{+1, -1\}$ is the spin variable at each site i and the second term is a nearest-neighbour interaction. Let us consider the case of a $L \times L$ square lattice, so that we can rewrite the hamiltonian as

$$H = -h \sum_{i,j} s_{i,j} - J \sum_{i,j} s_{i,j}(s_{i+1,j} + s_{i,j+1}). \tag{1.2}$$

Periodic boundary conditions (PBCs) are present, so that $s_{L+1,j} = s_{1,j}\ \forall j \in [1, L]$ and $s_{i,L+1} = s_{i,1}\ \forall i \in [1, L]$.

It is well known that in the 2D Ising model a phase transition from the paramagnetic state to the ferromagnetic state occurs spontaneously in zero magnetic field at

$$T_c = \frac{2}{\ln(1+\sqrt{2})} \frac{J}{K_B} \sim 2.269 \frac{J}{K_B}. \tag{1.3}$$

In absence of an external field, the 2D Ising model becomes appealing when $T \to T_c$. Indeed, considering the case $h = 0$, if $T < T_c$ the system is found to be in a ferromagnetic state with a non-zero magnetization density $\langle m(T) \rangle$, whereas if $T > T_c$ the system is in a paramagnetic phase with $\langle m \rangle = 0$. This transition can be reproduced throughout Monte Carlo simulations which provide a precise way to estimate numerically the critical temperature T_c and the critical exponents.

In this thesis the Monte Carlo simulations are made with the following ingredients:

- a square lattice of size $L \times L$;
- no external field ($h = 0$);
- a fixed temperature T at which the system evolves with the Glauber dynamics;
- a number of Monte Carlo time steps needed for equilibration;
- the total number of Monte Carlo time steps.

The time evolution of the system is ruled by the Glauber algorithm. The prescriptions are the following:

1. pick a spin at random and imagine to flip it;
2. accept the flip with probability

$$P = \frac{1}{1 + e^{\frac{\Delta E}{k_B T}}}, \tag{1.4}$$

   where ΔE indicates the energy difference between the state with that spin flipped and the one with the spin unchanged;

3. iterate the procedure for L^2 times and then update the magnetization density $m(t) = \frac{1}{N} \sum_{i=1}^{N} s_i$, where the (discrete) variable t denotes the *Monte Carlo time*, corresponding to the time occurred to attempt $t \cdot L^2$ spin flips;
4. go back to the first point and iterate the procedure for the total number of settled MC time steps, the first T_{eq} of which are needed to equilibrate the system.

The Glauber algorithm described here is an example of a *Markov chain Monte Carlo* method. If we want to simulate the equilibrium properties of the system, the algorithm must satisfy ergodicity and detailed balance. Indeed, without the ergodic hypothesis, we cannot use the time-average of an observable to compute the corresponding average over the ensemble. Besides, without detailed balance, you are not guaranteed to reach, after some time steps, the equilibrium distribution. The Glauber algorithm previously described satisfies both conditions: ergodicity is achieved if we run the simulation for a time enough to let the system explore all the configurations in the phase space. Detailed balance is also satisfied by fixing the Boltzmann distribution as the equilibrium one, so that

$$w(s \to s') P_{eq}(s) = w(s' \to s) P_{eq}(s') \tag{1.5}$$

holds for any configuration pair s, s' in the phase space. $w(s \to s')$ indicates the transition rate (that is proportional to the acceptance probability given by Eq. (1.4)) from the configuration s to the configuration s' and is equal to $\frac{1}{1+e^{\beta(E'-E)}}$, where E and E' are the corresponding energies. $P_{eq}(s)$ denotes the Boltzmann distribution $\frac{1}{Z} e^{-\beta E}$.

1.2 Critical exponents

Many observables can be computed throughout Monte Carlo simulations. The most interesting ones are the magnetization density $\langle m \rangle$, the susceptibility χ and the correlation length ξ. Their dependence on the temperature T is highly non trivial: in particular, in the vicinity of the critical temperature T_c, these observables follow a power law with some universal critical exponents that can be also computed exactly. Indeed, we have that

$$m \sim (-t)^{\beta}, \quad \chi \sim |t|^{-\gamma}, \quad \xi \sim |t|^{-\nu}, \tag{1.6}$$

where $t = \frac{T-T_c}{T_c}$ is the *reduced temperature* and the exponents are $\beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ and $\nu = 1$.

In Monte Carlo simulations the square lattice has a finite size and, for this reason, the divergence of χ and ξ in correspondence of the critical temperature appear as a sharp maximum. The divergence is indeed a direct consequence of the non analiticity of the free energy $-\frac{1}{\beta} \ln Z$. It is also worth mentioning that the candidate value for the critical temperature is slightly larger than the theoretical one: this fact is well known in the literature [17] and is due to a finite size effect. Indeed, if you want to estimate the critical temperature by measuring the maximum of the susceptibility, you would get

$$T_c(L) = T_c(\infty) + bL^{-\frac{1}{\nu}}. \tag{1.7}$$

The correlation function $C(r)$ at the critical temperature follows a power law with respect to the distance r, instead of being an exponential. This effect is simply due to the divergence of the correlation length ξ. In particular, above and below T_c, the correlation function is given asymptotically by

$$C(r) = \langle s_i s_{i+\vec{r}} \rangle \sim \frac{e^{-\frac{r}{\xi(T)}}}{r^{\eta+d-2}} + \langle m \rangle^2 , \tag{1.8}$$

where d indicates the dimension of the space and $\eta = 0.25$ is a critical exponent. We see that this expression reduces to a simple power law at $T = T_c \simeq 2.269$ ($J = k_B = 1$) because of the divergence of the correlation length.

1.2.1 Time evolution in Monte Carlo simulations

In Monte Carlo simulations time is necessarily discrete. Nevertheless, one can decide how to define the time unit, i.e. the minimum time step. To this end, the best choice is to fix the so called *Monte Carlo time step* as the time occurred to attempt N spin flips. The reason behind this choice is that, in average, within this time window, each spin of the system attempts one flip. Hence, this can be viewed as the physical time.

We may want to study the time evolution of the system and of its average magnetization density $\langle m(t) \rangle$ in Monte Carlo simulations. This analysis may be of great interest, because the evolution ruled by the Glauber algorithm (which satisfies detailed balance) can be a good candidate for the actual evolution experienced by a ferromagnet (or paramagnet) at a fixed temperature T.

To this end, an interesting object of study is the dynamics of the 2D Ising model that, starting from a certain configuration (say, all spins up), evolves with the Glauber dynamics at the critical temperature $T_c \simeq 2.27$. The system undergoes a *critical* *dynamics*, characterized by a power law decay in time of the magnetization density towards the equilibrium magnetization $m_{eq} = 0$.

It is possible to derive this result from simple scaling arguments. Indeed, for $T \to T_c$, we know that $m \sim |T - T_c|^\beta$ and $\xi \sim |T - T_c|^{-\nu}$. Hence, $m \sim \xi^{-\frac{\beta}{\nu}}$. Besides, it is well known [18] that the typical length of cluster domains grows with time as $l(t) \sim t^{\frac{1}{z}}$, where z is called the *dynamical* critical exponent. Since $l(t) \sim \xi(t)$ for small times, we obtain

$$m(t) \sim t^{-\frac{\beta}{\nu z}}. \tag{1.9}$$

This power law behavior of the magnetization density holds in the thermodynamical limit, when the lattice size $L \to \infty$. Instead, if the system size is finite, then, after some time, ξ becomes larger than L and finite size effects arise. Hence, from time $\tau = L^z$, called the *relaxation time*, the magnetization decays exponentially as $m(t) \sim e^{-\frac{t}{\xi(T)}}$ [19].

Another interesting phenomenon at the critical temperature that is worth analyzing is the *initial slip* of $\langle m(t) \rangle$ when the starting configuration has a finite magnetization $m(0) \neq 1$. In this case, scaling arguments [20] lead to a peculiar short-time behaviour of $\langle m(t) \rangle$ as

$$\langle m(t) \rangle \sim \langle m(0) \rangle \, t^\theta, \tag{1.10}$$

where θ is a critical exponent. Therefore, the average magnetization of the system increases up to a *crossover time* $t_c = \langle m(0) \rangle^{-1/\left(\theta + \frac{\beta}{\nu z}\right)}$, after which the dynamics is again given by the usual power law decay. In particular, t_c increases as $\langle m(0) \rangle$ decreases.

1.3 A simple example: diffusion with stochastic resetting

Before dealing with the main topic of this thesis, we want to present an interesting analysis of the resetting phenomenon in the case of a 1-particle diffusion process in a 1D space [1]. In this simple setting, the introduction of a non zero resetting rate r strongly changes the dynamics of the system.

To fix ideas, let us denote with x_0 and $x(t)$ the position of the particle at time $t = 0$ and time t respectively. If the resetting rate $r = 0$, then the particle will perform the ordinary Brownian motion, in which $p(x, t|x_0)$, i.e. the probability that the particle is at position x at time t having began the motion at x_0 at time $t = 0$, satisfies the usual diffusion equation

$$\frac{\partial p(x, t|x_0)}{\partial t} = D \frac{\partial^2 p(x, t|x_0)}{\partial x^2}, \tag{1.11}$$

where D is the diffusion constant. The solution of this equation is given by

$$p(x, t|x_0) = \frac{e^{-\frac{(x-x_0)^2}{4Dt}}}{\sqrt{4\pi D t}}, \tag{1.12}$$

a gaussian distribution with mean $\mu = x_0$ and variance $\sigma_t^2 = 2Dt$.

Let us consider now the case in which a non-zero resetting rate r is present in the dynamics. If we imagine to discretize the time in little time steps Δt, we have that, at each time step Δt, the particle, which is at position $x(t)$, comes back to the initial position x_0 with probability $r\Delta t$, while, with probability $1 - r\Delta t$, it performs a simple

Brownian motion going to $x(t+\Delta t) = x(t) + \eta(t)\Delta t$, where $\eta(t)$ is a gaussian noise. As a consequence, the master equation for $p(x,t|x_0)$ is now given by

$$\frac{\partial p_r(x,t|x_0)}{\partial t} = D\frac{\partial^2 p_r(x,t|x_0)}{\partial x^2} - rp_r(x,t|x_0) + r\delta(x - x_0), \tag{1.13}$$

with initial condition $p(x,0) = \delta(x - x_0)$.

The second and third term on the right side stand for a negative probability flux out of each point x and a corresponding positive probability flux into x_0 respectively. We can notice that we would get the usual diffusion equation as $r \to 0$. At large time ($t \to \infty$) there exists a stationary probability state given by

$$p_{st}(x|x_0) = \frac{1}{2}\sqrt{\frac{r}{D}}e^{-\sqrt{\frac{r}{D}}|x-x_0|}. \tag{1.14}$$

This probability state presents a cuspid at $x = x_0$ and, more importantly, it constitutes a non-equilibrium stationary state, since, even at long times, there is non-zero probability current at x_0: this clearly violates detailed balance.

Let us now consider more specifically the so called *searching problem*, in which a particle, diffusing from a starting position $x_0 \neq 0$, has to find a target at fixed position $x = 0$ (i.e. the origin). In this situation, the introduction of a non-zero resetting rate changes dramatically the dynamics. Indeed, if we denote with $T(x_0)$ the expected time needed by the particle (starting at x_0) to reach the origin for the first time, we know that, in absence of resetting, $T(x_0)$ diverges. In fact, this is nothing but the mean time to reach the origin in a simple random walk. Instead, in presence of $r \neq 0$, one finds

$$T(x_0) = \frac{1}{r}\left(e^{\sqrt{\frac{r}{D}}x_0} - 1\right), \tag{1.15}$$

which shows that in both cases $r \to 0$ and $r \to \infty$, $T(x_0)$ diverges. Indeed, in the first case, the result about the simple random walk is recovered, while, in the second case, the divergence is due to the fact that the particle tends to remain fixed at x_0 without moving. Hence, there must be an optimal value r^* which minimizes $T(x_0)$. In particular, by imposing $\frac{dT(x_0)}{dr} = 0$, one obtains $r^* = (z^*)^2 D/x_0^2$, with z^* being the solution of $\frac{z}{2} = 1 - e^{-z}$.

This example shows how the introduction of stochastic resetting can strongly modify the dynamics of a system. Nevertheless, one can analyze more complex systems and study the effects induced by stochastic resetting. Indeed, in this thesis, we will focus on the Ising model, which constitutes a paradigm for interacting many-body systems in physics and interdisciplinar subjects.

Chapter 2

Stochastic resetting in 1D Ising model

So far we have discussed the dynamics and the equilibrium properties of the 2D Ising model. Now we can add a resetting rate r such that, at each time step Δt, the system comes back to its initial configuration with probability $r\Delta t$. In this way a non equilibrium stationary dynamics arises at long times, leading to non trivial consequences.

It is well known that, differently from the two dimensional case, the Glauber dynamics of the 1D Ising model (without resetting) can be solved exactly [21]. For this reason, before analyzing the dynamics of the 2D Ising model under the influence of stochastic resetting, we start by solving exactly the equations of motion in the 1D case (including also the stochastic resetting process) and then compare the results with the ones obtained by Monte Carlo simulations.

2.1 Exact solution of the model

Let us consider a one dimensional spin chain of length L with periodic boundary conditions and spin variables $s_k \in \{+1, -1\}$ at each site k. The probability rate prescripted by the Glauber dynamics that the i^{th} the spin flips from the value s_i to $-s_i$ while the others remain momentarily fixed is given by

$$w(s_i \to -s_i) = \frac{1 - \tanh[\beta J s_i(s_{i+1} + s_{i-1})]}{2} \tag{2.1}$$

where $\beta = \frac{1}{k_B T}$ and J is the coupling constant between first nearest neighbours sites. One can prove that it is equal to $\frac{1}{1+e^{\beta \Delta E}}$. Indeed,

$$\frac{1 - \tanh[\beta J s_i(s_{i+1} + s_{i-1})]}{2} = \frac{1 - \frac{e^{\beta\frac{\Delta E}{2}} - e^{-\beta\frac{\Delta E}{2}}}{e^{\beta\frac{\Delta E}{2}} + e^{-\beta\frac{\Delta E}{2}}}}{2} = \frac{2e^{-\beta\frac{\Delta E}{2}}}{2(e^{\beta\frac{\Delta E}{2}} + e^{-\beta\frac{\Delta E}{2}})} = \frac{1}{1 + e^{\beta \Delta E}}, \tag{2.2}$$

with $\Delta E = 2\beta J s_i(s_{i+1} + s_{i-1})$.

Since the variable s_i can assume only the value $+1$ or -1, Eq. (2.1) can be further simplified as

$$\begin{aligned} w(s_i \to -s_i) &= \frac{1 - \tanh[\beta J s_i(s_{i+1} + s_{i-1})]}{2} = \frac{1 - s_i \tanh(2\beta J \frac{s_{i+1}+s_{i-1}}{2})}{2} \\ &= \frac{1 - s_i \frac{s_{i+1}+s_{i-1}}{2} \tanh(2\beta J)}{2} = \frac{1 - \gamma s_i \frac{s_{i+1}+s_{i-1}}{2}}{2}, \end{aligned} \tag{2.3}$$

where $\gamma = \tanh(2\beta J)$ and $0 < \gamma < 1 \; \forall T > 0$. This simplification, which is fundamental to obtain an exact solution of the model, is peculiar for the one dimensional case. Unfortunately, since in the 2D Ising model the coordination number is 4, it is not possible to reduce the hyperbolic tangent function to a number γ: this is the reason why we will make use of renewal theory to obtain an approximate solution.

Let us indicate with $\sigma = \{s_1, \ldots, s_i, \ldots, s_L\}$ a generic state of the system and with $\sigma^i = \{s_1, \ldots, -s_i, \ldots, s_L\}$ the same state but with the i^{th} spin flipped. Then, the time evolution of the ensemble-average of any observable $f(\sigma(t))$ is given by the solution of

$$\frac{d}{dt} \langle f(\sigma(t)) \rangle = \sum_{i=1}^{L} \left\langle [f(\sigma^i(t)) - f(\sigma(t))] w(\sigma \to \sigma^i) \right\rangle, \tag{2.4}$$

where $w(\sigma \to \sigma^i) = w(s_i \to -s_i)$.

Considering in particular the two observables $f(\sigma(t)) = s_i(t)$ and $f(\sigma(t)) = s_i(t)s_j(t)$, we obtain

$$\frac{d \langle s_i(t) \rangle}{dt} = -2 \left\langle s_i(t) w(\sigma \to \sigma^i) \right\rangle \tag{2.5}$$

and

$$\frac{d \langle s_i(t) s_j(t) \rangle}{dt} = -2 \left\langle s_i(t) s_j(t) [w(\sigma \to \sigma^i) + w(\sigma \to \sigma^j)] \right\rangle. \tag{2.6}$$

In particular, since $w(\sigma \to \sigma^i) = \frac{1 - \gamma s_i \frac{s_{i+1}+s_{i-1}}{2}}{2}$, Eq. (2.5) becomes

$$\frac{d \langle s_i(t) \rangle}{dt} = - \langle s_i(t) \rangle + \frac{1}{2} \gamma [\langle s_{i+1}(t) \rangle + \langle s_{i-1}(t) \rangle]. \tag{2.7}$$

Summing this equation over the L sites (exploiting the translational invariance of the system due to the presence of periodic boundary conditions) and dividing both members by L, we obtain the equation of motion for the average magnetization density $m(t) = \frac{1}{L} \sum_{i=1}^{L} \langle s_i(t) \rangle$:

$$\frac{dm(t)}{dt} = (\gamma - 1) m(t) \tag{2.8}$$

and its solution is

$$m(t) = m(0) e^{-(1-\gamma)t}, \tag{2.9}$$

with an initial magnetization $m(0)$.

The time evolution of the magnetization has some interesting properties. First of all, since $1 - \gamma > 0$, $m(t)$ decays exponentially to 0 for any positive temperature, as expected for the 1D Ising model. Besides, recalling that $\gamma = \tanh(2\beta J)$, we see that the decay becomes faster at higher temperatures.

Now we can introduce a resetting rate r such that, at each time step Δt, the system has a probability $r\Delta t$ to return back to its initial configuration with magnetization $m(0)$. Our goal is to determine the non-equilibrium stationary distribution of the magnetization and its first moment, in order to compare them with Monte Carlo simulations. To this end, it is convenient to solve the dynamics with discrete time steps Δt and then consider the limit $\Delta t \to 0$. The dynamics of the magnetization is therefore defined by

$$m(t+\Delta t) = \begin{cases} m(0) & \text{with prob.} \quad r\Delta t \\ m(t) + (\gamma - 1)m(t)\Delta t & \text{with prob.} \quad 1 - r\Delta t \end{cases}. \tag{2.10}$$

Then the probability $P(m,t)$ that the magnetization of the system is m at time t satisfies

$$P(m,t+\Delta t) = (1 - r\Delta t)\, P(m - \Delta m, t)\,(1 + c\Delta t) + r\Delta t\, \delta(m - m(0)), \tag{2.11}$$

where $c = 1 - \gamma$ is a positive number between 0 and 1. Keeping up to first order terms in Δt and $\Delta m = -cm\Delta t$ in the Taylor expansion of $P(m,t)$, we obtain the master equation

$$\frac{\partial P(m,t)}{\partial t} = cm\frac{\partial P(m,t)}{\partial m} + (c - r)P(m,t) + r\delta(m - m(0)). \tag{2.12}$$

The stationary distribution $P(m)$ can be obtained by setting $\frac{\partial P(m,t)}{\partial t} = 0$ and solving the differential equation

$$m\frac{dP(m)}{dm} = \frac{r-c}{c}P(m) - \frac{r}{c}\delta(m - m(0)). \tag{2.13}$$

In the following I will present the full solution of this differential equation since the procedure for the 2D case will be similar.

Denoting the magnetization of the initial configuration (to which the system resets) as $m(0) = m_0$ and integrating both members of the differential equation between m_0^- and m_0^+, we have

$$\int_{m_0^-}^{m_0^+} m\frac{dP(m)}{dm}dm = \frac{r-c}{c}\int_{m_0^-}^{m_0^+} P(m)dm - \frac{r}{c}. \tag{2.14}$$

Integrating by parts, we obtain

$$\lim_{\epsilon\to 0}\left\{\Big[mP(m)\Big]_{m_0-\epsilon}^{m_0+\epsilon} - \int_{m_0-\epsilon}^{m_0+\epsilon} P(m)dm\right\} = \frac{r-c}{c}\lim_{\epsilon\to 0}\left\{\int_{m_0-\epsilon}^{m_0+\epsilon} P(m)dm\right\} - \frac{r}{c}. \tag{2.15}$$

Let us restrict our analysis to the case $m_0 > 0$. Since between two consecutive resets the system evolves towards a state with zero magnetization in a deterministic way (with an exponential decay as shown in Eq. (2.9)), the magnetization of the system will be always between 0 and m_0. Based on this observation, we can safely put $P(m_0 + \epsilon) = 0$ and hence

$$\begin{aligned} -\lim_{\epsilon\to 0}\Big[(m_0-\epsilon)P(m_0-\epsilon)\Big] - \lim_{\epsilon\to 0}\int_{m_0-\epsilon}^{m_0+\epsilon} P(m)dm = \\ = \frac{r-c}{c}\lim_{\epsilon\to 0}\left\{\int_{m_0-\epsilon}^{m_0+\epsilon} P(m)dm\right\} - \frac{r}{c}. \end{aligned} \tag{2.16}$$

Since $P(m)$ is a probability distribution function, any integral of the type $\int_{m_0-\epsilon}^{m_0+\epsilon} P(m)dm$ is finite and, in particular, vanishes as $\epsilon \to 0$. So we have

$$\lim_{\epsilon \to 0} \Big[(m_0 - \epsilon) P(m_0 - \epsilon) \Big] = \frac{r}{c} \tag{2.17}$$

that provides the boundary condition needed to solve the differential equation (2.13):

$$\lim_{\epsilon \to 0} P(m_0 - \epsilon) = \frac{r}{c \cdot m_0} = P(m_0^-). \tag{2.18}$$

Dividing both members of Eq. (2.13) by $mP(m)$ (notice that m and $P(m)$ cannot be equal to 0), we obtain

$$\frac{dP(m)}{P(m)} = \frac{r-c}{c} \frac{dm}{m}, \tag{2.19}$$

since the condition $0 < m \leq m_0^-$ cancels the contribution due to the delta function, which is in fact contained in the boundary condition $P(m_0) = \frac{r}{c \cdot m_0}$. Integrating both members between m and m_0 we get

$$\ln \left(\frac{P(m_0)}{P(m)} \right) = \frac{r-c}{c} \ln \left(\frac{m_0}{m} \right), \tag{2.20}$$

which leads to

$$\boxed{P(m) = \frac{r}{c \cdot (m_0)^{\frac{r}{c}}} \cdot m^{\frac{r-c}{c}}}. \tag{2.21}$$

This is the (normalized) stationary probability distribution of the magnetization density m of the 1D Ising model in presence of a resetting rate r. Its first moment, which is of great interest, is given by

$$\boxed{\langle m \rangle} = \int_0^{m_0} mP(m)dm = \frac{r}{c \cdot (m_0)^{\frac{r}{c}}} \int_0^{m_0} m^{\frac{r}{c}} dm = \boxed{\frac{r}{r+c} m_0}. \tag{2.22}$$

2.2 Monte Carlo simulations

These theoretical results can be evaluated throughout appropriate Monte Carlo simulations. I have chosen the following parameters:

- $L = 10000$;
- $m(0) = m_0 = 0.992$ (most of the spins is +1 at time 0): this is the configuration to which the system is reset;
- $T = 3.5$;
- MC time steps = 300000;
- list of resetting rates used in the simulations $r = \{1000, 300, 100, 30, 10, 6, 3, 1, 0.6, 0.3, 0.2, 0.1, 0.06, 0.03, 0.02, 0.01, 0.006, 0.003, 0.002, 0.001, 0.0006, 0.0003, 0.0002, 0.0001\}$. The resetting probability p_r is defined as $r\Delta t$, which is the probability to reset the system at each time step Δt. In our simulation the time step Δt is given by $\Delta t = \frac{1}{L}$.

We should notice that the time step Δt is different from the Monte Carlo (MC) time step. Indeed, the time step Δt used in our simulation and in the theoretical derivation is the time occured to attempt one spin flip, while the MC time step is the time occured to attempt L spin flips. For this reason, the time step Δt is defined as $\Delta t = \frac{1}{L}$.

It is clear that, in absence of resetting, the long time probability distribution of m would be a delta function centered around the equilibrium magnetization, which is $m_{eq} = 0$ in the case of the 1D Ising model. Notice that the variable m is defined as the ensemble-average magnetization density $m(t) = \frac{1}{L}\sum_{i=1}^{L} \langle s_i(t) \rangle$: it is therefore a deterministic variable whose dynamics is governed by 2.9. On the other hand, when $r \to \infty$, i.e. the system is reset with probability one, the probability distribution of m becomes a delta function centered around the initial magnetization density $m(0)$. Only when the stochastic resetting mechanism is switched on (with a non zero but finite resetting rate r), the variable m becomes a *random* variable: in particular, the stationary PDF of m is not given by a delta function, but by a function with support $[m_{eq}, m(0)]$. In principle, we can imagine that, gradually switching on the resetting mechanism, the PDF "trasforms" from a delta function (centered around $m_{eq} = 0$) to the other delta function (centered around $m(0)$), eventually becoming a uniform distribution at a certain value r^*.

Fig. (2.1), Fig. (2.2) and Fig. (2.3) show the comparison between the theoretical stationary probability distribution (2.21, red line) and the one obtained by Monte Carlo simulations (blue line) with a resetting rate $r = 0.2$, $r = 0.4836$ and $r = 3$ respectively. The two lines are in perfect agreement in all the plots.

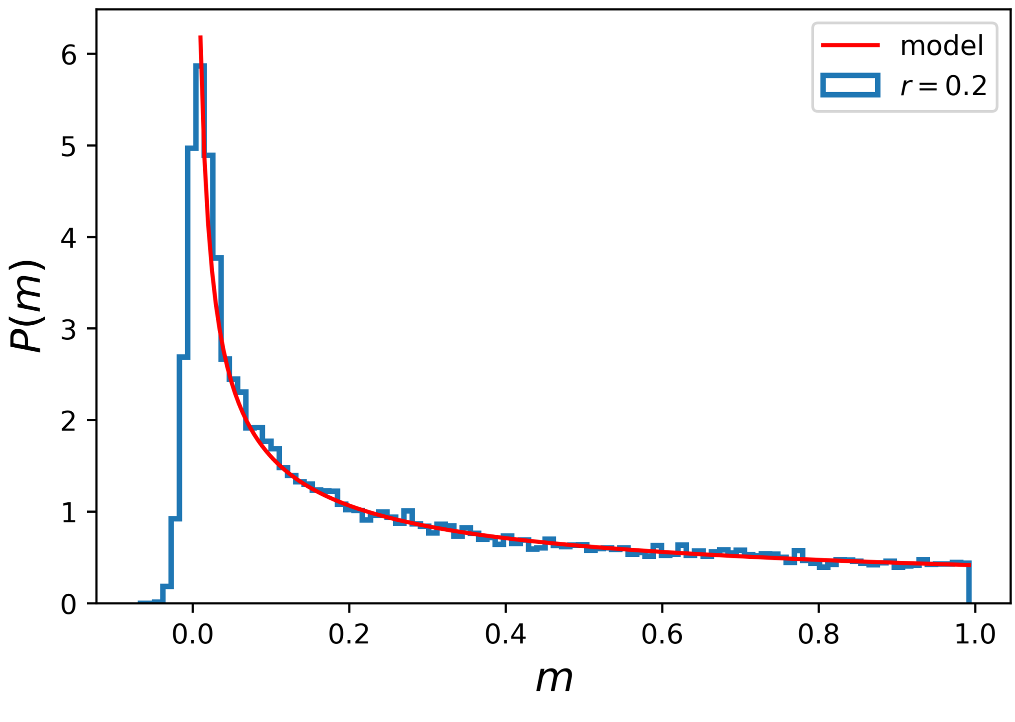

Figure 2.1: Stationary probability distribution of m with $r = 0.2$.

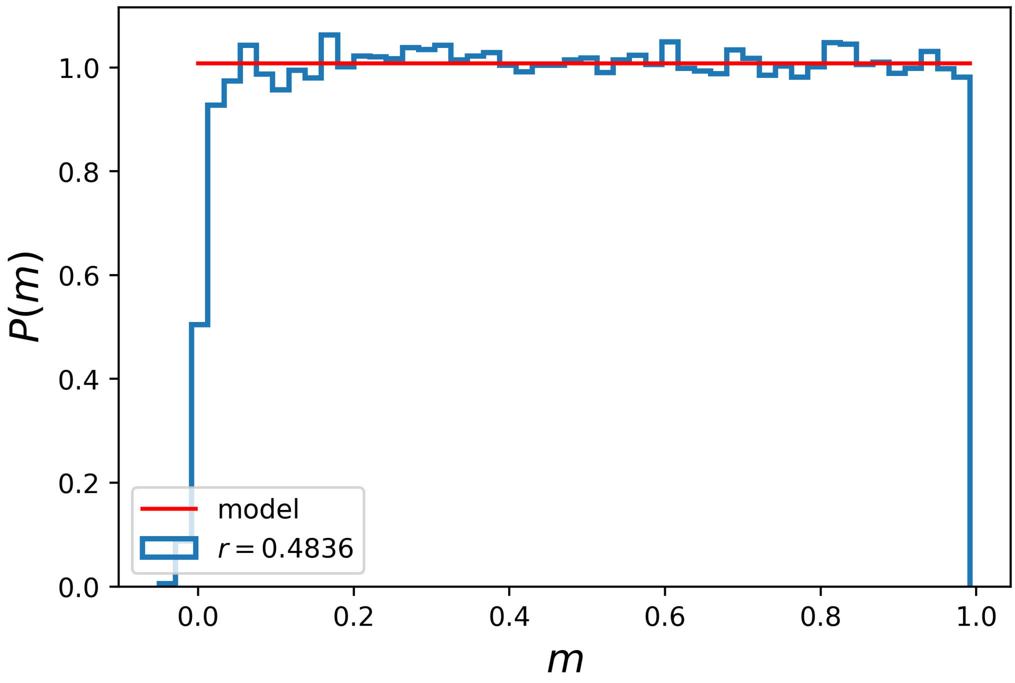

Figure 2.2: Stationary probability distribution of m with $r^* = 1 - \tanh(2/T) = 0.4836$.

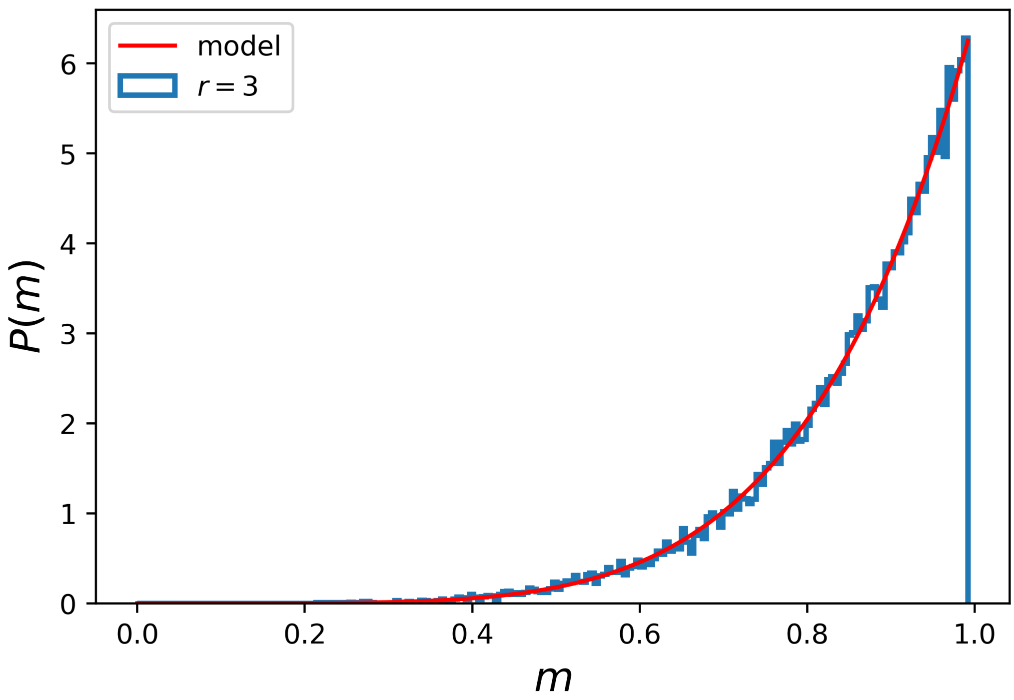

Figure 2.3: Stationary probability distribution of m with $r = 3$.

The three figures are ordered in such a way that the resetting rate r applied to the system is increasing. In particular, in Fig. (2.1), where the resetting rate is $r = 0.2$, the stationary PDF of m shows that the value $m = 0$ is the most probable value of the magnetization. Fig. (2.3) instead plots the PDF of m with $r = 3$: in this case, the system tends to be in the initial configuration with magnetization $m(0) = 0.992$.

More interestingly, at a resetting rate $r^* = c$, a crossover between the two situations appears, leading to a uniform probability distribution (see Fig. (2.2)). All these properties are perfectly described by Eq. (2.21), on which it is worth to spend some additional words. Indeed we can notice that there is a kind of competition between two probability rates, c and r. The first is exactly the rate of the (deterministic) exponential decay (Eq. (2.9)) of the magnetization density, while the second is the rate of resetting. For this reason the crossover appears exactly when $r = c$.

Fig. (2.4) shows the behaviour of the crossover resetting rate r^* as a function of temperature. The red dashed line shows the theoretical prediction given by $r^* = 1 - \tanh(2\beta J)$ (in our case $k_B = J = 1$). The blue dots instead are results coming from Monte Carlo simulations and correspond to the value of the resetting rate at which the stationary PDF of m becomes uniform at a given temperature T. In order to find these values, it was of course necessary to do several simulations until I could find, for each temperature T, the right resetting rate r for which the stationary PDF of m is uniform. The agreement between theory and numerics is excellent.

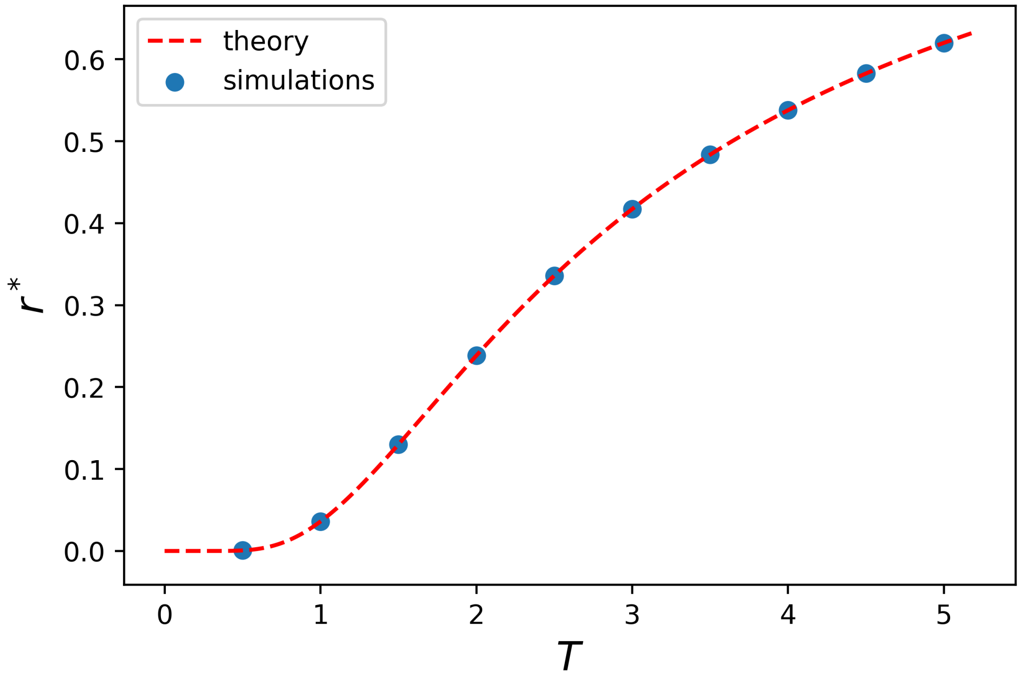

Figure 2.4: Crossover resetting rate r^* as a function of temperature.

The plot in Fig. (2.4) represents a sort of "phase diagram". Indeed, we can interpret this function $r^*(T)$ as a line that separates two "phases". One phase is above the line and corresponds to the case when the PDF of m is equal to 0 at $m_{eq} = 0$. In this phase the resetting rate is so strong that the equilibrium state cannot be reached, i.e. the system is never found with zero magnetization. On the other hand, the phase that is below the line corresponds to the appearance of a divergence of $P(m)$ at $m = m_{eq} =$

0. Indeed, in this phase, the equilibrium magnetization ($m_{eq} = 0$) is still the most probable configuration of the system, because the resetting rate is sufficiently small. Interestingly, when $r = r^* = 1 - \tanh(2\beta J)$, we have an intermediate condition between the two cases, since the the stationary PDF becomes a uniform function with a non zero finite value at $m = 0$.

Let us now focus on the average magnetization density $\langle m \rangle$, which has been derived in Eq. (2.22). Again, the theoretical curve is in perfect agreement with the results of the simulations, as shown in Fig. (2.5).

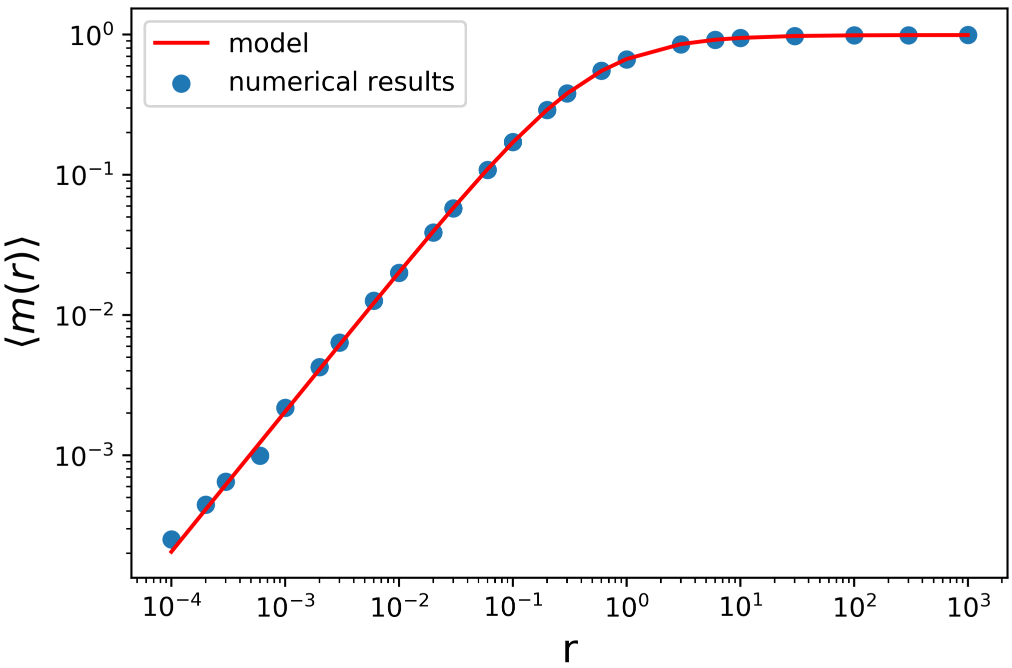

Figure 2.5: Average magnetization density $\langle m \rangle$ as a function of r ($m_0 = 0.992$).

2.3 Alternative theoretical solution

Let us conclude this analysis with an alternative procedure to obtain the stationary probability distribution function $P(m)$ (2.21).

As previously stated, the dynamics of the magnetization density m between two consecutive resets is perfectly deterministic and is given by Eq. (2.9). Then, if we denote with τ the time elapsed since the last reset, the probability distribution of m at time τ is

$$P(m, \tau) = \delta(m - m_0 e^{-c\tau}). \tag{2.23}$$

In this case τ is a real-valued random variable, which is exponentially distributed as $P(\tau) = re^{-r\tau}$. Hence, the stationary probability distribution of the magnetization density is obtained by averaging over τ:

$$
\begin{aligned}
P(m) &= \int_0^{+\infty} re^{-r\tau}\delta(m - m_0 e^{-c\tau})d\tau = \int_0^{+\infty} re^{-r\tau}\frac{\delta(\tau + \frac{1}{c}\ln(\frac{m}{m_0}))}{cm}d\tau = \\
&= \frac{r}{cm}e^{-r(-\frac{1}{c})\ln(\frac{m}{m_0})} = \frac{r}{cm}\left(\frac{m}{m_0}\right)^{\frac{r}{c}},
\end{aligned}
\tag{2.24}
$$

which is indeed equal to Eq. (2.21). Notice that in the second equality we have used $\delta(f(\tau)) = \frac{\delta(\tau-\tau_0)}{|f'(\tau_0)|}$ (which is valid for any continuously differentiable function f with f' nowhere zero), where τ_0 is the (unique) real root of f.

Similarly, it is possible to obtain the average magnetization density $\langle m \rangle$ as

$$
\langle m \rangle = \int_0^{+\infty} re^{-r\tau} m_0 e^{-c\tau} d\tau = -\frac{rm_0}{r+c}\left[e^{-(r+c)\tau}\right]_0^{+\infty} = \frac{rm_0}{r+c}, \tag{2.25}
$$

which is exactly Eq. (2.22).

Chapter 3

Stochastic resetting in 2D Ising model

In the two dimensional case the study of the consequences induced by the introduction of a resetting rate r in the time evolution of the system is still possible. However, Eq. (2.7) is no more valid because in the 2D Ising model the time evolution of $\langle s_{i,j}(t)\rangle$ is affected by four nearest neighbours. Hence a different approach is needed to tackle the two dimensional problem.

What we expect is that, without resetting, the fixed points of the dynamics would be $0, +m_{eq}, -m_{eq}$, depending on the temperature T. Instead, if we introduce a resetting rate $r \neq 0$, then two situations are possible. If $T \geq T_c$ (paramagnetic phase and at the critical temperature), then the support of the stationary probability distribution of m is $[0, m(0)]$. If $T < T_c$ (ferromagnetic phase), then the support is $[m(0), m_{eq}]$ (if $m(0) < m_{eq}$) or $[m_{eq}, m(0)]$ (if $m(0) > m_{eq}$) (we focus on positive m_{eq} and $m(0)$, but the generalization to negative values of the magnetization density is straightforward).

3.1 Main results

In the 2D Ising model subject to stochastic resetting, the non equilibrium stationary PDF of m assumes three different forms depending on the temperature T at which the Glauber dynamics is performed. In particular, the three different forms are found in the paramagnetic phase, ferromagnetic phase and at the critical temperature. Therefore, we briefly present the expressions of $P(m)$ in the three regimes.

For simplicity we denote by a, b and c the whole set of parameters that play a role in the time evolution of the magnetization density with the Glauber dynamics at any temperature T. Their meaning will be given in detail later. In particular, when $T > T_c$, the stationary PDF of the magnetization density with support $[0, a]$ takes the form of a power law

$$\boxed{P_r^{\mathrm{par}}(m) = \frac{r}{\lambda a^{\frac{r}{\lambda}}} m^{\frac{r}{\lambda}-1}}, \tag{3.1}$$

where λ is the decay rate in the Glauber evolution of the magnetization in absence of resetting. Moreover, since the 2D Ising model in the paramagnetic phase and the 1D Ising model share the same time evolution of the ensemble-average magnetization m, i.e. in both cases it decays exponentially to 0, the PDF of m given by Eq. (3.1) takes exactly the same form of Eq. (2.21). However, this analogy is strictly true only at temperatures much larger than the critical temperature: as we will see later, the reason is that m does not decay to 0 as a simple exponential when T is greater but close to T_c.

In the ferromagnetic phase, when $T < T_c$, the stationary PDF of the magnetization density is given by

$$\boxed{P_r^{\text{ferr}}(m) = \frac{r}{bc[f(m)]^{\frac{c-1}{c}}|m - m_{eq}|} \exp\left\{ -r[f(m)]^{\frac{1}{c}} \right\}}, \tag{3.2}$$

where $f(m) = \frac{1}{b} \ln\left(\left|\frac{a}{m-m_{eq}}\right|\right)$ and m_{eq} is the equilibrium magnetization. In particular, its support is $[m_{eq}, m_{eq} + a]$ if $m(0) > m_{eq}$ or $[m_{eq} - a, m_{eq}]$ if $m(0) < m_{eq}$, as we will show later.

At the critical temperature T_c, the stationary PDF of m has a support $[0, +\infty]$ and reads

$$\boxed{P_r^{\text{crit}}(m) = \frac{ra^{\frac{1}{b}}}{bm^{\frac{b+1}{b}}} \exp\left[-r\left(\frac{a}{m}\right)^{\frac{1}{b}} \right]}. \tag{3.3}$$

Focussing on the m dependence of the three stationary PDFs, we indeed see that the latter strongly depends on the phase of the system. In particular, in the paramagnetic phase, when the resetting rate takes the value $r^* = \lambda$, the stationary PDF given by Eq. (3.1) undergoes a transition that dramatically changes its behaviour in the vicinity of the equilibrium magnetization $m_{eq} = 0$. As we will show in more detail, all these theoretical predictions are confirmed by Monte Carlo simulations.

3.2 Stochastic resetting as a renewal process

The Glauber dynamics of the 2D Ising model cannot be solved exactly. Hence we exploit a general correspondence between resetting processes and renewal theory to obtain an alternative description of the nonequilibrium stationary state when a resetting rate r is introduced in the system.

In a general framework, a resetting event simply interrupts the deterministic evolution of a system. Therefore, the state of the system at time t uniquely depends on the time τ elapsed since the last resetting event. In particular, since the resetting process is a Poisson process with rate r, $e^{-r\tau}$ is the probability that no reset occurs in the time interval $[t - \tau, t]$ and $rd\tau$ is the probability that a reset occurs in $d\tau$. Hence, the probability that the time elapsed since the last resetting event is τ is simply given by $p(\tau)d\tau = re^{-r\tau}d\tau$. Of course there is also the possibility that the system evolves until time t without experiencing any reset: this event occurs with probability e^{-rt}, that is the probability that no reset occurs in the whole time interval $[0, t]$.

Therefore, denoting by $P(m, t)$ the probability density that the 2D Ising model has magnetization m at time t, we can write the same probability density in presence of a resetting rate r, $P_r(m, t)$, as a sum of two contributions:

$$P_r(m, t) = \int_0^t re^{-r\tau} P(m, \tau)d\tau + e^{-rt}P(m, t). \tag{3.4}$$

The first term collects all the contributions due to the occurrence of a reset process at time $t - \tau$ for any possible $\tau \in [0, t]$. The second term represents the probability that no reset has occurred up to time t. In the limit $t \to \infty$, the second term clearly vanishes and we obtain the nonequilibrium stationary PDF of the magnetization density:

$$\boxed{P_r^{\text{stat}}(m) = \int_0^\infty re^{-r\tau}P(m,\tau)d\tau}. \tag{3.5}$$

Hence, estimating the PDF $P(m,t)$ in absence of resetting, we easily obtain $P_r^{\text{stat}}(m)$ through Eq. (3.5), thus characterizing the stationary state of the 2D Ising model in presence of resetting. Remarkably, Eq. (3.5) is a general expression that links the PDF of a random variable with the PDF of the same random variable in presence of a stochastic resetting process.

Our strategy consists in the evaluation of $P(m,t)$ without explicitly solving the master equation associated to it. Indeed, since m is defined as the ensemble-average magnetization $m(t) = \frac{1}{L^2}\sum_{i,j=1}^{L}\langle s_{i,j}(t)\rangle$, then m has a deterministic time evolution. Actually, the unique source of stochasticity comes from the introduction of a non zero resetting rate r. Therefore, we can simply substitute $P(m,t)$ with a delta function of the form $\delta(m-g(t))$, where $g(t)$ is the (deterministic) time evolution of the ensemble-average magnetization density m. In this way, we costrain m to be equal to $g(t)$ $\forall t$. The approximation in this reasoning is only due to the fact that the expression of $g(t)$, together with its parameters (a, b, c that we previously mentioned), needs to be estimated throughout Monte Carlo simulations.

In fact, the expression of $g(t)$ (and of course also of $P(m,t)$) depends on the temperature T at which the Glauber dynamics is performed. The details of the estimate of $P(m,t)$ in the two phases (ferromagnetic and paramagnetic) and at the critical temperature are described in the following sections.

3.2.1 General expression of the cumulative distribution

Since it is generally more convenient to plot cumulative distributions than probability distributions, we want to compute exactly the CDF

$$G(m) = \int_{-\infty}^{m} P_r^{\text{stat}}(m'), \tag{3.6}$$

where $P_r^{\text{stat}}(m)$ is the non equilibrium stationary PDF of m in presence of a resetting rate r (see Eq. (3.5)). We can write $P_r^{\text{stat}}(m)$ as a sligthly more general expression

$$P_r^{\text{stat}}(m) = \int_0^\infty f(t)\delta(m-g(t))dt, \tag{3.7}$$

where of course in our case $f(t) = re^{-rt}$. Then, making the change of variable $z = g(t)$, we can write the previous integral as

$$P_r^{\text{stat}}(m) = \int_{g(0)}^{g(\infty)} \frac{1}{|g'(g^{-1}(z))|} f(g^{-1}(z))\delta(m-z)dz = \frac{1}{|g'(g^{-1}(m))|} f(g^{-1}(m))). \tag{3.8}$$

Since $f^{'}(t) = -rf(t)$, the corresponding CDF is given by

$$G(m) = \pm\frac{1}{r} f(g^{-1}(m)) + const, \tag{3.9}$$

where the sign is $+$ and $const = 0$ if g is a decreasing function of time, while the sign is $-$ and $const = 1$ if g is an increasing function of time. Since in our case $f(t) = re^{-rt}$, Eq. (3.9) reduces to

$$G(m) = \pm e^{-r(g^{-1}(m))} + const. \tag{3.10}$$

In the following we will plot this function in the three different temperature regimes to compare it with the numerical results coming from Monte Carlo simulations.

3.3 Case $T > T_c$

In absence of resetting, at $T > T_c$, the magnetization density of the 2D Ising model decays exponentially in time to $m_{eq} = 0$ with a decay rate that depends on the temperature [18], as in the one dimensional case. Its large-time behavior is therefore given by

$$m(t) \sim ae^{-\lambda t}, \tag{3.11}$$

where a and the decay rate $\lambda = \lambda(T)$ need to be estimated with Monte Carlo simulations, since they cannot be computed exactly as in the one dimensional case. In presence of a constant resetting rate r, using Eq. (3.5) with $P(m, \tau) = \delta(m - ae^{-\lambda\tau})$ (justified by the fact that m is the ensemble-average of the magnetization density), we obtain the stationary PDF of the magnetization density given by Eq. (3.1), which, being equivalent to the one obtained in Eq. (2.21) for the 1D Ising model, shows two different behaviors in the case $r > \lambda$ and $r < \lambda$. Its first moment is given by

$$\langle m(r)\rangle = \int_0^{+\infty} re^{-rt}ae^{-\lambda t}dt = a\frac{r}{r+\lambda}. \tag{3.12}$$

It is important to note that, differently from the one dimensional case, the time evolution of the magnetization density m described by Eq. (3.11) is not exact, since it is valid only asymptotically for large t. As a consequence, in presence of a resetting rate r, its validity is guaranteed only for small values of r. Hence, as we will show later, we expect our derivation of the nonequilibrium stationary PDF of the magnetization density to be valid only in the small r regime.

As shown in Fig. (3.1), for the paramagnetic phase, we perform Monte Carlo simulations on a 256×256 square lattice at temperature $T = 3.5$ with $J = k_B = 1$ (with this choice the critical temperature is $T_c = 2.269$). The parameters $a = 0.889$ and $\lambda = 0.117$ of Eq. (3.11) are estimated in the long time limit. The main plot in Fig. 3.2 shows the transition between the stationary cumulative distribution function of the magnetization density in presence of a resetting rate $r = 0.01966$ (blue line) and the one with $r = 0.655$ (green line). The fixed initial configuration to which the system is reset has magnetization $m_0 = 0.9905$. The CDF $F_r^{\text{par}}(m)$ obtained by taking the primitive of Eq. (3.1) is also plotted for the two values of r (red dotted lines). The discrepancy at $m \simeq 0$ in the case of $r = 0.01966$ is due to finite size effects: in Monte Carlo simulations, m can assume negative values because the total number of lattice sites is finite, while in the thermodynamic limit this fact does not happen. The worse agreement between theory and numerics in the other case is due to the large value of $r = 0.655$, since Eq. (3.11) is correct only in the long time limit. Notice that for $r^* = \lambda = 0.117$ we would have a straight line, corresponding to the CDF of a uniform distribution. The inset plot shows the behaviour of the average magnetization density with respect to r.

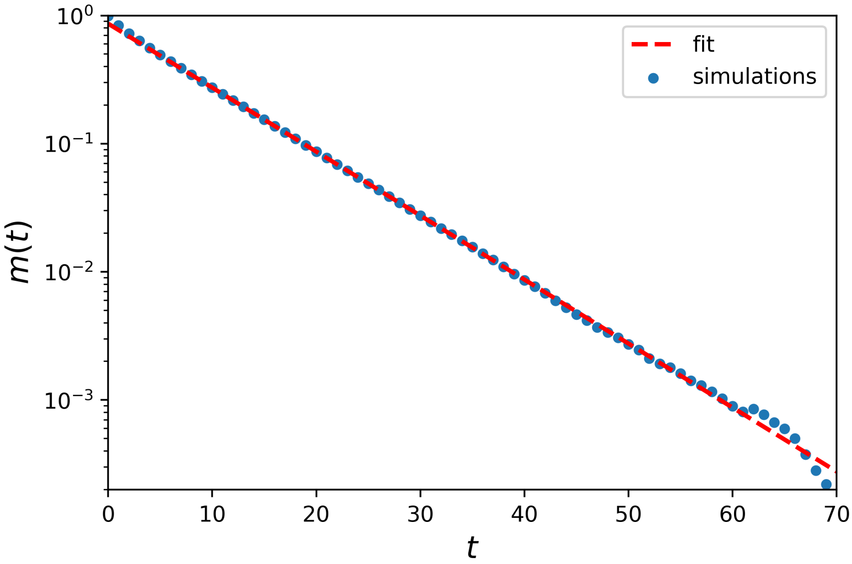

Figure 3.1: Numerical simulations and fit for the time evolution of $m(t)$ in the paramagnetic phase ($T = 3.5$).

Note that for $r \to 0$, where the use of the time evolution of m in Eq. (3.11) is well justified, the agreement between simulations (blue dots) and Eq. (3.12) (red line) is excellent, because the fluctuations due to finite size effects balance perfectly and do not affect the average $\langle m(r) \rangle$.

3.4 Case $T < T_c$

In the ferromagnetic phase and in absence of resetting, the magnetization density of the 2D Ising model reaches a nonzero equilibrium value m_{eq} with a stretched exponential decay in time [18]. In particular, in the large time limit, the dynamics is given by

$$m(t) \sim m_{eq} \pm a e^{-bt^c}, \tag{3.13}$$

with a, b and $0 < c < 1$ to be determined through numerical simulations. The $+$ and $-$ signs are used in the case $m(0) > m_{eq}$ and $m(0) < m_{eq}$ respectively. When a constant resetting rate r is introduced in the system, the stationary PDF of the magnetization density is given by Eq. (3.2). The behavior of the average magnetization density as a function of r is given by

$$\langle m(r) \rangle = m_{eq} \pm ra \int_0^{+\infty} e^{-rt-bt^c} dt, \tag{3.14}$$

where the last integral can be computed numerically for arbitrary values of r.

As shown in Fig. (3.3), we also perform Monte Carlo simulations of the 2D Ising model in the ferromagnetic phase. At temperature $T = 2$, we estimate the parameters $a = 0.61763$, $b = 0.159$ and $c = 0.5955$ of Eq. (3.13) in the long time limit. The CDF $F_r^{\text{ferr}}(m)$ (obtained by taking the primitive of Eq. (3.2)) is well reproduced by Monte Carlo simulations as shown in the main plot of Fig. 3.4, which refers to the case of

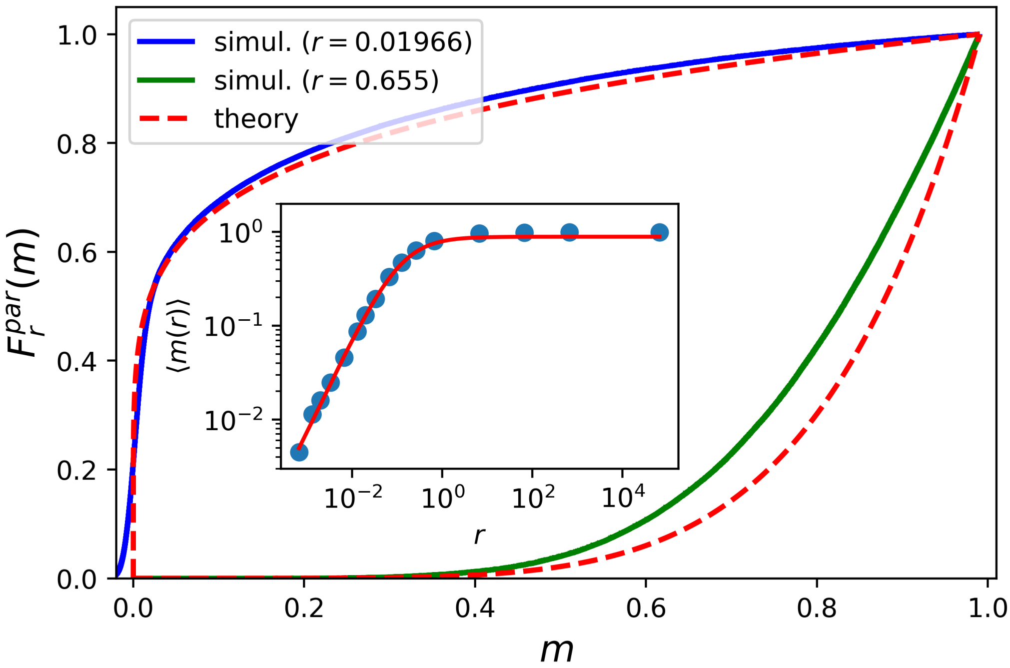

Figure 3.2: The nonequilibrium stationary CDF of m with $r = 0.01966$ and $r = 0.655$ in the paramagnetic phase ($T = 3.5$) obtained from simulations (blue and green lines) are compared to the theoretical prediction given by the primitive of Eq. (3.1) (red dotted line). Inset: Average magnetization $\langle m(r) \rangle$ evaluated from simulations for different values of r (blue dots) and from Eq. (3.12) (red line).

$r = 0.0131$ and $m_0 = 0.2937$. In particular, we see that the support of the $F_r^{\text{ferr}}(m)$ is $[m_0, m_{eq}]$, with $m_{eq} = 0.91132$ being the equilibrium magnetization at temperature $T = 2$. Since $m_0 < m_{eq}$, we take the minus sign in the previous equations. The inset shows the excellent agreement between theory and simulations about the function $\langle m(r) \rangle$.

3.5 Case $T = T_c$

At the critical temperature $T = T_c$ and in absence of resetting, the magnetization density of the 2D Ising model has a peculiar time evolution [19]:

$$m(t) \sim \begin{cases} m_0 t^{\theta} & \text{for} \quad 0 < t < t_c \\ t^{-\frac{\beta}{\nu z}} & \text{for} \quad t_c < t < L^z \\ e^{-\frac{t}{\tau}} & \text{for} \quad t > L^z \end{cases} , \tag{3.15}$$

where the crossover time $t_c \sim m_0^{-1/\left(\theta + \frac{\beta}{\nu z}\right)}$ is the time at which the magnetization density reaches its maximum value. Hence, if we choose $m_0 \simeq 1$ and we consider big enough resetting rates to prevent to fall into the exponential regime, then the time evolution of $m(t)$ is given only by the power law $t^{-\frac{\beta}{\nu z}}$. This condition is verified in our simulations since $\frac{1}{r_{min}} \ll L^z$, where r_{min} is the lowest value of resetting rate used in

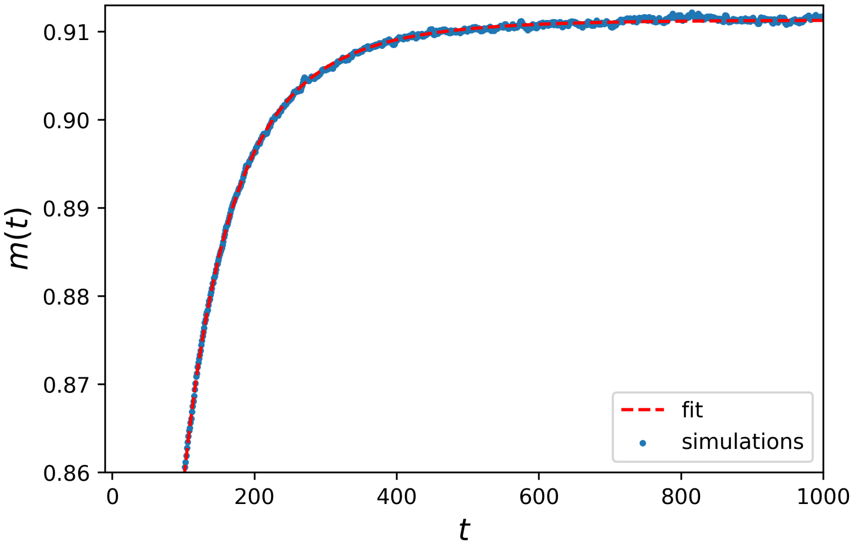

Figure 3.3: Numerical simulations and fit for the time evolution of $m(t)$ in the ferromagnetic phase ($T = 2$).

our simulations. If we consider $m(t) = at^{-b}$ as the time evolution of the magnetization density without resetting, then the PDF of m in presence of a resetting rate r is given by Eq. (3.3), where $b = \frac{\beta}{\nu z}$. In particular, we show that it takes a scaling form as

$$P_r^{\text{crit}}(m) \sim \frac{1}{r^{\frac{\beta}{\nu z}}} G\left(\frac{m}{r^{\frac{\beta}{\nu z}}}\right), \tag{3.16}$$

with

$$G(y) = y^{-1-\frac{\nu z}{\beta}} \exp\left[-\left(\frac{a}{y}\right)^{\frac{\nu z}{\beta}}\right]. \tag{3.17}$$

As shown in Fig. (3.5), Monte Carlo simulations have been done at the critical temperature $T_c = 2.269$ (with $J = k_B = 1$), while we choose $m_0 = 0.9905$ as the fixed initial magnetization density to which the system is reset at a constant rate r. We estimate the parameters $a = 0.9576$ and $b = 0.0576$, leading to a value of the dynamical critical exponent of $z = 2.17$ (if we take $\beta = \frac{1}{8}$ and $\nu = 1$), which is in excellent agreement with previous large-scale simulations. In Fig. 3.6 the CDFs $F_r^{\text{crit}}(m)$ for three different resetting rates ($r = 0.01966$, $r = 0.0131$, $r = 0.00655$) obtained from Monte Carlo simulations are plotted in the rescaled variable $\frac{m}{r^{\frac{\beta}{\nu z}}}$ together with the primitive of the scaling function in Eq. (3.17).

The behavior of the average magnetization density as a function of r, which is plotted in the inset of Fig. 3.6, is given by

$$\langle m(r) \rangle = \int_0^{+\infty} re^{-rt} at^{-\frac{\beta}{\nu z}} dt = a\,\Gamma\left(1 - \frac{\beta}{\nu z}\right) r^{\frac{\beta}{\nu z}}, \tag{3.18}$$

where Γ is the gamma function.

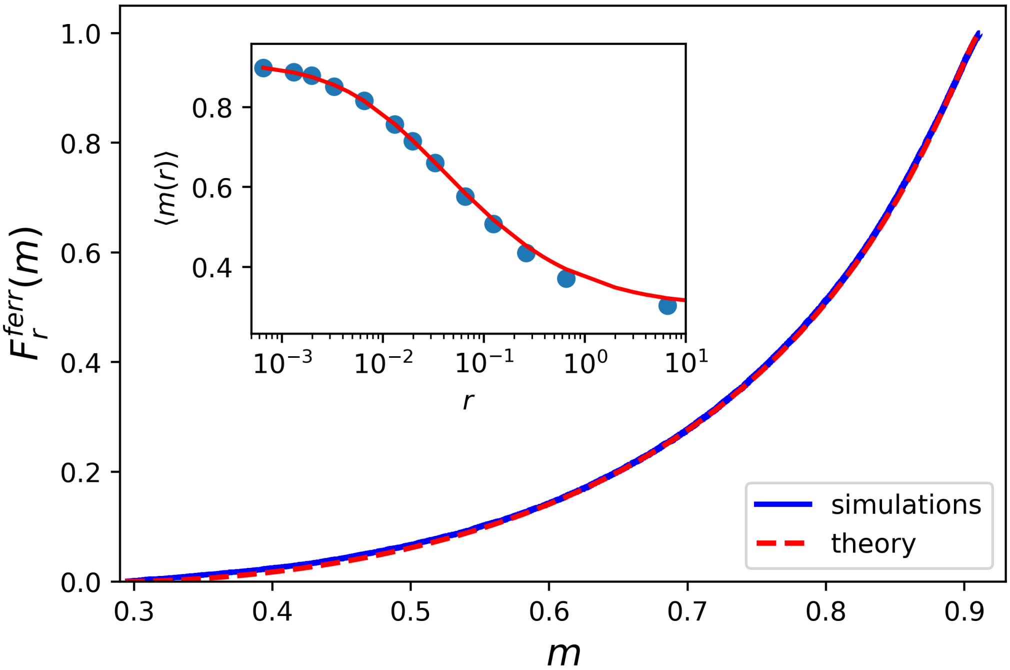

Figure 3.4: The nonequilibrium stationary CDF of m with $r = 0.0131$ and $m_0 = 0.2937$ in the ferromagnetic phase ($T = 2$) obtained from simulations (blue line) is compared to the theoretical prediction given by the primitive of Eq. (3.2) (red dotted line). Inset: Average magnetization $\langle m(r) \rangle$ evaluated from simulations for different values of r (blue dots) and from Eq. (3.14) (red line).

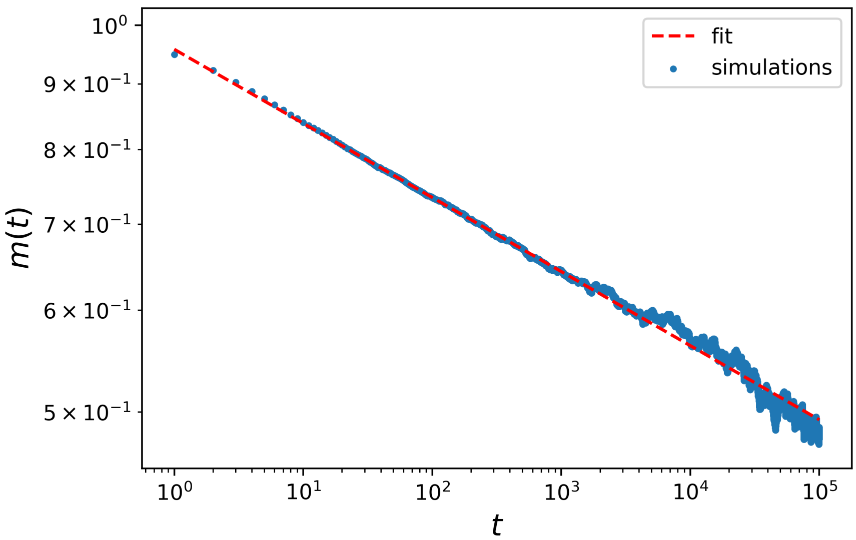

Figure 3.5: Numerical simulations and fit for the time evolution of $m(t)$ at the critical temperature ($T_c = 2.269$).

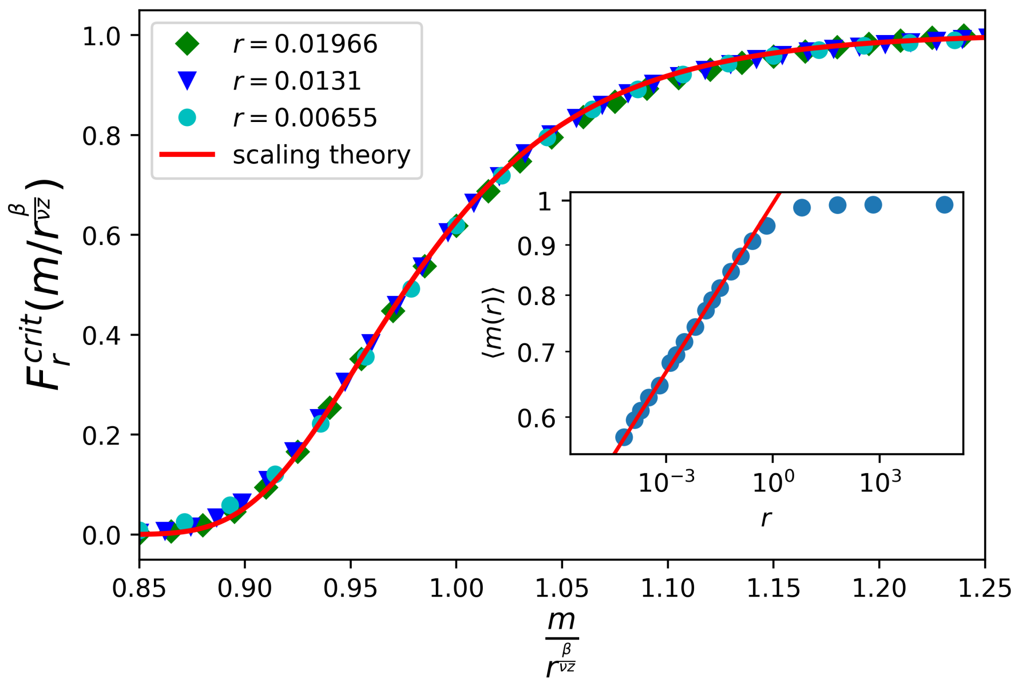

Figure 3.6: The scaling relation given by Eq. (3.16) is confirmed by numerical simulations, performed with three different values of r. Inset (in log-log scale): Average magnetization $\langle m(r) \rangle$ evaluated from simulations for different values of r (blue dots) and from Eq. (3.14) (red line).

Conclusions

This thesis presents an original study on the dynamics of the Ising model. In the first part the equilibrium properties of the 2D Ising model have been studied with Monte Carlo simulations, focusing in particular on the determination of the critical exponents. The dynamical behaviour of the 2D Ising model has also been explored, giving rise to interesting properties, like the the presence of a dynamical exponent z and an initial slip at the critical temperature.

The second part of the thesis is an original study on the Glauber dynamics of the Ising model with a stochastic resetting rate r, that gives rise to a non equilibrium stationary state, as expected. In particular, in the one dimensional case, the theoretical derivation of the out of equilibrium stationary distribution of the magnetization density m has been carried out in an exact way. Eq. (2.21) shows the presence of a crossover when the resetting rate r is equal to the exponential decay rate c. This property has been perfectly verified with Monte Carlo simulations.

The presence of a stochastic resetting rate induces non trivial effects also in the two dimensional case. The stationary probability distribution and its first moment have been derived in the paramagnetic and ferromagnetic phase and at the critical temperature. The theoretical predictions have been tested with Monte Carlo simulations giving promising outcomes.

The results presented in this thesis can be applied to all the fields in which the Ising model is used to model the dynamics of a system. Moreover, particular systems in which any kind of resetting may play an important role, like searching algorithms and target problems, can be decribed by the models proposed in this thesis.

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