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On Shocks in Economic Networks

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Summary

Networks describe complex systems composed by a multitude of agents and tangled interactions. An economic system is an example of an interaction network: a set of economic entities, such as banks or production companies, are linked by a network of interdependencies. The network could works as a filter that propagates and expands microeconomic shocks until they become potentially destructive for the whole macroeconomic system. In this thesis we provide a way of characterizing the aggregate level through the use of local measures such as the Katz-Bonacich centrality. We first introduce two theoretical models of shocks over economic network, describing how agents interact on a network and how the aggregate level is affected by the microeconomic states. Then, we examine how Katz-Bonacich centrality and other network's statistics resume the effects of shocks over the macro state of the economy. We show these results using two particular economies and we analyze some optimization problems. This last part allows us to comparing different networks topologies and enables us to rank economic structures in terms of their macroeconomic state.

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Chapter 1

Introduction

Today's interlinked world has bring the public attention to describe the complex connectedness of modern society. The key argument of this revolution is the idea of *network*: a mathematical model of interconnections between a set of entities. In the most basic sense, a network is any collection of objects in which some pairs of these objects are connected by links.

The first and most known example of network is the society. The sum of all human relations are called the *social networks* and originally described "the core of the society and determines the spread of knowledge, goods and resources" [4]. Nowadays, due to technological advances, the term social network defines all the revolutionary technologies, as Facebook or Twitter, that makes the spread of information even faster than in the past, facilitating distant travel, global communication and digital interaction. Networks also refer to artificial systems. It is the case of our economic system that has become dependent on networks of enormous complexity. An *economic network* is a a set of economic entities, such as banks or production companies, that are linked by a network of interdependencies. Consider for example production firms, each of which producing different outputs that could be used both from consumers and as input from other units. The tangled nature of interactions has made the economic networks susceptible to disruptions that spread through the underlying network structures, sometimes turning local shocks into cascading failures or financial crises.

With the current thesis we want to analyze economic networks and a problem that arises considering interdependencies between firms: the systemic failure and the amplification of shocks. Differently from social networks where sciences try to understand some hidden natural law, the contradiction of economic networks is that they are systems constructed by humans but not understood at all. A brilliant explanation for the starting challenge of economic networks is given in article [20]:

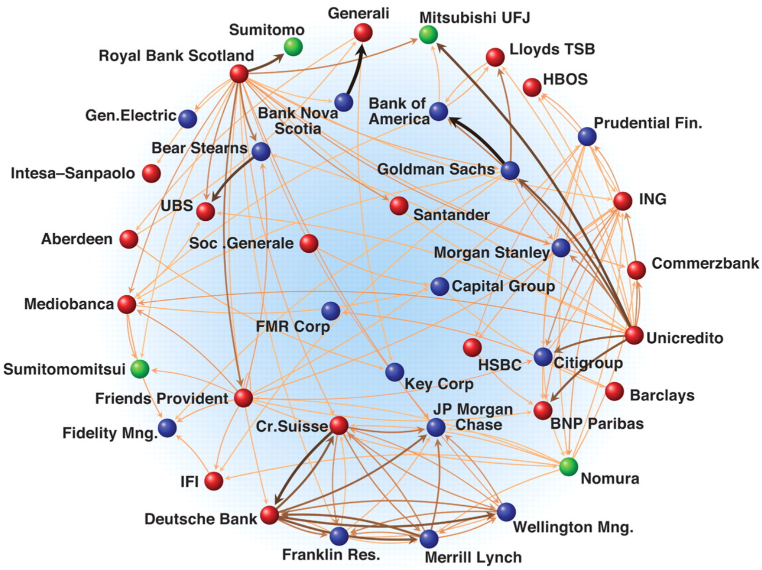

Figure 1.1. A sample of the international financial network, where the nodes represent major financial institutions and the links are both directed and weighted and represent the strongest existing relations among them. Node colors express different geographical areas: European Union members (red), North America (blue), other countries (green) [20].

> We need, therefore, an approach that stresses the systemic complexity of economic networks and that can be used to revise and extend established paradigms in economic theory. This will facilitate the design of policies that reduce conflicts between individual interests and global efficiency, as well as reduce the risk of global failure by making economic networks more robust.

Economy reflects the problem of a large number of interacting agents who influence each other. The resulting aggregate behavior, i.e. the totality of the micro levels, often shows consequences that are hard to predict, as illustrated by the 2008 crisis, which cannot only be explained by the failure of a few major agents. Economic networks are subject to amplifications that may result from the redistribution of agent's failure. In this contest, connections could act as channels of amplification and propagation of *shocks*: if a single node fails, it may force other nodes to fail as well; this may eventually lead to failure cascades and the breakdown of the system, denoted as *systemic risk*. However, it is not well understood how the underlying interaction network affects the chance of a systemic failure.

The staring point of the thesis is the article “Networks, Shocks, and Systemic Risk” [2]; this paper joined with “The Network origins of Aggregate Fluctuations” [1] and “From Micro to Macro via Production Networks” [9] constitute a complete survey on the field of shocks over economic networks with a specific attention spent on production systems based on the Cobb-Douglas production technologies. The main purpose of these papers is to formalize a general theory of how *idiosyncratic* shocks shape the economic outcome. Formally, shocks are represented as stochastic variables and the adjective idiosyncratic means that they are independent and identically distributed.

Our aim in this thesis is to complete the analysis done in [2] spending more attention on the role of network measures. Instead of studying a general model we have considered two macroeconomic state: the total *activity* and the total *welfare*. The first focuses on the sum of players' actions while the second studies the sum of utilities. In the thesis we have focused on the analysis of *Katz-Bonacich* centrality measure, a concept that emerged in social science but now used also in economic networks. In fact, this term represents both the social importance of an agent in a network of information than the economic power of a firm in network of production. Compared to the analysis of idiosyncratic shocks, we consider more complicated shock natures. Both from a deterministic and stochastic point of view our goal is to characterize the worst possible shock for a given network. The theoretical part is flanked by a part of simulation in which the results obtained are verified. The networks taken into account were those extensively studied by graph theory, which, even if simpler than the real ones, manage to summarize some patterns present in reality.

The rest of the thesis is organized as follows. Chapter 2 presents basic facts about graph theory and network measures of agents' centrality; in this chapter are described all the instruments to understand further chapters. In Chapter 3, we provide our two models of economic networks flanked by the general framework presented in [2]. In Chapter 4 we explain the *ex-post* analysis focusing on which measures are useful to describe the macro states. Chapter 5 describes the *ex-ante* analysis of our models reflecting on optimizations problems that arise considering extremum values of the macro state of the economy. Chapter 6 concludes.

Chapter 2

Networks

The fundamental mathematical concept to describe a network is that of a graph. In this chapter we introduces some of the basic ideas behind *graph theory*, the study of network structure. This will allow us to formulate basic network properties in a unifying language, independently from the applications. A single mathematical model is than used for many real objects of extremely heterogeneous nature.

Firstly, we introduce graph theory explaining basic facts and definitions, that are necessary to understand economic networks. We also introduce a particular class of graphs to give easy and understandable examples. In the second part, we describe centrality measures explaining how could be used to summaries networks properties. In this part we define the *Katz-Bonacich* centrality, the most important parameter to describe agents' importance. In the last part we focus on the *Leontief matrix*; this matrix demonstrates how goods from one industry are consumed in other industries but it is also a good approach to describe interdependences.

2.1 Graphs

The skeleton of networks are *graphs*: a graph is a mathematical model that specifies the relationships among a collection of items. A graph consists of a set of *nodes* $\mathcal{V}$ which represent the units participating in the network. Nodes are also called vertices, agents, or players, depending on the context. The set of *links*, or edges, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ expresses the interaction between units, that is, between pairs of nodes. The *weight* of interaction between agents i and j is captured by $W_{ij} \geq 0$ that is equal to 0 only when there is no interaction between agents.

Formally, we define a graph as a triple $(\mathcal{V}, \mathcal{E}, \mathbf{W})$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of links, and $\mathbf{W} \in \mathbb{R}_+^{\mathcal{V} \times \mathcal{V}}$ is the weight matrix [13]. $\mathbf{W}$ is a nonnegative square matrix that expresses the strength of each interaction, and whose entries are W_{ij}. If $W_{ij} \in \{0,1\}$ for all nodes i, j in the graph, then we call this matrix *adjacency* matrix.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ is called:

- *unweighted* if $\mathbf{W}$ is the adjacency matrix. In this case we could construct a graph from matrix $\mathbf{W}$ and vice-versa;
- *undirected* if $\mathbf{W}^\top = \mathbf{W}$, where $\mathbf{W}^\top$ means the *transpose* matrix of $\mathbf{W}$. Link between agent i and agent j exists if and only if exists also the link of the same strength in the opposite direction. We express undirected links as unordered pairs $\{i, j\}$. In term of network, it is when an agent i could interacts with agent j only if also the other wants (ex. network of friendship relationships is an undirected network, but network of web pages is not);
- *simple* if it is unweighted, undirected, and does not contain self loops, $W_{ii} = 0$.

Given a simple graph $\mathcal{G}$ we define the *degree* w_i of the node $i \in \mathcal{V}$ as the number of links connected to him, i.e $w_i = \sum_{j \in \mathcal{V}} W_{ij}$ that in compact notation becomes

$$\mathbf{w} = \mathbf{W}\mathbb{1},$$

where $\mathbb{1}$ is the column vector of all 1 of the order of the number of nodes.

We say that two nodes are *neighbors* if they are connected by an edge; the degree of a node also expresses the number of neighbors, and we define the neighborhood of node $i \in \mathcal{V}$ as $\mathcal{N}_i = \{j \in \mathcal{V} | \{i, j\} \in \mathcal{E}\}$. In a simple graph $\mathcal{G}$ with n nodes we define the average degree as $\overline{w} = \frac{1}{n} \sum_{i \in \mathcal{V}} w_i$, and consequently we call $\mathcal{G}$ *regular* if all its nodes have the same degree $\overline{w}$.

The notion that defines the concept of connectivity between units of a network is that of walk. A *walk* from node i to node j is a finite sequence of nodes $\gamma = (i_0, i_1, ..., i_l)$ such that $i_0 = i$, $i_l = j$ and $(i_{h-1}, i_h) \in \mathcal{E}$ for all $h = 1, ..., l$. The length of a walk is l, and we say that a node j is reachable from node i if there exists a walk from i to j.

When there is a walk between any two nodes i, j of the graph $\mathcal{G}$, we say that the graph is *strongly connected*.

2.1.1 Linear Algebra of Graphs

Given that a graph is described by its matrices, graph theory intends to understands property of graphs using linear algebra. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the powers of adjacency matrix contain interesting information on the walks over $\mathcal{G}$.

Remark. *For every $l \geq 0$, the number of length-l walks from node i to a node j in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is equal to the (i, j)-entry of the matrix power $\mathbf{W}^l$ [13].*

Another important matrix associated to the graph is the *normalized weight* matrix $\mathbf{P}$, also called *simple random walk* SRW, defined as

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{W}, \quad \mathbf{D} = \mathrm{diag}(\mathbf{w}),$$

where $\mathbf{D}$ is an invertible diagonal matrix which entries are the degrees. All the entries of matrix $\mathbf{P}$ are nonnegative and moreover each row sums up to 1, i.e. $\mathbf{P}\mathbb{1} = \mathbb{1}$ where $\mathbb{1}$. Matrices that satisfy these properties are called *stochastic*.

Considering a simple graph $\mathcal{G}$, another important property is that of its eigenvalues and eigenvectors. The *spectrum* of $\mathcal{G}$ is the spectrum of its associated matrices, that is its set of eigenvalues together with their multiplicities. Since $\mathcal{G}$ is simple, adjacency matrix is nonnegative and symmetric; important informations about these kind of matrix are declared by *spectral theorem*, that here is presented and whose proof can be found in [22].

Theorem 1 (Spectral theorem)**.** *Let $\mathbf{W} \in \mathbb{R}^{n\times n}$ be a nonnegative and symmetric matrix. Then there are n real numbers $\lambda_1, ..., \lambda_n$ and n orthonormal real vectors $\mathbf{x}_1, ..., \mathbf{x}_n$ such that $\mathbf{x}_i$ is an eigenvector of λ_i.*

One of the main result on nonnegative matrices is expressed by a theory stated by *Perron* and *Frobenius*. The main result states that largest-in-module eigenvalue λ_W of a nonnegative matrix $\mathbf{W}$ is a nonnegative real number with a corresponding eigenvector with nonnegative entries. Moreover, any other eigenvalues μ of $\mathbf{W}$ is smaller in absolute value, $|\mu| \leq \lambda_W$. The largest eigenvalue of a graph is also known as *spectral radius* or *dominant eigenvalue*.

These results have important consequences on the normalized weight matrix $\mathbf{P}$ which are resumed by the proposition below and that are proved in [13].

Proposition 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ be a simple graph and let $\mathbf{P} = \mathbf{D}^{-1}\mathbf{W}$ be its normalized weight matrix. Then*

1. *the dominant eigenvalue is equal to 1, $\lambda_P = 1$;*
2. *there exist a nonnegative vector π such that $\pi^\top \mathbb{1} = 1$ and $\mathbf{P}^\top \pi = \pi$;*
3. *the degree vector $\mathbf{w}$ is an eigenvector of $\mathbf{P}^\top$ relative to λ_P;*
4. *the ones vector $\mathbb{1}$ is an eigenvector of $\mathbf{P}^\top$ relative to λ_P if $\mathcal{G}$ is regular;*
5. *if $\mathcal{G}$ is strongly connected, then $\lambda_P = 1$ is geometrically and algebraically simple and there exist $\mathbf{x} > 0$ such that $\mathbf{P}^\top \mathbf{x} = \mathbf{x}$.*

2.1.2 Examples

The following simple graphs are basic examples which frequently recur in the theory and its applications; these graphs will be used in next chapters as academic representations of simple networks. Here, every graph is described and represented, and later also some spectrum are calculated.

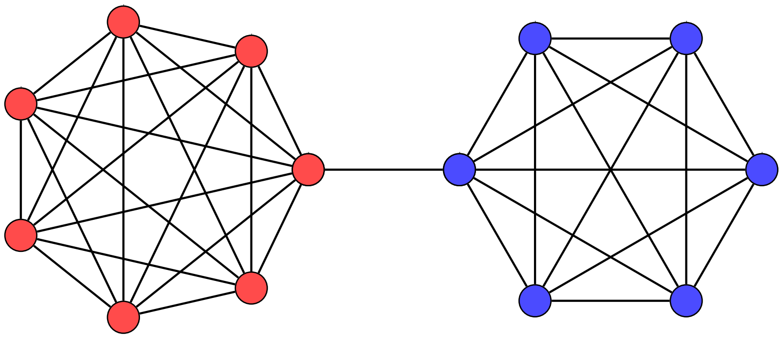

Figure 2.1. Barbell graph with $n_1 = 7$ and $n_2 = 6$

Complete graph

The *complete* graph K_n is a graph consisting of n nodes, each connected to every other node. Defining m as the number of undirected links in the graph, for K_n this results $m = n(n-1)/2$. The complete graph is regular and the average degree, common to all the nodes, is $\overline{w} = n - 1$.

Barbell graph

The *barbell* graph B_{n_1,n_2} is obtained by connecting two distinct complete graphs, respectively with n_1 and n_2, with only one link, called *bridge*, between one node in each side. The resulting number of undirected links is $m = 1 + [n_1(n_1-1) + n_2(n_2-1)]/2$.

Ring graph

The *ring* graph C_n, also called cycle graph, is formed by n nodes all of which of degree 2; consequently the cycle graph is a 2-regular graph. The number of undirected links is simply $m = n$.

Star graph

The *star* graph S_n is a graph where a central node, called *hub*, is connected with all the other n marginal nodes and the marginal nodes are connected only with the hub; the hub has degree $n-1$ while the marginal nodes have all degree 1. The number of undirected links is $m = n - 1$.

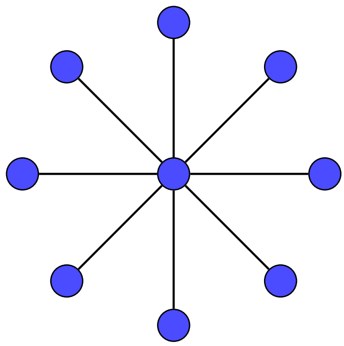

Figure 2.2. Star graph with 8 marginal nodes

Product of graphs

Let $\mathcal{G}^i = (\mathcal{V}^i, \mathcal{E}^i)$ for $i = 1{,}2$, be two simple graphs. The *Cartesian product* of graphs $\mathcal{G}^i$ is a graph $\mathcal{H}$ such that:

- the vertex set of $\mathcal{H}$ is the Cartesian product $\mathcal{V}^1 \times \mathcal{V}^2$;
- any two vertices (v^1, v^2) and (w^1, w^2) are connected by and edge in $\mathcal{H}$ if and only if either

$$\begin{cases} v^1 = w^1, & (v^2, w^2) \in \mathcal{E}^2 \\ v^2 = w^2, & (v^1, w^1) \in \mathcal{E}^1 \end{cases}$$

The number of nodes of $\mathcal{H}$ is $n = |\mathcal{V}^1|\,|\mathcal{V}^2|$, and the number of edges results $m = |\mathcal{E}^1|\,|\mathcal{V}^2| + |\mathcal{E}^2|\,|\mathcal{V}^1|$.

2.2 Cayley graphs

A generalization of regular graphs is described by a large class of graph that originate from groups. Recall that a *group* $(\Gamma, *)$ is defined by a set Γ and an operation $*$ that satisfy these following properties:

- the operation $*$ is *associative*;
- for all $\forall x, y \in \Gamma$, also $x * y$ is an element of Γ;
- there exists an element 1_Γ, called *identity* or *neutral* element, such that $\forall x \in \Gamma$, $1_\Gamma * x = x * 1_\Gamma = x$;
- for any element $x \in \Gamma$ there exists an *inverse* element $x^{-1} \in \Gamma$, such that $x * x^{-1} = x^{-1} * x = 1_\Gamma$.

Moreover, if the operation $*$ is also commutative, the group $(\Gamma, *)$ is called *Abelian*. From now on we will use additive notation: the group operation is denoted by $+$, the neutral element is denoted by 0, and the inverse element is denoted by -1 [16]. Let $(\Gamma, +)$ be a finite Abelian group of order $|\Gamma| = N$, and let S be a subset of Γ with the property that if $x \in S$, then $-x \in S$ and $0 \notin S$; such a subset is called *symmetric*. The subset S is also called the *generating set* of an Abelian group Γ if every element $x \in \Gamma$ could be expressed as the combination of finitely many elements of the subset. If the subset S has only one element, we say that the group Γ is *cyclic* [21].

Suppose that n is a positive integer. We say that two integers x, y are *congruent modulo* n, and writes $x \equiv y \,(\mathrm{mod} n)$, if $x - y$ is divisible by n. This relation is an equivalence relation and gives rise to n equivalence classes, that are called the residues and are denoted by $0,1,...,n-1$. Fix now $n \geq 2$ and consider the set $\mathbb{Z}_n$ of all *residues modulo* n, with the operation $+$. Then $(\mathbb{Z}_n, +)$ is an Abelian group and is a *cyclic* group generated by 1, since any element $x \,(\mathrm{mod} n)$ is a sum of x ones.

Cayley graphs on finite Abelian groups are an useful tool to describe regular graphs. In particular we will consider Cayley graphs on the additive group $\Gamma = \mathbb{Z}_n$ with edge generating set S.

Definition 1. *Let* Γ *be an Abelian group and let* $S \subset \Gamma$ *be a symmetric subset of* Γ*. The Cayley graph* $\mathcal{G}(\Gamma, S)$ *is the graph whose nodes set is* Γ *and edges set is*

$$\mathcal{E} = \{(x, y) \in \Gamma \times \Gamma : y - x \in S\}$$

Notice that a Cayely graph generated by the subset S is simple. The graph is undirected because the existence of link (x, y) implies also (y, x); it contains no self-loops because $0 \notin S$, and is obviously unweighted.

Before giving some examples, we introduce some notions on group characters that will be used to calculate eigenvalues and eigenvectors of Cayley graphs.

Definition 2. *Let* Γ *be a finite Abelian group of order* N*, and let* $\mathbb{C}^\top$ *be the multiplicative group of the nonzero complex numbers. A character on* Γ *is a group homomorphism* $\chi : \Gamma \to \mathbb{C}^\top$*, namely a map* χ *from* Γ *to* $\mathbb{C}^\top$ *such that* $\chi(g + h) = \chi(g)\chi(h)$ *for all* $g, h \in \Gamma$*.*

Since we have that $\chi(g)^N = \chi(Ng) = \chi(0) = 1$ for any $g \in \Gamma$, it follows that χ takes values on the N^{th}-roots of unity [8]. The character that maps all the elements of a group in 1 is called the *trivial* character χ_0. Given a group Γ, the set of all characters of the group forms an Abelian group $\hat{\Gamma}$ with respect to the point-wise multiplication. The unit of the group is the trivial character χ_0, and the inverse of a character is the point-wise conjugate of the character. Moreover, $\hat{\Gamma}$ is isomorphic to Γ and its cardinality is N [8].

Consider the cyclic group $\Gamma = \mathbb{Z}_N$, and suppose $r, x \in \Gamma$. We define

$$e_r(x) := e^{2\pi i r x/N}, \tag{2.1}$$

for every $r \in \{0,1,...,N-1\}$ [21]. Each such a function is clearly a character because:

- is a group homomorphism from the additive group Γ to the multiplicative group of nonzero complex number, $e_r : \Gamma \to \mathbb{C}^\top$;
- maps 0 into trivial character 1, and the $e_r(-x)$ is the multiplicative inverse of $e_r(x)$;
- $e_r(x + y \bmod n) = e^{2\pi i r x/N} \cdot e^{2\pi i r y/N}$;
- moreover, maps into N-th roots of unity.

For every r we have different values of $e_r(1)$, so it follows that for the cyclic group we have N different characters. Defining the inner product of two functions $f_1, f_2 : \Gamma \to \mathbb{C}$ as

$$\langle f_1,\, f_2 \rangle := \sum_{g \in \Gamma} f_1(g) f_2^\top(g).$$

it is possible to prove that the N characters of Γ are orthogonal respect to the inner product, i.e. this means that they are linearly independents [21]. This property is valid for any set of characters from a finite group Γ to $\mathbb{C}$ [22].

Now that we have stated a lot of properties about characters, we could use them for calculating eigenvalues of Cayley graphs. The orthogonality of the characters seems to be equal to the orthogonality of eigenvector and, in fact, there is a nice identity between characters of Abelian groups and eigenvectors and eigenvalue of adjacency matrix associated to a regular graph.

Consider the Cayley graph $\mathcal{G} = (\mathbb{Z}_N, S)$ defined above, and its adjacency matrix $\mathbf{W}$. If we construct a vector $\mathbf{x}_r \in \mathbb{C}^N$ such that its elements are $e_r(s), s \in \Gamma$, is possible to prove that

$$\mathbf{W}\mathbf{x}_r = \left(\sum_{s \in S} e^{2\pi i r s/N} \right) \mathbf{x}_r, \tag{2.2}$$

that is, $\mathbf{x}_r$ is the r-th right eigenvector of matrix $\mathbf{W}$ associated to the eigenvalue $\left(\sum_{s \in S} e_r(s)\right)$ [22]. Than, if we want to obtain the spectrum of a Cayley graph with N nodes, we do not have to compute the characteristic polynomial of matrix $\mathbf{W}$, but we just need to compute some exponential sums.

2.2.1 Eigenvalues of regular graphs

Using the shortcut of eigenvector and eigenvalues of Cayley graphs we now express the spectrum of regular graphs introduced in the previous paragraph 2.1.2. We are interested in eigenvalues of normalized weighted matrix $\mathbf{P}$, that, in case of regular graphs, has a particular linkage with adjacency matrix.

Remark. *Consider a regular graph with n nodes, and its adjacency matrix $\mathbf{W}$ and normalized weight matrix $\mathbf{P}$. Eigenvalues of these two matrices are connected by the relationship*

$$\lambda \in \rho(\mathbf{W}) \Rightarrow \frac{\lambda}{\overline{w}} \in \rho(\mathbf{P}), \quad \lambda \in \rho(\mathbf{P}) \Rightarrow \overline{w}\lambda \in \rho(\mathbf{W}),$$

where $\rho(\mathbf{W})$ indicates the spectrum of $\mathbf{W}$, and $\rho(\mathbf{P})$ the spectrum of $\mathbf{P}$.

Complete graph

Let $G = \mathbb{Z}_n$ and consider the symmetric set $S = \mathbb{Z}_n \setminus \{0\}$; it follows that every two distinct elements $x, y \in \mathbb{Z}_n$ are connected by an edge. The resulting Cayley graph is the complete graph with n nodes K_n.

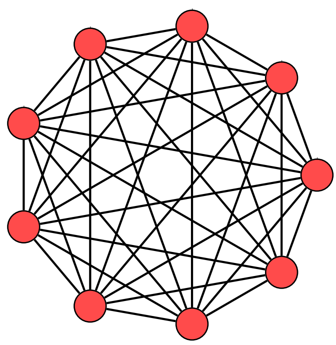

Figure 2.3. Complete graph with 9 nodes

For every $k \in \{0,1,...,n-1\}$, the eigenvalue λ_k of the Adjacency matrix is simply the summation of the characters

$$\lambda_k = \sum_{s \in S} \chi_k(s) = \sum_{s \in S} \exp\left(\frac{2\pi}{n} iks\right),$$

that results

$$\lambda_k = \begin{cases} \cos(\pi k) + 2\sum_{s=1}^{\frac{n}{2}-1} \cos\left(\frac{2\pi}{n}ks\right) \text{for} \quad n \text{ even} \\ 2\sum_{s=1}^{\frac{n-1}{2}} \cos\left(\frac{2\pi}{n}ks\right) \text{for} \quad n \text{ odd.} \end{cases}$$

Assuming λ_0 as the dominant eigenvalue of $\mathbf{W}$, is easy to see that its value is $n-1$ and that all the others $\lambda_k, k = 1, ..., n-1$ are -1. The resulting eigenvalues of $\mathbf{P}$ are derived from the above remark:

$$\lambda \in \rho(\mathbf{W}) \Rightarrow \mu = \frac{\lambda}{n-1} \in \rho(\mathbf{P}) \Rightarrow \rho(\mathbf{P}) = \left\{1, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right\}.$$

Cycle graph

Let $G = \mathbb{Z}_n$ and consider the symmetric set $S = \{+1, -1\}$. Each node $k = 0,1, ..., n-1$ has two neighbors. For example, 0 has the neighbors 1 and $n-1$. Thus, the resulting Cayely graph is the n-cycle graph C_n.

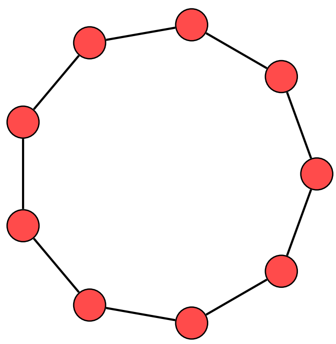

Figure 2.4. Cycle graph with 9 nodes

As done before, we calculate the eigenvalues of the adjacency matrix $\lambda_k \in \sigma(\mathbf{W})$, $k \in \{0,1, ..., n-1\}$ as

$$\lambda_k = \sum_{s=\pm 1} \chi_k(s) = \exp\left(-\frac{2\pi}{n}ik\right) + \exp\left(+\frac{2\pi}{n}ik\right) = 2\cos\left(\frac{2\pi}{n}k\right).$$

Given that the cycle graph is 2-regular, spectrum of matrix $\mathbf{P}$ could be defined as

$$\rho(\mathbf{P}) = \left\{\cos\left(\frac{2\pi}{n}k\right), k = 0, ..., n-1\right\}.$$

Toroidal graph

The toroidal graph is a d-dimensional extension of the cycle graph. More formally, toroidal graph $T_n^{(d)}$ is a *product graph* of d cycle graph where each one has n nodes: $T_n^{(d)} = C_n \times C_n \times ... \times C_n$, d times.

We have seen before that the cycle graph could be rewritten as a Cayley graph $\mathcal{G}(\mathbb{Z}_n, \{+1, -1\})$. Let $G = \mathbb{Z}_n^d = \mathbb{Z}_n \times ... \times \mathbb{Z}_n$, that consists of d-tuples $(x_1, ..., x_d)$ of residues modulo n, that is, each x_i is $0, ..., n-1$. Let S consist of all elements $(x_1, ..., x_d)$ such that only one x_i is equal to 1 or -1 and all others are 0. Then the resulting graph $\mathcal{G}(\mathbb{Z}_n^d, S)$ is a d-dimensional toroidal graph.

Eigenvalues of product graphs have a particular form, expressed by this remark [13]:

Remark. *Be λ an eigenvalue of adjacency matrix $\mathbf{W}$ of $G = (\mathcal{V}, \mathcal{E})$, and μ an eigenvalue of adjacency matrix $\mathbf{V}$ of $G' = (\mathcal{V}', \mathcal{E}')$, each one with all diagonal elements equal to 0. Then $\lambda + \mu$ is an eigenvalue of adjacency matrix $\mathbf{S}$ of product graph $H = G \times G'$.*

Remembering that the spectrum of cycle graph is $\rho(\mathbf{W}_{C_n}) = \{2\cos(2\pi k/n), k = 0,1, ..., n-1\}$ it is possible to calculate the spectrum of $T_n^{(d)}$ as

$$\rho(\mathbf{W}) = \left\{ \lambda_{k_1 \dots k_d} = \sum_{i=1}^{d} 2\cos\left(\frac{2\pi}{n} k_i\right), \quad (k_1, ..., k_d) \in \{0,1, ..., n-1\}^d \right\}.$$

Therefore, noting that the graph is d-regular, the spectrum of $\mathbf{P}$ is

$$\rho(\mathbf{P}) = \left\{ \mu_{k_1 \dots k_d} = \frac{\lambda_{k_1 \dots k_d}}{d}, \quad (k_1, ..., k_d) \in \{0,1, ..., n-1\}^d \right\}.$$

2.3 Centrality measures

Consider a network represented by a graph. In many cases, we might be interested in micro measures that allow us to compare nodes and to describe which role a given node has related to the entire network. The *centrality* of a node resumes this ideas and somehow captures the importance of node's position.

Measures of centrality differentiate depending on the types of statistics on which they are based, for example degree centrality or closeness centrality. We will focus only on *neighborhood* centrality, that determines the importance of a node as a function of how important its neighbors are. This concept go beyond the number of neighbors and accounts for the fact that a node is more central if is connected with other important nodes. Being chosen by a popular individual should add more to one's popularity. Being nominated as powerful by someone seen by others as powerful should contribute more to one's perceived power. Having power over someone who in turn has power over others makes one more powerful [10].

The difficulty is that such a definition is recursive because define the centrality of a node as function of the centrality of its neighbors, and so forth.

2.3.1 Eigenvector centrality

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple graph with n nodes, and $\mathbf{W}$ the associated adjacency matrix. The *eigenvector* centrality e_i of a vertex i is proportional to the sum of the centralities of the vertices which is connected

$$\lambda e_i = \sum_{j=1}^{n} W_{ji} e_i,$$

where λ is a proportional factor. Above equation written in matrix form better explain the name of the measure

$$\lambda \mathbf{e} = \mathbf{W}\mathbf{e}, \tag{2.3}$$

where $\mathbf{e} \in \mathbb{R}^n$ is the centrality vector [5]. Thus, $\mathbf{e}$ is an eigenvector of $\mathbf{W}$ and λ is its corresponding eigenvalue. Given that we look for nonnegative measures, the largest eigenvalue is usually the preferred one because, as mentioned in section 2.1.1, Perron-Frobenius theorem ensures the existence of a nonnegative vector associated to it.

2.3.2 Katz prestige

Katz *prestige* is defined by its author as:

> A new method of computing status, taking into account not only the number of direct "votes" received by each individual but, also, the status of each individual who chooses the first, the status of each who chooses these in turn, etc. Thus, the proposed new index allows for who chooses as well as how many choose [18].

Rephrasing, the prestige of a node, i.e. the centrality, is a weighted sum of its connections. A link of length 1 weight α, a walk of length 2 weight α^2, and so on. Formally, the parameter α is a decay factor that reduces the weight of longer paths.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be again a simple graph with n nodes, and $\mathbf{W}$ the associated adjacency matrix. As seen in the remark 2.1.1, the (i, j) entry of k-th power of $\mathbf{W}$ expresses the number of indirect walks from node i to node j of length k. Given a scalar $0 \leq \alpha < 1$, we define the vector $\mathbf{t} \in \mathbb{R}^n$ of prestige of nodes as

$$\begin{aligned}\mathbf{t}(\alpha) &= \alpha\mathbf{W}\mathbb{1} + \alpha^2\mathbf{W}^2\mathbb{1} + \alpha^3\mathbf{W}^3\mathbb{1} + ... \\ &= \left(\mathbb{I} + \alpha\mathbf{W} + \alpha^2\mathbf{W}^2 + ...\right)\alpha\mathbf{W}\mathbb{1} \\ &= \left(\sum_{i=0}^{\infty} \alpha^i\mathbf{W}^i\right)\alpha\mathbf{W}\mathbb{1}.\end{aligned}$$

A sufficient condition for the above series to be finite is that α be smaller than 1 over the norm of the largest eigenvalue of $\mathbf{W}$, that is, $0 \leq \alpha < 1/(n-1)$ [12] [10]. For an α that satisfy this condition, and using the *Neuman series* of a matrix [19],the prestige vector could be rewritten as

$$\mathbf{t}(\alpha) = (\mathbb{I} - \alpha\mathbf{W})^{-1}\alpha\mathbf{W}\mathbb{1}. \tag{2.4}$$

This values of α also ensure that $(\mathbb{I} - \alpha\mathbf{W})^{-1}$ exists and is nonnegative.

2.3.3 Bonacich centrality

Let $\mathbf{P}$ be the normalized weighted matrix of the simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and consider the transpose $\mathbf{P}^\top$ in order to consider a column-stochastic matrix. Given that the largest eigenvalue of $\mathbf{P}$ is equal to 1, we could rewrite equation (2.3) as

$$\mathbf{e} = \mathbf{P}^\top\mathbf{e}. \tag{2.5}$$

Finding the unit eigenvector of $\mathbf{P}$, namely calculating the eigenvector associated to the largest eigenvalue, is equal to calculating the Bonacich eigenvector centrality. If $\mathcal{G}$ is strongly connected than the Bonacich eigenvector centrality vector is unique up to normalization. In this case $\mathbf{e}$ is also known as *invariant distribution* for $\mathbf{P}$, that has a crucial role in *flow dynamics* on graph.

Note that, this measure is equal to the eigenvector centrality when we consider a weighted matrix instead of the adjacency matrix.

Bonacich has also defined another centrality measure, that could be thought as an extension of the Katz prestige. This last measure introduce "a parameter β that reflects the degree to which an individual's status is a function of the statuses of those to whom he or she is connected" [5].

Considering the graph $\mathcal{G}$ defined before, the Bonacich centrality vector is

$$\mathbf{c}(\gamma, \beta) = (\mathbb{I} - \beta\mathbf{W})^{-1}\gamma\mathbf{W}\mathbb{1}, \tag{2.6}$$

where $\gamma, \beta \in \mathbb{R}$ are positive scalar and β is sufficiently small so that the above equation is well defined, as seen before. There is an important relation between the Katz prestige, or Bonacich centrality with $\gamma = \beta$, and eigenvector centrality; in fact, $\mathbf{c}(\gamma, \beta)$ usually approaches $\mathbf{e}$ as β approaches its extreme values, i.e. the reciprocal of the largest eigenvalue of $\mathbf{W}$.

2.3.4 Katz-Bonacich centrality

We now define *Katz-Bonacich* centrality, an affine transformation of the original Bonacich and Katz centralities that will be used later on in describing economic networks. Given a scalar $\alpha \geq 0$, a simple graph $\mathcal{G} = (\mathcal{V}, \mathbf{E})$, and the associated SRW $\mathbf{P}$, we define the *Leontief* matrix

$$\mathbf{L}(\alpha) = (\mathbb{I} - \alpha\mathbf{P})^{-1}. \tag{2.7}$$

Once again, this equation is well defined when $\alpha < 1$, i.e. smaller than the inverse of the largest eigenvalue of $\mathbf{P}$. The entry ℓ_{ij} captures the direct and indirect connections between agents i and j.

Definition 3. *Consider a network with a normalized weighted matrix* $\mathbf{P}$ *and a scalar* α *such that the Leontief matrix* $\mathbf{L}(\alpha)$ *is well defined and nonnegative. The Bonacich centrality vector* $\mathbf{v}(\alpha)$ *of parameter* α *is*

$$\mathbf{v}(\alpha) = \mathbf{L}^\top \mathbb{1}. \tag{2.8}$$

The centrality of agent i is $v_i = \sum_{j=1}^{n} \ell_{ji}$, that is, the total influence of i on the rest of the agents of the network.

The Katz-Bonacich centrality is obtained from Bonacich's measure by an affine transformation [3]. Remembering the previous measures, but changing $\mathbf{W}$ with $\mathbf{P}^\top$ we find that

$$\begin{aligned}
\mathbf{v}(\alpha) &= \mathbb{1} + \mathbf{t}(\alpha) \\
&= \mathbb{1} + \alpha\left(\mathbb{I} - \alpha\mathbf{P}^\top\right)^{-1}\mathbf{P}^\top\mathbb{1} \\
&= \left(\mathbb{I} + \alpha\mathbf{P}^\top + \alpha^2(\mathbf{P}^\top)^2 + \ldots\right)\mathbb{1} \\
&= \left(\mathbb{I} - \alpha\mathbf{P}^\top\right)^{-1}\mathbb{1}.
\end{aligned}$$

A recursive definition of the Katz-Bonacich centrality is

$$\mathbf{v}(\alpha) = \mathbb{1} + \alpha\mathbf{P}^\top\mathbf{v}(\alpha), \tag{2.9}$$

which help us to understand how the interconnectivity of the network is a fundamental actor in the analysis. This point of view defines Katz-Bonacich centrality as a fixed point, the unique solution of the above equation [10]. This expression shows that i has a higher centrality if it is connected with agents that are themselves central

$$v_i(\alpha) = 1 + \alpha\sum_{j=1}^{n} P_{ji}v_j(\alpha) = 1 + \alpha\sum_{j\in\mathcal{N}_i} P_{ji}v_j(\alpha)$$

In a network of n agents the sum of all agents' Bonacich centrality not sum to 1 but to $n/(1-\alpha), \alpha \in [0,1)$. From now on we will call Katz-Bonacich centrality simply *KB* centrality, and we will omit the fact that $\mathbf{v}(\alpha)$ depends on α writing $\mathbf{v}$.

Regular,Star and Barbell graphs

Consider the complete graph, the star graph, and the barbell graph all with n nodes; our aim is to compare Bonacich centrality and the *KB* centrality in this different cases.

Complete graph and star graph represent opposite topology. In the first, each node has clearly the same importance because is connected with all the others. We recall that any agent in a regular graph have the same *KB* centrality; hence, following results could be used for any k-regular graph.

Interest for the star graph is motivated by the presence of the hub; this particular agent seems to play a more important role than the marginal ones due to its central position. Here, we want to quantify the consequences of this inequality between agents' centrality.

The curiosity for barbell graph is given by the existence of the bridge. Even if it is similar to two separated complete graphs we want to understand if the additional link generates sensible difference in *KB* centralities. In all the thesis we will consider a barbell graph formed by two complete graph of the same quantity of agent, i.e. $B_{n/2}$.

In section 2.3.3, we have seen that the Bonacich centralities vector for simple graph simply is the degree vector divided by the total number of links in the graph. For regular graph, as the complete graph, the Bonacich centrality vector is a constant vector where each element is one divided by the number of nodes. For the star graph, except for the central node, all the others have the same Bonacich centrality. In the barbell graph bridge agents have a different Bonacich centrality than other agents.

Precisely, we define the Bonacich centrality vectors of these graphs as

$$\mathbf{e}_{K_n} = \frac{1}{n}\mathbb{1}, \quad \mathbf{e}_{S_n} = \frac{1}{2(n-1)}\begin{pmatrix} n-1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{e}_{B_{n/2}} = \frac{1}{n(n-2)+4}\begin{pmatrix} n \\ n \\ n-1 \\ \vdots \\ n-1 \end{pmatrix}.$$

Calculating the *KB* centrality means to understand the behavior of Leontief matrix. Instead of finding the inverse of $(\mathbb{I}-\alpha\mathbf{P})$ we could use the Newmann series to define $\mathbf{L}$ as $\sum_{i=0}^{\infty}\alpha^i\mathbf{P}^i$ and rewrite equation (2.8) as

$$\mathbf{v} = \mathbb{I}\mathbb{1} + \alpha\mathbf{P}^\top\mathbb{1} + \alpha^2(\mathbf{P}^\top)^2\mathbb{1} + \alpha^3(\mathbf{P}^\top)^3\mathbb{1} + \dots \quad . \tag{2.10}$$

Given that the complete graph is regular, matrix $\mathbf{P}_{K_n}$ results doubly stochastic, i.e. both rows and columns sum to one, and the resulting *KB* centralities vector is a constant vector dependents only on the value of alpha

$$\mathbf{v}_{K_n} = \mathbb{1} + \alpha\mathbb{1} + \alpha^2\mathbb{1} + \alpha^3\mathbb{1} + \dots = \frac{1}{1-\alpha}\mathbb{1} \tag{2.11}$$

For the star graph is not possible to proceed in the same way because the matrix $\mathbf{P}_{S_n}$ is not symmetric. Fortunately the powers of the SRW are particularly easy because the matrices with odd powers are all equal to $\mathbf{P}_{odd}$ and with even powers are all equal to $\mathbf{P}_{even}$, where

$$\mathbf{P}_{odd} = \begin{pmatrix} 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} \\ 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad \mathbf{P}_{even} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} \\ \vdots & & \ddots & \\ 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} \end{pmatrix}.$$

This particular behavior allows us to write the Leontief matrix as

$$\begin{aligned} \mathbf{L}_{S_n} &= \mathbb{I} + (\alpha + \alpha^3 + \dots)\mathbf{P}_{odd} + (\alpha^2 + \alpha^4 + \dots)\mathbf{P}_{even} \\ &= \mathbb{I} + \frac{\alpha}{1-\alpha^2}\mathbf{P}_{odd} + \frac{\alpha^2}{1-\alpha^2}\mathbf{P}_{even}. \end{aligned}$$

Considering that $\mathbf{P}_{even}$ is symmetric and stochastic, and $\mathbf{P}_{odd}^\top \mathbb{1}$ has an easy result we rewrite equation (2.10) as

$$\begin{aligned} \mathbf{v}_{S_n} &= \mathbb{I}\mathbb{1} + \frac{\alpha}{1-\alpha^2}\mathbf{P}_{odd}^\top \mathbb{1} + \frac{\alpha^2}{1-\alpha^2}\mathbf{P}_{even}^\top \mathbb{1} \\ &= \frac{1}{1-\alpha^2}\mathbb{1} + \frac{\alpha}{1-\alpha^2}\begin{pmatrix} n-1 \\ \frac{1}{n-1} \\ \vdots \\ \frac{1}{n-1} \end{pmatrix}. \end{aligned}$$

Given this last results we conclude that the star graph has two different values of *KB* centrality, one for the hub and one for marginal nodes

$$\mathbf{v}_{S_n} = \begin{cases} v_i = \dfrac{(n-1)\alpha + 1}{(1-\alpha^2)}, \; i = \text{hub} \\ v_j = \dfrac{(n-1)+\alpha}{(n-1)(1-\alpha^2)}, \; \forall j \neq i \end{cases} \tag{2.12}$$

The easiest way for calculating the *KB* centrality vector for the barbell graph is the recursive method. We could write the recursive system where we consider the two typologies of centralities

$$\begin{cases} v_i = 1 + \alpha \left[v_j + \dfrac{1}{n/2} v_i \right], \; i = \text{bridge} \\ v_j = 1 + \alpha \left[\dfrac{(n/2-2)}{(n/2-1)} v_j + \dfrac{1}{n/2} v_i \right], \; \forall j \neq i \end{cases}.$$

Node i	e_i Bonacich eigenvector	v_i, $\alpha = 0.1$	v_i, $\alpha = 0.5$	v_i, $\alpha = 0.9$
hub	0.5000	0.1727	0.3667	0.4788
marginal	0.0556	0.0919	0.0704	0.0579

Table 2.1. Comparison of centrality measures in a star graph with $n = 10$ nodes

The solution of previous system is

$$\begin{cases} v_i = n\dfrac{(1+\alpha\, v_j)}{(n/2-\alpha)},\ i = \text{bridge} \\ v_j = \dfrac{n(n-2)}{(1-\alpha)(n^2-2n+4\alpha)}\ \forall j \neq i \end{cases},$$

that for $n \to \infty$ becomes

$$v_j = \frac{1}{(1-\alpha)}, \quad v_{bridge} = \frac{1}{(1-\alpha)}. \tag{2.13}$$

This result highlight that the added link does not influence the vector of *KB* centralities.

We now illustrate these results considering a complete network and a star network, both with $n = 10$ nodes. Comparing the *KB* centrality with the Bonacich one is possible if we make the first sum to 1, and this could be done multiplying the vector $\mathbf{v}$ for the inverse of its sum $(1-\alpha)/n$.

As Bonacich states in the article [5], the parameter α " reflects the degree to which authority or communication is transmitted locally or to the structure as a whole". In the case of star graph is perfectly clear that the central agent plays a predominant role in the network: it is the authority.

Table 2.1 resumes the comparison between Bonacich eigenvector and *KB* centralities vector of the star graph when both sum to 1. Katz-Bonacich values represent the portion of total influence, and as $\alpha \to 1$ they equal Bonacich eigenvector values. As $\alpha \to 1$ the sum of centralities is divided into two parts: one for the hub and the other half for all the marginal nodes. *KB* centrality and the Bonacich centrality of regular graph are equal, because equation (2.11) becomes $\mathbf{v}_{K_n} = n^{-1}\mathbb{1}$ when we make the first sum to 1. The interpretation is clear: any agent plays an equal role in the network because each agent has the same amount of neighbors and the weight of all connections is the same. There is not an authority and each node has the same influence.

Figure 2.5 presents the behavior of Katz-Bonacich centrality in a regular network and in a star network.

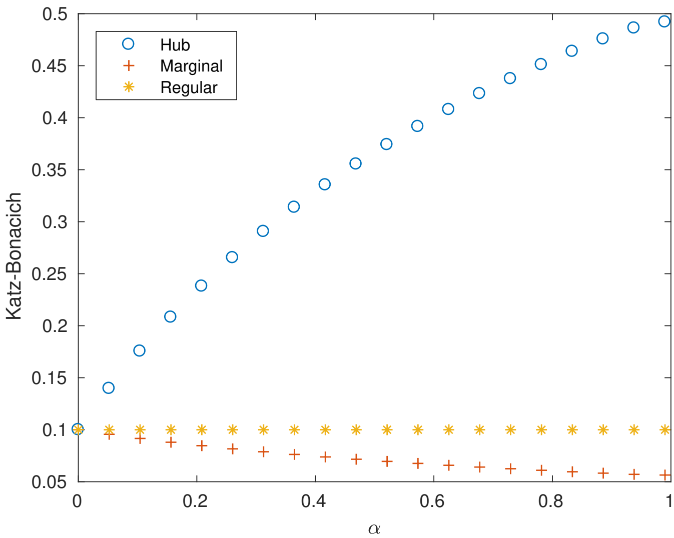

Figure 2.5. Katz-Bonacich centrality values of hub and marginal nodes of the star, and of any node belonging to a regular graph.

2.4 Input-Output matrix

Wassily Leontief published a pair of articles that laid the groundwork for input-output economics. He formulated a set of mathematical equations based on the interdependence of the different parts of an economy and defined this new kind of table, the *input-output* matrix, followed by an empirical implementation of the mathematical model.

"This table records all transactions taking place in the economy in a specific period of time. The capacity to absorb a substantial level of detail, and the conceptual simplicity and transparency of the framework, make the input-output table and the models that manipulate it well suited to evaluating strategies for sustainable development." [11]

The input-output matrix has n rows and n columns, and the (i, j)-th entry represents the amount of product from industry i delivered to industry j in a particular time. Dividing that quantity by the total output of industry j we obtain a coefficient measuring input per unit of output. In this way the input-output matrix of coefficients is column stochastic, and the j-th column represents all the inputs needed to produce one unit of output of industry j.

In the previous section we have used the SRW matrix $\mathbf{P}$ as the input-output matrix. We just have said that the sum of columns of the input-output matrix represent the output degree. The authors intended to have a row stochastic matrix and that the sum of each column expresses the out-degree of agents. Hence, the SRW is defined as $\tilde{\mathbf{P}} = \mathbf{W}\mathbf{D}^{-1}$. For our study, this SRW is simply the transpose of the old one because we consider simple graph with symmetric adjacency matrix. $\tilde{\mathbf{P}}$ is a column stochastic matrix and the rows sum are given by the out-degree. Considering the $\tilde{\mathbf{P}}^\top$ we obtain what authors of article [2] want. For the sake of exposure we will always denote with $\mathbf{P}$ this last matrix $\tilde{\mathbf{P}}^\top$; therefore, the direct edge from vertex j to vertex i is represented by $P_{ij} > 0$ and remarks that the state of agent i is directly affected by the state of agent j.

As we have seen before, the Leontief matrix derived from the input-output matrix and is defined as stated in equation (2.7). It is useful to to link the properties of the SRW to the Leontief matrix. All properties or facts that we now enunciate will be used in the next chapters. When the network is regular, both SRW matrix than Leontief matrix are symmetric. In the general case the Leontief matrix is not always symmetric. A factorization that links the Leontief matrix to its spectral properties is the *singular value decomposition* (SVD). Formally, a SVD of a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ is defined to be a tuple $(\mathbf{V}, \boldsymbol{\Gamma}, \mathbf{U})$ satisfying

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Gamma}\mathbf{U}^\top, \tag{2.14}$$

where

- $\boldsymbol{\Gamma} \in \mathbb{R}^{n\times m}$ is a diagonal matrix with nonnegative real number γ on the diagonal called *singular values*; these non-zero elements are the square roots of the non-zero eigenvalues of $\mathbf{A}^\top\mathbf{A}$ or $\mathbf{A}\,\mathbf{A}^\top$;
- $\mathbf{V} \in \mathbb{R}^{n\times n}$ is an orthonormal matrix whose columns are eigenvector of $\mathbf{A}\,\mathbf{A}^\top$ (a matrix $\mathbf{U}$ is orthonormal if $\mathbf{U}^\top\mathbf{U} = \mathbb{I}$);
- $\mathbf{U} \in \mathbb{R}^{m\times m}$ is an orthonormal matrix whose columns are eigenvector of $\mathbf{A}^\top\mathbf{A}$.

Columns of $\mathbf{V}$ are called *left-hand singular vectors* of $\mathbf{A}$ and columns of $\mathbf{U}$ are called *right-hand singular vectors*. If we think of the columns of $\mathbf{A}$ as n data points we can define the *first principal component* of $\mathbf{A}$ as a “fictitious vector that best summarizes the data set of $\mathbf{A}$” [14]. Is a fact that principal components coincide with left singular vectors.

Recalling that Bonacich centrality $\mathbf{e}$ is defined as the eigenvector of $\mathbf{P}^\top$ associated to 1, is easy to see that it is also an eigenvector of $\mathbf{L}^\top$ associated to eigenvalue $1/(1-\alpha)$. When the SRW matrix is symmetric, the SVD of $\mathbf{P}$ coincides with the eigenvalues decomposition. In this special case, the SVD of $\mathbf{M}$ is simply given by $\left(\mathbb{I} - \alpha\mathbf{V}\boldsymbol{\Lambda}\,\mathbf{V}^\top\right)$, where $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues of $\mathbf{P}$. The resulting columns of $\mathbf{V}$ are the eigenvector of $\mathbf{P}$ and, as a conclusion, Bonacich centrality coincide with the first principal component. We know that when the network is symmetric, Bonacich centrality $\mathbf{e}$ is equal to KB centrality; unfortunately we also know that for not symmetric **P** this is true only when $\alpha \to 1$. However, when we will consider the first principal component of a Leontief matrix we have to remember that it is something similar to KB centralities vector.

Chapter 3

Shocks in Economic Networks

In this chapter we introduce the economic model on networks.

Agents are the active parts of the system. Representing an agent means to define how it interprets its interactions within the network: "player's well-being depends on own action as well as on the actions taken by his or her neighbors" [15]. Any agent is represented by his/her *state* that resumes his/her choice of action or some other economic or social variable of interest. The totality of all agents' state defines the aggregate level of the system. Considering the alteration of the status of some agent we would like to understand how the network and the agents influence the shape of whole system. The study of modeling agents' actions on network belongs to the theory of *network games*, that captures a wide variety of problems, from economic and finance networks to criminal or educational ones [15] [14] [6] [17]. When we talk about studying shocks on economic network, illustrative examples are:

- a production network where agents are competitive sectors each of which producing a distinct product [9];
- a financial market where agents are financial institutions which are linked via debt contracts.

This chapter is organized as follows. In the first part, we introduce the two economic models that we will use during the thesis; we describe which are the measures of interest and the ways to analyze the whole network or the importance of its agents. Given that our models derive from a more general theory, in the second part we describe the general model defined by Acemoglu, et. all in [2], for the study of shocks over economic networks.

3.1 Economic networks models

Consider a system in which n economic agents are interacting. The connections between a finite set of agents $\mathcal{V} = \{1, ..., n\}$ are described by a fixed network. This network is represented by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathbf{W}\}$, where the set of nodes represents the set of agents, and the set of links represents the interaction between agents. The matrix that describes the graph is the adjacency matrix.

Each network's entity i is represented by its state $x_i \in \mathbb{R}$ which captures the agent's choice of action or other economic variable of interest. The vector of all states is denoted $\mathbf{x} = (x_1, ..., x_n)^\top \in \mathbb{R}^n$. In a network the states of different agents are interconnected; the extent of interaction between agent j and agent i is captured by the SRW matrix $\mathbf{P}$ of the graph. The state of any given agent i depends on the states of its neighbors and on a stochastic variable $S_i \in \mathbb{R}$ that represents the *shock* that hits agent i. The multivariate vector of all stochastic shocks is denoted $\mathbf{S} = (S_1, ..., S_n)^\top \in \mathbb{R}^n$, while a realization of shocks is denoted $\mathbf{s} = (s_1, ..., s_n)^\top \in \mathbb{R}^n$. A parameter $\alpha \geq 0$, also known as *state of the world* [23], gives the magnitude of the effect of agents' actions on their neighbors.

Formally, we define an economy of n agents as a triple $\mathfrak{E}_n(\alpha, \mathbf{P}, \mathbf{S})$ using a parameter α, the matrix of interactions $\mathbf{P}$, and the vector of shocks $\mathbf{S}$.

3.1.1 Agents' state

To explicitly model the behavior of individuals we refer to *network games*. Agent i chooses an action x_i simultaneously with others; let $\mathbf{x}_{\mathcal{N}_i}$ denote the profile of actions taken by the neighbors of agent i. Payoffs of agent i depend on his/her own action x_i, others' actions $\mathbf{x}_{\mathcal{N}_i}$, a realization of shocks $\mathbf{s}$, matrix of interactions $\mathbf{P}$, and the parameter α:

$$u_i(x_i, \mathbf{x}_{\mathcal{N}_i}; \alpha, \mathbf{P}, \mathbf{s}) = \alpha s_i x_i - \frac{1}{2} x_i^2 + \alpha x_i \sum_{j \in \mathcal{N}} P_{ij} x_j \tag{3.1}$$

Quadratic payoffs are highly used in many papers that talk about network games [23] [17] [6]. The payoff function has two parts. The first two terms describe the individual part where the marginal benefits are given by $s_i x_i$. The last term of the function describes the *local-aggregate* effect of interactions since the state of agent i is weighted by the sum of efforts of its neighbors. We only consider games of *strategic complements*, that is, where the increase in the actions of other players leads also to an increase of the action of a given player. This specificity is underlined by the fact that the parameter α is positive.

Each agent chooses his/her action to maximize individual payoffs. Let $x_i = f_i(\mathbf{x}_{\mathcal{N}_i}; \alpha, \mathbf{P}, \mathbf{s})$ be the *best reply* of agent i to other agents' action given α, $\mathbf{P}$, and a realization of shocks $\mathbf{s}$. The first-order necessary condition for each agents i's choice of action to maximize his/her payoff is

$$\frac{\partial u_i(x_i, \mathbf{x}_{\mathcal{N}_i}; \alpha, \mathbf{P}, \mathbf{s})}{\partial x_i} = \alpha \left(\sum_{j \in \mathcal{N}_i} P_{ij} x_j + s_i \right) - x_i = 0. \tag{3.2}$$

A *Nash equilibrium* is a vector $\mathbf{x}^* = (x_1^*, ..., x_n^*)$ that satisfies the system of best replies. The above condition leads to

$$x_i^* = \alpha \left(\sum_{j=1}^{n} P_{ij} x_j^* + s_i \right), \quad \forall i, \tag{3.3}$$

that written in matrix notation results

$$\mathbf{x}^* = \alpha \left(\mathbf{P}\, \mathbf{x}^* + \mathbf{s} \right). \tag{3.4}$$

The uniqueness of Nash equilibrium of the game is characterized by the following proposition.

Proposition 2. *Let $\mathfrak{E}_n(\alpha, \mathbf{P}, \mathbf{s})$ be an economy defined by a network of interactions $\mathbf{P}$, parameter α and a realization of shocks $\mathbf{s}$. Then, a unique Nash equilibrium exists if $\alpha < 1$, and the equilibrium states are determined by*

$$\mathbf{x}^* \;=\; \alpha\, \mathbf{L}\, \mathbf{s}, \tag{3.5}$$

where $\mathbf{L} = (\mathbb{I} - \alpha \mathbf{P})^{-1}$ is the Leontief matrix of the economy.

As showed in [7], the equilibrium depends on the largest in module eigenvalue of interactions matrix $\mathbf{P}$. The main result states that when $|\alpha\, \lambda_{max}(\mathbf{P})| < 1$, there is a unique Nash equilibrium. In fact, the Nash equilibrium is simply a solution to the system of linear equations defined by (3.4) and a unique equilibrium exists if $\det(\mathbb{I} - \alpha\, \mathbf{P}) \neq 0$, that is, if the Leontief matrix is invertible. Given that the largest eigenvalue of a stochastic matrix $\mathbf{P}$ is 1, we conclude that $\alpha < 1$.

The same condition on the spectral radius of $\alpha\, \mathbf{P}$ ensures that this equilibrium is asymptotically stable according to the standard conditions for the stability of a system of linear differential equations [6]. The stability condition imposes a joint restriction on the parameter α and the network structure $\mathbf{P}$, which jointly give what is called the network effects. The equilibrium is stable only when these network effects are small enough.

We have seen that this matrix collects information of direct and indirect interactions. Defining the equilibrium of agents' states with the Leontief matrix of the economy means to consider all its interactions as factors of the equilibrium.

Suppose that one shock affects only agent j, how much an agent $i \neq j$ is also influenced? Considering only the i-th agent, we rewrite equation (3.9) emphasizing all the connections as

$$\begin{aligned} x_i^* &= \alpha\, L_{ij} s_j \\ &= \alpha\, s_j \left(\alpha\, P_{ij} + \alpha^2 \sum_{k=1}^{n} P_{ik} P_{kj} + \ldots \right), \end{aligned}$$

where $L_{ij} \in \mathbb{R}$ is the (i, j) entry of matrix $\mathbf{L}$. Even if the two agents, i and j, are not directly connected, the right side of the second equation states that the shock to agent j propagate to the agent i. The term α turns out to be a parameter that weighs the level of interconnections, that is, it gives less importance to longer walks.

3.1.2 Macro states

Aggregation of whole agents' states define the *macro state* y of the system. This variable represents some macroeconomic outcome of interest that is obtained by aggregating the individual states of all agents; it is also an observable measure that help us to classify networks' structure in terms of how the macro level consider the micro ones. The economic theory behind this vision derives on the idea of Nobel price winner Robert Lucas. His theory is based on a new classical approach to macroeconomics that argued how macroeconomic model have to be built as aggregated version of microeconomic model.

We define two different models, each of which exhibit a nonlinearity on agents' state. Formally, we could think the macro state as the output of the economy whose input requirement is the equilibrium vector of state, i.e. $y = \Psi(\mathbf{x}^*)$, $\Psi : \mathbb{R}^n \to \mathbb{R}$. Given that the macro state works on the equilibrium vector, from now on we will simply use the term $\mathbf{x}$ indeed $\mathbf{x}^*$ to indicate the equilibrium configuration.

- The first macro state aims to emphasize the total utility, also called the total welfare [14], at the equilibrium. Using the fact that equilibrium actions satisfy (3.3), two times the sum of the utilities $y = 2\sum_{i=1}^{n} u_i$ becomes

$$y_W = \sum_{i=1}^{n} x_i^2 = \mathbf{x}^\top \mathbf{x}. \tag{3.6}$$

  During the thesis we will refer to this macro state as the *total welfare*.

- The second macro state is defined as the square of equilibrium states' sum

$$y_A = \left(\sum_{i=1}^{n} x_i \right)^2 = \left(\mathbb{1}^\top \mathbf{x} \right)^2, \tag{3.7}$$

This summation represents the aggregate level of activity and for this reason will be called *total activity*.

The relation between micro, macro, and stochastic shocks defines two different analysis. The *ex-post* analysis study the behavior of the network from a deterministic point of view. The vector of shocks $\mathbf{s}$ is considered an observable quantities. The equilibrium definition is then an *ex-post* equilibrium notion that enables to study how the equilibrium varies as a function of the shock realizations. The first analysis mainly focuses on the role that different agents have in shaping the macro state. Since we know a particular realization of shocks, our study will focus on who the central agents are. This problem, in network games theory, is called the *key players* problem. The article [23] gives a definition of key player negative-problem referring to a network of criminals: " a key player is the individual (criminal) to be removed from the network so that total crime is minimized".

The *ex-ante* analysis aims to study the expected value of the macro state, $\mathbb{E}[y]$. As the previous investigation, this study highlights the differences between structure of networks but also depends on the nature of shocks. Given that $\mathbf{S}$ is a multivariate random vector is important to understand the behavior of the macro state in terms of this vector.

3.2 Networks, Shocks, and Systemic Risk

Our work belongs to a more general framework described in the article "Networks, Shocks, and Systemic Risk" [2]. In this paper, the authors consider general functions both for the agents' state and the macro economic state. Their goal is to develop a general theory that could explain how idiosyncratic shocks model different networks of interactions. In the article are always considered *idiosyncratic* shocks, that is, shocks that are independent and identically distributed (IID); a lot of examples are described, from production and financial networks to networks games.

Using the same notation of our thesis, the starting point of their article are two functions. The collection of agents' states is described by the relation

$$\mathbf{x} = f\left(\mathbf{P}\,\mathbf{x} + \mathbf{s}\right),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and increasing function called economy's *interaction function*: it describes the nature of interaction between agents. Given that shocks are idiosyncratic, the multivariate variable $\mathbf{S}$ has also a simple form of variance-covariance matrix, i.e. $\mathbf{\Omega} = \sigma^2\mathbb{I}$. The element P_{ij} still represents the influence that agent j has on agent i, and the matrix $\mathbf{P} \in \mathbb{R}_s^{n\times n}$ has the same characteristic than our interaction matrix.

Following the same definition of equilibrium (3.3), the set of equilibria depends on the economy's network and on the properties of the interaction function; what follows establish the existence and generic uniqueness of the equilibrium.

Theorem 2. *Suppose that there exists $\beta \leq 1$ such that $|f(z) - f(\tilde{z})| \leq \beta |z - \tilde{z}|$ for all $z, \tilde{z} \in \mathbb{R}$. Furthermore, if $\beta = 1$, then there exists $\delta > 0$ such that $|f(z)| < \delta$ for all $z \in \mathbb{R}$. Then, an equilibrium always exists and is generically unique [2].*

The macro state of the economy is defined as

$$y = g\left(h(x_1) + \cdots + h(x_n)\right), \tag{3.8}$$

where $g, h : \mathbb{R} \to \mathbb{R}$. The function g is called the economy's *aggregation function* [2]. The nature of the three functions f, g, and h defines the details of the economy and plays a key role in how the network translates microeconomic shocks into macroeconomic outcomes. The assumption $f(0) = h(0) = g(0) = 0$ ensures that in the absence of shocks the equilibrium state of all agents and the economy's macro state are equal to zero.

The main purpose of the article is to define an explicit dependence of the macro state y of the economy on idiosyncratic shocks and on the curvatures of the functions. To simplify the analysis, it is assumed that the economy is *smooth*, in the sense that the functions f, g, and h are at least twice differentiable and the agent-level shocks are small. These assumptions allows to develop an approximation of the equilibrium state of each agent to the first and second order of their Taylor expansions.

Starting from the interaction function, it results that the equilibrium states, around the point of null shocks, can be *linearly* approximated as

$$\mathbf{x} = f'(0)\mathbf{L}\,\mathbf{s}, \tag{3.9}$$

where $\mathbf{L} = \mathbf{L}\left(f'(0)\right) \in \mathbb{R}^{n \times n}$ is the Leontief matrix of the economy with parameter $f'(0)$. The same consideration are done for our case.

The linear approximation of the macro state of the economy, as a function of its underlying structure and the agent level shocks, results

$$y^{1st} = f'(0)g'(0)h'(0)\mathbf{v}^\top \mathbf{s}, \tag{3.10}$$

where the vector $\mathbf{v} \in \mathbb{R}^n$ represents the *KB* centrality; it summarizes how shocks shape the macro state of the economy given a first-order approximation. The last equation means that if a shock impacts the most central node, from a *KB* point of view, then the impact on the macro state will be more pronounced. Therefore, Leontief matrix of a network and its associated *KB* centralities resume the tangled nature of interconnections and the role of each agent in the whole economy.

Unfortunately, the equilibrium approximated only to the first term is not informative from an *ex-ante* point of view. The expected value of the macro state is clearly equal to 0 because is linearly dependent on $\mathbf{s}$, that has mean equal to $\mathbf{0}$. In this case, the macro state shows a *certainty equivalence*, that is, $\mathbb{E}[y^{1st}] = 0$ regardless of the underlying network of the economy.

Given that the goal is to compare and evaluate economies from an *ex-ante* point of view, also the second-order approximation has to be defined. In order to consider a more detailed analysis, taking the second-order effects into account provides a more refined characterization of how agent-level shocks shape the macro state of the economy.

As done for the approximation to the first term, it is possible to write the macro state of the economy using second-order approximation. The following theorem summarizes the most important result of article [2].

Theorem 3. *Suppose that $f'(0) < 1$. Then, the second-order approximation to the macro state of the economy is given by*

$$\begin{aligned} y^{2nd} &= f'(0)g'(0)h'(0)\,\mathbf{v}^\top \mathbf{s} + \\ &+ \frac{1}{2} g''(0) \left(f'(0)h'(0)\,\mathbf{v}^\top \mathbf{s}\right)^2 + \\ &+ \frac{1}{2} g'(0)\mathbf{s}^\top \Big(h'(0)f''(0)\mathbf{L}^\top \mathbf{V}\,\mathbf{L} + \left(f'(0)\right)^2 h''(0)\mathbf{L}^\top \mathbf{L}\Big)\,\mathbf{s}, \end{aligned} \tag{3.11}$$

where $\mathbf{L}$ is the Leontief matrix with parameter $f'(0)$, $\mathbf{v}$ is the corresponding KB centrality, and $\mathbf{V} = diag(\mathbf{v})$ is a square diagonal matrix which entries are described by the vector $\mathbf{v}$. [2].

The first term of the equation simply is the first-order approximation; it expresses the dependences of the macro state to the *KB* centralities. The second term represents some curvature in the aggregation function, while the third one describes the non-linearity of the interaction function and h. It is due to these last two terms that the expected value of y is not equal to 0. In fact, are non-linearities that ensure an expected value different to 0. Therefore, if we want to understand the role of microeconomic shocks in different economies, we have to consider functions of the economy that are at least quadratics.

Clearly, our two macro states are part of this equation; however, it should be noted that even if losing a bit of generality, only considering some parts of this aggregate state is more explanatory as well as easier to read.

Chapter 4

Ex-Post analysis

We now propose the *ex-post* analysis for both the economic models. Recall that this point of view tries to understand the behavior of the macroeconomic output after a realization of shocks. We focus our attention on the agents' role and we define who is the most systematically important agent within each structure. The macro states are quadratic functions of the equilibrium vector $\mathbf{x}$; therefore, sample variance of network measures results a useful statistic to describe both aggregate levels.

Macro states depend both on network structure and on a realization of shocks; as a first description of the economies, we provide a characterization in terms of the Leontief matrix and *KB* centralities. After this introduction, we describe two classes of optimization problems. The first help us to understand which is the *worst* shock that could affects a network, interpreted as the one that maximizes y; the second aims to find which is the best structure that minimizes the worst shock.

As a result of this representation we rank the structures defined in the first chapter in terms of their macroeconomic output. The most important results of this chapter are the Theorems 4 and 5 and following propositions; these statements resume how to compare different networks in terms of both macro states.

4.1 Total Welfare

Referring to equation (3.6), the *ex-post* value of y_W depends on the nature of interactions matrix and on a realization of shocks; we make this relationship explicit by writing $y_W = \Psi_W(\mathbf{P}, \mathbf{s})$. Given the equilibrium vector defined in Proposition

3.5, the total welfare results

$$\begin{aligned} y_W &= \sum_{i=1}^{n} (x_i)^2 \\ &= \alpha^2 \sum_{j=1}^{n} \sum_{k=1}^{n} s_j s_k \left(\sum_{i=1}^{n} L_{ij} L_{ik} \right). \end{aligned}$$

Using matrix notation becomes

$$y_W = \alpha^2 \mathbf{s}^\top \mathbf{L}^\top \mathbf{L}\, \mathbf{s}. \tag{4.1}$$

We first consider the special case when just one agent is affected by the shock. We are interested in finding which is the most important agent in the network following the definition below.

Definition 4. *Agent i is said to be systemically more important than agent j if $y^{(i)} > y^{(j)}$, where $y^{(i)}$ denotes the macro state of the economy when only agent i is hit with a shock.*

We analyze agents' systemically importance in both the aggregate level. Higher is the macro state of the economy when a shock affects agent i and more the agent i is considered systemically important. We expect that this parameter is strictly related to a centrality measure.

To this aim we take $\mathbf{s} = s\mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with all zeros but a 1 in position i, and we elaborate on the previous equation:

$$\begin{aligned} y_W^{(i)} &= (\alpha\, s)^2 \mathbf{e}_i^\top \mathbf{L}^\top \mathbf{L}\, \mathbf{e}_i \\ &= (\alpha\, s)^2 \sum_{j=1}^{n} L_{ji}^2. \end{aligned} \tag{4.2}$$

This equation captures the value of the macroeconomic outcome when a shock affects only the agent i. Comparing the role of each agent means to focus the analysis on the last parameter $\sum_{j=1}^{n} L_{ji}^2$. In fact, we say that agent i is systemically more important than agent k if

$$\sum_{j=1}^{n} L_{ji}^2 > \sum_{j=1}^{n} L_{jk}^2.$$

Using the definition of population variance[1] of a finite population of size n the term $\sum_{j=1}^{n} L_{ji}^2$ results

$$\sum_{j=1}^{n} L_{ji}^2 = n\mathrm{Var}(L_{1i}, ..., L_{ni}) + \frac{v_i^2}{n}. \tag{4.3}$$

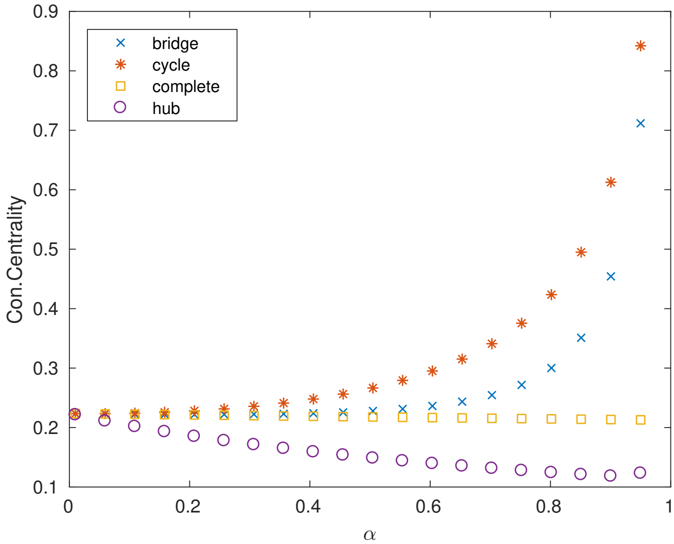

Figure 4.1. Concentration centrality of most significant agents in each networks when the parameter α varies in its interval of values.

This summation essentially measures the variation in the extent to which agent i influences other agents in the economy. The standard deviation of the i-th column of Leontief matrix is called *concentration centrality* and expresses "how evenly the agent i influence is distributed across the rest of the agents" [2].

The above equation states that *KB* centrality is different from systemically importance. In fact, they represent the one-norm and the euclidean norm of Leontief matrix columns. An interesting consequence concerns regular networks where all the agents have the same *KB* centrality. For example, consider the undirected complete network and the undirected ring. Even if they are both regular economies, agents in the first network uniformly distribute their influence over a larger number of agents than in the latter. As a result, the concentration centrality of an agent in the cycle graph is higher than the one of an agent in the complete graph.

Figure 4.1 represents the concentration centralities of distinctive nodes in different networks of 20 agents. The hub is the central node in the star graph, the bridge is the node of barbell graph that links the two complete graph, cycle is any node of the ring graph, and complete is any node of the complete graph. The higher

[1] $\mathrm{Var}(X) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu_x^2$ where $\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean of the population.

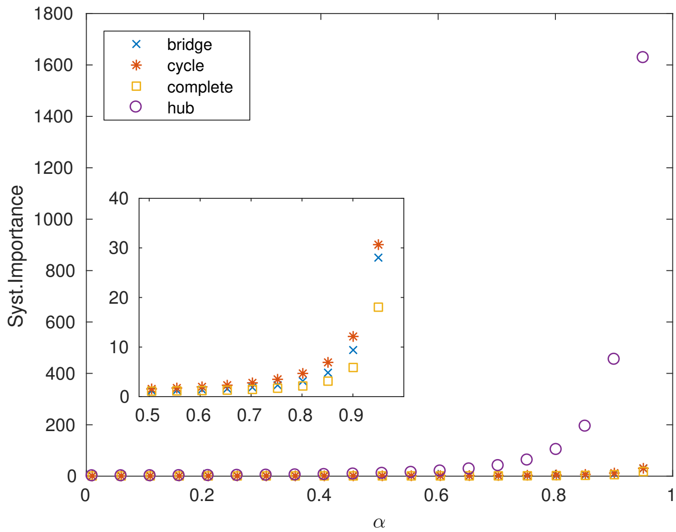

Figure 4.2. Systemically importance of most significant agents in each networks when the parameter α varies in its interval of values.

values of concentration centrality are present in networks where agents' influence is less distributed, that is, the cycle and the barbell graph.

The parameter α expresses how much important the interactions are. The concentration centrality in the complete network does not show a dependence on α. In cycle and barbell graph, higher α makes the influence of agent i propagates stronger in the whole network, also arriving to opposite agents. Then, the influence of i varies from agent to agent making its variance growing. Only the concentration centrality of the hub is descending in α. When α grows the influence of the hub on marginal nodes becomes equal to the one that it has on itself.

Even if the variance of agent's interactions is useful to understand differences between regular economies, it is not clear how much the systemically importance depends on it and on *KB* centrality. Therefore, it is interesting to rewrite equation (4.3) using the max-norm and the one-norm

$$\sum_{j=1}^{n} L_{ji}^2 \leq (\max_j L_{ji}) \left(\sum_{j=1}^{n} L_{ji} \right) = (\max_j L_{ji})\, v_i. \tag{4.4}$$

Even if this equation defines an upper bound, it is clear that the *KB* centrality plays again a key role in the definition of systemically importance.

Figure 4.2 represents the systemically importance of previous nodes in networks of 20 agents. In contrast to the previous figure, the hub of the star is clearly more systemically important than any other node. This means that equation (4.3) is informative if we aims to study agents with the same *KB* centralities. Instead, when we compare nodes with very different *KB* centralities it is better to consider (4.4).

We now consider the case when all the agents are affected by a shock. Considering the macro state (4.1), it is useful to rewrite Leontief matrix using the *singular value decomposition* (SVD) defined in Section 2.4. In this way it is possible to relate the macro state y_W to eigenvalues of $\mathbf{L}^\top \mathbf{L}$, that is, to singular values of $\mathbf{L}$.

The singular value decomposition of the Leontief matrix is defined as a tuple $(\mathbf{V}, \mathbf{\Gamma}, \mathbf{U})$ satisfying

$$\mathbf{L} = \mathbf{V}\,\mathbf{\Gamma}\,\mathbf{U}^\top, \;\; \mathbf{U},\, \mathbf{V},\, \mathbf{\Gamma} \in \mathbb{R}^{n\times n}, \tag{4.5}$$

where matrices $\mathbf{U}$, $\mathbf{V}$ are orthonormal and $\mathbf{\Gamma}$ is a diagonal matrix whose entries are called *singular values*.

Using the SVD, the outcome of y_W becomes

$$\begin{aligned} y_W &= \alpha^2 \left[\mathbf{L}\,\mathbf{s}\right]^\top \left[\mathbf{L}\,\mathbf{s}\right] \\ &= \alpha^2 \left[\mathbf{V}\,\mathbf{\Gamma}\,\mathbf{U}^\top \mathbf{s}\right]^\top \left[\mathbf{V}\,\mathbf{\Gamma}\,\mathbf{U}^\top \mathbf{s}\right] \\ &= \alpha^2 \left[\mathbf{s}^\top \mathbf{U}\,\mathbf{\Gamma}\,\mathbf{V}^\top\right] \left[\mathbf{V}\,\mathbf{\Gamma}\,\mathbf{U}^\top \mathbf{s}\right] \\ &= \alpha^2 \mathbf{s}^\top \mathbf{U}\,\mathbf{\Gamma}^2 \mathbf{U}^\top \mathbf{s}. \end{aligned}$$

Summarizing, the output of the economy results

$$y_W = \alpha^2 \sum_{i=1}^{n} \gamma_i \, \left(\mathbf{s}^\top \mathbf{u}_{\cdot i}\right)^2, \tag{4.6}$$

where $\mathbf{u}_{\cdot i} \in \mathbb{R}^n$ is the i-th right-hand singular vector. Columns of $\mathbf{U}$ are orthonormal and generate the vector space $\mathbb{R}^n$; given a vector of shocks belonging to $\mathbb{R}^n$ we are interested in finding an upper bound for the macro state of the economy y_W.

Theorem 4. *Let $\mathfrak{E}_n$ be an economy described by its associated interactions matrix $\mathbf{P}$ and the parameter α. For every vector of shocks $\mathbf{s} \in \mathbb{R}^n$, we have that*

$$y_W \;\leq\; \gamma_{max} (\alpha \, \|\mathbf{s}\|)^2, \tag{4.7}$$

where γ_{max} indicates the highest singular value of the Leontief matrix $\mathbf{L}$.

Proof. Using Schwartz inequality we write an upper bound for equation (4.1):

$$\alpha \, \mathbf{s}^\top \mathbf{L}^\top \mathbf{L}\,\mathbf{s} \leq \alpha \, \|\mathbf{L}\,\mathbf{s}\|^2.$$

Using the definition of matrix norm[2]follows that

$$\|\mathbf{L}\,\mathbf{s}\| \leq \|\mathbf{L}\|\|\mathbf{s}\|, \forall\, \mathbf{s} \in \mathbb{R}^n.$$

The matrix norm induced by the euclidean vector norm is

$$\|\mathbf{L}\| = \sqrt{\rho(\mathbf{L}^\top \mathbf{L})},$$

where $\rho(\cdot)$ denote the spectral radius. In that case, the spectral radius is equal to largest singular value γ_{max} of $\mathbf{L}$. Hence, we have found that

$$\begin{aligned} y_W &\leq \alpha^2(\|\mathbf{L}\,\mathbf{s}\|)^2 \\ &\leq \alpha^2(\|\mathbf{L}\|\|\mathbf{s}\|)^2 \\ &= \gamma_{max}(\alpha\,\|\mathbf{s}\|)^2. \end{aligned}$$

□

Given that our goal is to classify networks in terms of their macroeconomic state, we analyze which is the nature of shock that gives back the highest value of y_W. We assume that the vector $\mathbf{s}$ has a bounded norm, and we define the maximization problem

$$\Psi^*_W(\mathbf{P}) \;=\; \max_{\|\mathbf{s}\|\leq 1,\, \in \mathbb{R}^n} \quad \Psi_W(\mathbf{P}, \mathbf{s}). \tag{4.8}$$

Proposition 3. *Let $\mathfrak{E}_n$ be an economy described by its associated interactions matrix $\mathbf{P}$ and the parameter α. Assuming a fixed norm for the vector of shocks, the maximum value of the macro state y_W, defined by problem (4.8), is*

$$y^*_W \;=\; \alpha^2 \gamma_{max}, \tag{4.9}$$

where γ_{max} indicates the largest singular value of the Leontief matrix $\mathbf{L}$. This maximum is achieved if and only if the vector of shocks is proportional to column $\mathbf{u}_{\cdot max} \in \mathbf{U}$ associated to γ_{max}.

Proposition 3 says that when the vector of shock is restricted to have a fixed norm, the highest in singular value is the resuming parameter of the Leontief matrix of an economy. Therefore, we could classify networks' structure just using singular value. The maximum is an extreme value for categorizing the macro states but give us an idea of how large are the fluctuations of the aggregate level.

[2]A vector norm $\|\mathbf{x}\| \in \mathbb{R}^n$ *induces* a matrix norm on $\mathbb{R}^{n\times n}$ by setting$\|\mathbf{A}\| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\,\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\,\mathbf{x}\|$, $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Proof. From Theorem 4 immediately follows that $y_W \leq \alpha^2 \gamma_{max}$; we have to understand when the equality holds and for which kind of vector $\mathbf{s}$. We assume a non-increasing order of singular values, that is $\gamma_{max} = \gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n \geq 0$.

(If) Not considering parameter α, Schwartz inequality ensures that equation (4.6) could be written as

$$\begin{aligned}
\sum_{i=1}^{n} \gamma_i \left(\mathbf{s}^\top \mathbf{u}_{\cdot i}\right)^2 &= \gamma_1 (\mathbf{s}^\top \mathbf{u}_{\cdot 1})^2 + \sum_{i=2}^{n} \gamma_i (\mathbf{s}^\top \mathbf{u}_{\cdot i})^2 \\
&\leq \gamma_1 \left(\|\mathbf{s}\| \|\mathbf{u}_{\cdot 1}\|\right)^2 + \sum_{i=2}^{n} \gamma_i \left(\mathbf{s}^\top \mathbf{u}_{\cdot i}\right)^2 \\
&\leq \gamma_{max} + \sum_{i=2}^{n} \gamma_i \left(\mathbf{s}^\top \mathbf{u}_{\cdot i}\right)^2 .
\end{aligned}$$

If the vector of shocks is proportional to vector $\mathbf{u}_{\cdot max}$ then we have that the equation holds for the last inequality. Moreover, $\mathbf{s}$ has to be orthogonal to all the other columns of $\mathbf{u}_{\cdot j}$, $j \neq 1$, due to the orthonormality of matrix $\mathbf{U}$. This ensures that the summation of the last inequality is equal to 0 and therefore we have that $y_W^* = \alpha^2 \gamma_{max}$.

(Only if) Considering equation (4.6), we have to prove that the only possible solution for

$$\gamma_{max} = \sum_{i=1}^{n} \gamma_i (\mathbf{s}^\top \mathbf{u}_{\cdot i})$$

is when $\mathbf{s}$ is proportional to $\mathbf{u}_{\cdot max}$. Given that the Leontief matrix is invertible its rank is n. Hence, all its eigenvalues are different from 0; as a direct consequence, also its singular values are different from 0. As a conclusion, the only terms in the summation that have to be null are the scalar products. This happens when the vector $\mathbf{s}$ is orthogonal to all columns except for the one associated to γ_{max}; given that the columns of $\mathbf{U}$ generate $\mathbb{R}^n$ the only possible solution in $\mathbb{R}^n$ is that $\mathbf{s}$ is proportional to $\mathbf{u}_{\cdot max}$. □

We have found that the singular value is the resuming value when we consider the worst shock (intended as the one that generates the highest y_W). Now we focus on which is the network's structure that minimizes the worst shock, that is, we want to minimize the highest singular value. Formally we define a minimization problem on all fully connected networks described by a positive row-stochastic matrix $\mathbf{P}$

$$\min_{\mathbf{P} \in \mathbb{R}^{n \times n}} \quad \Psi_W^*(\mathbf{P}), \tag{4.10}$$

where $\Psi_W^*(\mathbf{P})$ is defined by (4.8).

Proposition 4. *Consider the economic networks $\mathfrak{E}_n$ described by their interactions matrix* $\mathbf{P}$ *and the parameter* α*. Then,*

$$\Psi^*_W(\mathbf{P}) \geq \left(\frac{\alpha}{1-\alpha}\right)^2, \tag{4.11}$$

and the equality holds if and only if matrix $\mathbf{P}$ *is symmetric.*

Proof. (If) We start considering networks that have a symmetric $\mathbf{P}$, for example regular and undirected networks such as the complete graph. If the SRW matrix is symmetric also $\mathbf{L}$ is symmetric too. As a direct consequence, the product $\mathbf{L}^\top\mathbf{L}$ is the square of the Leontief matrix and the singular values coincide with squared eigenvalues. When the matrix $\mathbf{P}$ is symmetric, square root of the largest singular value of $\mathbf{L}$ is equal to its largest eigenvalue and columns of matrix $\mathbf{U}$ are equal to its eigenvectors. Then, equation (4.9) becomes

$$y^*_W = \alpha^2\lambda^2_{max},$$

where λ_{max} indicates the largest in module eigenvalue. Remembering that the largest eigenvalue of $\mathbf{P}$ is 1, we obtain the new form

$$y^*_W = \left(\frac{\alpha}{1-\alpha}\right)^2.$$

Now we have to prove that if we consider a not symmetric $\mathbf{P}$ we always obtain an higher value of γ_{max}.

When the Leontief matrix is not symmetric the square root of its largest eigenvalue is different from largest singular value, and the eigenvector $\mathbf{u}_{\cdot 1}$ is different from the eigenvector centrality. Even if is not possible to explicitly define the largest singular value it is possible to prove that it is always greater than any eigenvalue of $\mathbf{L}$. The largest in module eigenvalue of $\mathbf{L}$ could be written

$$|\lambda_{max}| = \|\lambda_{max}\mathbf{x}\|_2,$$

where $\mathbf{x}$ is the eigenvector with norm 1 that is associated to λ_{max}. Using the definitions of eigenvalue the above equation becomes

$$|\lambda_{max}| = \|\mathbf{L}\,\mathbf{x}\|_2.$$

Using again the definition of matrix norm we have that

$$|\lambda_{max}| \leq \sqrt{\gamma_{max}},$$

and, consequently, a network with a symmetric $\mathbf{P}$ always has a lower or equal value of y^*_W than any other network.

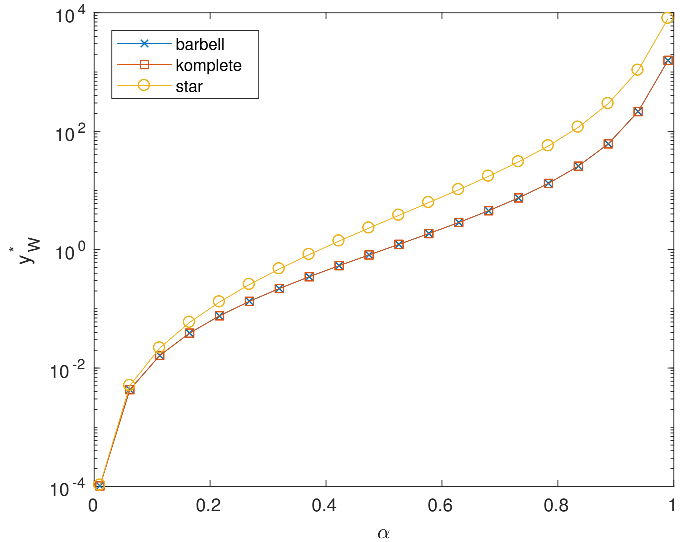

Figure 4.3. Values of y^*_W when the number of agents is fixed to $n = 20$ and α is in [0,1). The y-axis is plotted in a logarithmic base 10 scale.

(Only if) From the equation $\Psi^*_W(\mathbf{P}) = [\alpha/(1-\alpha)]^2$ we easily obtain that γ_{max} has to be equal to $1/(1-\alpha)^2$. In this case the above inequality becames an equality and therefore $\gamma_{max} = \lambda^2_{max}$. When eigenvalues and the square root of singular values coincide it means that the SVD is equivalent to the eigenvalue decomposition and that matrix $\mathbf{L}$ is symmetric. Finally, if $\mathbf{L}$ is symmetric also $\mathbf{P}$ has to be too. □

There are two important facts to note from the last proposition. The first is that the lower bound of y^*_W does not depend on the number of nodes; it is not important the number of agents that forms the network rather than the parameter α. The second fact is that symmetric matrices $\mathbf{P}$ represent regular and undirected networks, such as the complete and ring graphs that we have considered in the examples. We have seen that when $\mathbf{L}$ is symmetric, its first principal component represents the eigenvector Bonacich centrality that is equal to the constant vector; this proposition highlights that having all agents with the same importance brings to lower values of y^*_W.

Figure 4.3 represents y^*_W in a semi-log plot with the change in α for different networks with $n = 20$ agents. We have considered simple network as underlying structures of the economies and therefore we have plotted the complete graph as representative of undirected and regular networks. Complete and barbell networks result almost equal showing that the bridge link does not work as a bottleneck if it has almost the same weight of all the other links. Star network clearly dominates all the others due to the difference between its agents' importance.

4.2 Total Activity

Consider now the total activity of an economy. We recall that y_A is defined as the square of the sum of agents' state (3.7). As the previous macro state we could consider y_A as a quadratic function that for a fixed value of α depends on the nature of the network and on the nature of shocks $y_A = \Psi_A(\mathbf{P}, \mathbf{s})$. In this way we can proceed as done for y_W; after a brief analysis on which are the important variables we classify networks in terms of the macro state y_A or some measures directly connected to it.

Given the equilibrium configuration (3.5) the *ex-post* value of y_A is defined as

$$\begin{aligned} y_A &= \left(\sum_{i=1}^{n} x_i \right)^2 \\ &= \left(\alpha \sum_{i=1}^{n} v_j s_j \right)^2 , \end{aligned}$$

that written in matrix notation becomes

$$y_A = \left(\alpha\, \mathbf{s}^\top \mathbf{v} \right)^2 . \tag{4.12}$$

Thus, instead of considering the entire matrix $\mathbf{L}$, we could describe the macro state y_A only with the vector of *KB* centralities. The role of each agent in the economy is described by v_i and makes easy to understand how a shock to distinct agents differently shapes the aggregate level.

We now identify which is the systemically most important agent in each economy. We first consider the special case when just one agent is affected by the shock; as for y_W we take $\mathbf{s} = s\mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with all zeros but a 1 in position i, and we elaborate on the previous equation

$$\begin{aligned} y_A^{(i)} &= \left(\alpha\, s\, \mathbf{e}_i^\top \mathbf{v} \right)^2 \\ &= \left(\alpha\, s\, v_i \right)^2 . \end{aligned} \tag{4.13}$$

This equation states that a node with higher *KB* centrality has higher importance in shaping the macroeconomic output. In fact agent i is systemically more important than agent j if

$$v_i > v_j$$

and, as a result, a shock will propagate more if it affects a central node. In the star network the hub has an higher *KB* centrality than marginal nodes; hence, if a shock affects the central agent rather than a marginal one the macro state y_A will be higher. In any regular network, where each agent has the same centrality, the macro state has the same values and it does not matter which agent is affected.

We now consider the case of a whole shock, that is we assume that each agent is affected by a shock of intensity s_i, $\forall i$. Considering any fully connected economic network we aim to find an upper bound for y_A given a vector of shock $\mathbf{s} \in \mathbb{R}^n$.

Theorem 5. *Let $\mathfrak{E}_n$ be an economy represented by its interactions matrix $\mathbf{P}$ and the parameter α. For every realization of shocks $\mathbf{s} \in \mathbb{R}^n$, we have that*

$$y_A \leq (\alpha \|\mathbf{s}\| \|\mathbf{v}\|)^2, \tag{4.14}$$

where $\mathbf{v} \in \mathbb{R}^n$ represents the vector of Katz-Bonacich centralities of the economy.

We do not prove the theorem that immediately follows from equation (4.12) simply using the Schwartz inequality. Now consider the maximization problem

$$\Psi_A^*(\mathbf{P}) = \max_{\|\mathbf{s}\| \leq 1, \in \mathbb{R}^n} \Psi_A(\mathbf{P}, \mathbf{s}). \tag{4.15}$$

Proposition 5. *Let $\mathfrak{E}_n$ be an economy represented by its interactions matrix $\mathbf{P}$ and the parameter α. Then, the maximum value of problem (4.15) is*

$$y_A^* = \alpha^2 \|\mathbf{v}\|^2 \tag{4.16}$$

and is reached if and only if the vector of shocks is proportional to the vector of Katz-Bonacich centralities.

This proposition directly follows from Theorem 5 imposing the norm of the vector of shocks to 1. Given that the upper bound is $y_v \leq \alpha^2 \|\mathbf{v}\|^2$, is clear that the only vector of shocks that satisfies the equation has the form $\mathbf{s} = \mathbf{v}/\|\mathbf{v}\|$. As for Proposition 3, the worst shock is the one that hit agents proportionally to their importance; in this case the importance is defined by the *KB* vector.

The economic network that manifests the lower value of $\|\mathbf{v}\|$ is the one where all the agents have the same KB centrality. Defining the minimization problem

$$\min_{\mathbf{P} \in \mathbb{R}^{n \times n}} \Psi_A^*(\mathbf{P}), \tag{4.17}$$

where Ψ_A^* is defined in (4.15), the following proposition states which are optimum economic networks.

Proposition 6. *Consider economic networks $\mathfrak{E}_n$ described by their interactions matrix $\mathbf{P}$ and the parameter α. Then,*

$$\Psi_A^*(\mathbf{P}) \geq n \left(\frac{\alpha}{1 - \alpha} \right)^2, \tag{4.18}$$

and the equality holds if and only if matrix $\mathbf{P}$ is doubly stochastic.

Differently from the previous macro state, in this case y_A^* linearly depends on the size of the network. When the vector of shocks has a fixed norm, the two cases have in common the fact that regular and undirected networks are the ones with lower values of both macro states. When agents have the same importance the location of the shocks is not significant; this result is intuitive because agents appear equal to a shock that has to decide who to hit.

Proof. Using the definition of sample variance we can rewrite the squared norm of vector $\mathbf{v}$ as

$$\|\mathbf{v}\|^2 = n \operatorname{Var}(v_1, ..., v_n) + \frac{n}{(1-\alpha)^2}.$$

(If) When the SRW matrix of the network is doubly stochastic it means that the KB centralities vector is the constant vector $1/(1-\alpha)\mathbb{1}$; therefore, the term $\operatorname{Var}(v_1, ..., v_n)$ is null and y_A^* results equal to the second term of the above equation.

(Only if) When $\Psi_A^*(\mathbf{P}) = n\,[\alpha/(1-\alpha)]^2$ we have that $\|\mathbf{v}\|^2 = n/(1-\alpha)^2$ and hence the term $\operatorname{Var}(v_1, ..., v_n)$ has to be null. The variance of a vector is null if and only if all the element of the vector are equal; as a consequence, when all the agents have the same KB centrality we find that $v_i = 1/(1-\alpha)$, $i = 1, ..., n$. Using the definition of $\mathbf{v}$ in terms of $\mathbf{L}$ we find that

$$\begin{aligned}
\mathbf{v} = \mathbf{L}^\top \mathbb{1} \iff & \frac{1}{(1-\alpha)}\mathbb{1} = \mathbf{L}^\top \mathbb{1} \\
\iff & \left(\mathbb{I} - \alpha \mathbf{P}^\top\right)\mathbb{1} = (1-\alpha)\mathbb{1} \\
\iff & \mathbf{P}^\top \mathbb{1} = \mathbb{1}.
\end{aligned}$$

Thus, when the KB centralities vector is constant the matrix $\mathbf{P}$ is doubly stochastic.

□

Figure 4.4 represents y_A^* in a semi-log plot with the change in α for different networks with $n = 20$ agents. We have considered simple network as underlying structures of the economies and therefore we have plotted the complete graph as representative of undirected and regular networks. Again, the barbell networks result almost equal to a undirected and regular networks showing that the centrality of the two agents of the bridge does not create a visible difference. Star network clearly dominates all the others due to centrality of the hub.

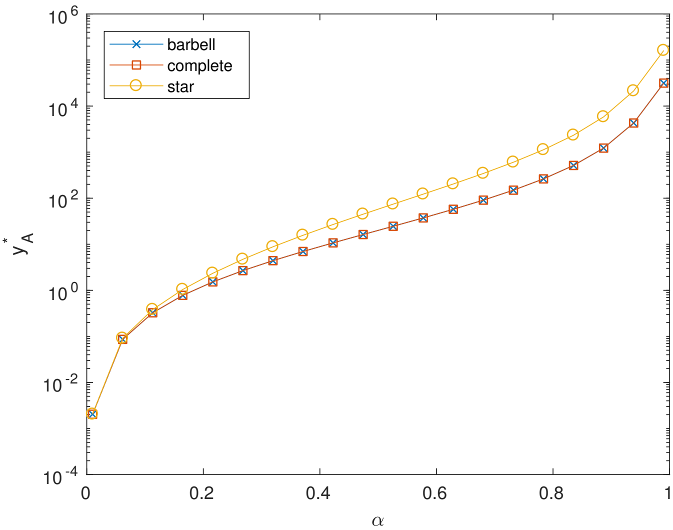

Figure 4.4. Values of y_A^* when the number of agents is fixed to $n = 20$ and α is in [0,1). The y-axis is plotted in a logarithmic base 10 scale.

Chapter 5

Ex-Ante analysis

This chapter is devoted to the *ex-ante* analysis of the aggregate effects of shocks in economic networks. In particular, we shall study the dependence of the expected macro-states of the network as a function of the jointly distribution of the shocks

In oder to emphasize the effects of shocks on both macro states we have divided the chapter in two parts: one for the total activity and one for the total welfare. Differently from what is done in article [2], we define the shocks column vector $\mathbf{S}$ as a random variable allowing for possible correlation. The case of idiosyncratic shocks is defined as a special case.

Analogously to the previous chapter, we aim to characterizing the worst possible shock distribution for a given network. We then consider the network design problem for the network structure that can optimally react to the worst shock. The final result that we want to achieve is to have simple macroeconomic measures with which it is easy to evaluate the performance of a network. This conclusions are stated in Theorem 8 for y_W and in Theorems 6 and 7 for y_A.

Formally we define the shocks vector

$$\mathbf{S} = (S_1 \dots S_n)^\top, \in \mathbb{R}^n$$

and we assume the normalization that expected value is $\mathbb{E}[\mathbf{S}] = \mathbf{0}$. Given that we consider squared functions of $\mathbf{S}$ the most important statistic is the variance-covariance matrix $\mathbf{\Omega}$.

5.1 Total Activity

We recall that the macro state y_A is equal to $\left(\mathbb{1}^\top \mathbf{x}\right)^2$. The performance measure is found by taking the expected value of y_A:

$$\begin{aligned}\mathbb{E}[y_A] &= \alpha^2 \mathbb{E}\left[(\mathbf{S}^\top \mathbf{v})^2\right] \\ &= \alpha^2 \mathbb{E}\left[\mathbf{v}^\top (\mathbf{S}\,\mathbf{S}^\top)\mathbf{v}\right] \\ &= \alpha^2 \left(\mathbf{v}^\top \mathbb{E}[\mathbf{S}\,\mathbf{S}^\top]\mathbf{v}\right).\end{aligned}$$

In other words,

$$\mathbb{E}[y_A] = \alpha^2 \mathbf{v}^\top \mathbf{\Omega}\, \mathbf{v}, \tag{5.1}$$

therefore, as in the *ex-post* analysis, *KB* centrality vector $\mathbf{v}$ plays key role in shaping the macro state. From now on we will denote $\mathbb{E}[y_A]$ with $\mu_A = \Phi_A(\mathbf{P}, \mathbf{S})$, where $\Phi_A : \mathbb{R}^n \to \mathbb{R}$ expresses the dependence on the stochastic vector and on the nature of the network.

As a first analysis we consider idiosyncratic shocks. If shocks are assumed to be independent and identically distributed (i.i.d) with expected value zero and variance σ^2, then $\mathbf{\Omega}$ is a diagonal matrix equal to $\sigma^2\mathbb{I}$.

Reformulating the performance metric, we state the following result:

Proposition 7. *Let $\mathfrak{E}_n$ be an economy defined by the interactions matrix $\mathbf{P}$, the parameter α, and the vector of stochastic shocks $\mathbf{S}$. Suppose that shocks are independent and identically distributed with mean 0 and finite variance σ^2.*

Then, the ex-ante performance of the macro state y_A is

$$\mu_A = (\alpha\sigma)^2 \sum_{i=1}^{n} v_i^2. \tag{5.2}$$

Proposition 7 states that when shocks are i.i.d. the performance metric μ_A does not show a dependency on which agent is affected. Everything is captured by the squared norm of $\mathbf{v}$ that resumes the topological properties of a network.

We have already seen that $\sum_{i=1}^{n} v_i^2$ could be rewritten using the definition of population variance:

$$\sum_{i=1}^{n} v_i^2 = n\mathrm{Var}(v_1, ..., v_n) + \frac{n}{(1-\alpha)^2}.$$

The term $\mathrm{Var}(v_1, ..., v_n)$ focus on the distribution of *KB* centralities. When shocks are i.i.d., population variance of Bonacich centralities is what makes μ_A changes.

The term $\mathbb{E}[y]$ is a function of the structure and is used as a measure of performance, giving us the possibility of classifying economies. Formally:

Definition 5. *An economy outperforms another if* $\mathbb{E}[y]$ *is larger in the former than the latter.*

Using this definition we could state that network in which agents play disproportional role *outperform* regular networks that have an equal distribution of *KB* centralities.

The effects of a shock spread homogeneously if the network is regular. It is not important which agent is affected because each one has the same role. At the aggregate level this results in a more efficient way of *washing out* the shock [2].

On the other end, a network with agents that play disproportional roles will result in a higher value of $\text{Var}(v_1, ..., v_n)$, and consequently in an higher value of the expected value of total activity. The effects of a shock are no more equal because ones on central node plays a more relevant impact on the performance metric.

For example, consider a regular network and a star network, both with n nodes. In a regular network, where each agent has the same Bonacich centrality $v = 1/(1-\alpha)$, the term $\text{Var}(v_1, ..., v_n)$ is null and the expect value of y_A becomes

$$[\mu_A]_{regular} = n\sigma^2 \left(\frac{\alpha}{1-\alpha}\right)^2.$$

In the star network we have the opposite situation. The term $\text{Var}(v_1, ..., v_n)$ is not null due to the difference between marginal agents and the hub. In this economy the expected value of y_A is

$$[\mu_A]_{star} = \alpha^2\sigma^2 \left\{ \frac{[n\alpha + (1-\alpha)]^2}{(1-\alpha^2)^2} + \frac{[n-(1-\alpha)]^2}{(n-1)(1-\alpha^2)^2} \right\}.$$

We now focus on μ_A in large scale networks, that is, when the number of agents is large. As $n \to \infty$, the expected total activity clearly scales as the squared Euclidean norm of $\mathbf{v}$. Then, the distribution of *KB* centralities is what shape the behavior of aggregate performance. Given that the sum of v_i depends on n, when $n \to \infty$ also $\|\mathbf{v}\|^2$ will tend too. Thus, the differences between structures is captured by the rate of divergence.

Consider the above example of regular and star network. It is clear that the limits of these two *ex-ante* performance, when $n \to \infty$ and α is fixed, are:

$$[\mu_A]_{star} \sim \left(n^2/(1-\alpha)^2\right)$$
$$[\mu_A]_{regular} \sim \left(n/(1-\alpha)^2\right).$$

Figure 5.1 compares the expected values of these opposite structure when shocks are i.i.d. and follow a standard normal distribution. The number of agents changes between 10 and 10^4 and we have fixed the parameter $\alpha = 0.8$.

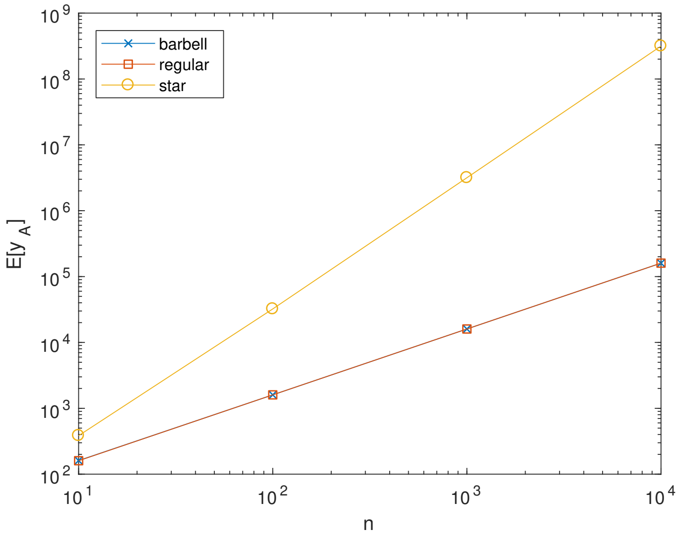

Figure 5.1. Expected total activity μ_A when shocks are i.i.d. as a standard normal distribution. A logarithmic (base 10) scale is used for the Y-axis.

We have also considered the barbell network for understanding if the bridge is relevant in shaping the expected performance. As we have seen in the *ex-post* analysis, the expected value of the macro state is not conditioned by the existence of the bridge between the two complete networks. When the number of agents tends to infinity, *KB* centralities of barbell network, defined in 2.3.4, tends to $1/(1-\alpha)$, the same of regular networks.

The same results were found in the article [1] where the macro state is given by $y = \mathbf{S}^\top \mathbf{v}$ and the focus was on the *aggregate volatility*, defined as

$$\nu_A = [\text{Var}(y)]^{1/2} = \sqrt{\sum_{i=1}^{n} \sigma_i^2 v_i^2}. \tag{5.3}$$

In fact it is clear that the aggregate volatility scales with the Euclidean norm of $\mathbf{v}$.

The only difference with our analysis is that the vector of *KB* centralities sum up to 1 instead to $n/(1-\alpha)$. Therefore, the resulting limits are equal up to some transformations. Consider again the example of regular and star network, now with a vector $\mathbf{v} = \mathbf{v}(1-\alpha)/n$. The resulting aggregate volatilities when $n \to \infty$

are

$$[\nu_A]_{star} \sim \frac{\alpha}{(1+\alpha)}$$
$$[\nu_A]_{regular} \sim \frac{1}{\sqrt{n}}.$$

In light of these last results is also easier to understand the behavior of μ_A when the networks are large. In regular networks the aggregate volatility vanishes when $n \to \infty$ and precisely decays with a rate $1/\sqrt{n}$. While this statement is clear for complete network because a shock on an agent is shared with all the other, is interesting for the ring network that is often classified as unstable. The aggregate volatility of the star network does not vanish even if the network is large. Given that this volatility is different to 0 is also called *systemic volatility* because a shock creates system wide co-movement.

5.1.1 Non-idiosyncratic shocks

The previous analysis focus on the role of *KB* centrality vector because the covariance matrix has the simple nature $\sigma^2\mathbb{I}$. Suppose now that the n shocks are not i.i.d. . The covariance matrix is a symmetric positive semidefinite[1] (PSD) matrix. Its entry σ_{ij} represents the covariance between shock i and shock j. The set of symmetric positive semidefinite matrix of order n is a subset of $\mathbb{R}^{n\times n}$ and is denoted with $\mathbb{S}_+$.

We now focus on the role of shocks; we are interested in finding which is the best covariance matrix that maximizes the expected value of y_A. Formally, we want to study the maximization problem defined as follows:

$$\begin{aligned} \Phi_A^*(\mathbf{P}) = \max_{\boldsymbol{\Omega}\,\in\,\mathbb{S}_+} \quad & \Phi_A(\mathbf{P},\mathbf{S}) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \Omega_{ii} = 1. \end{aligned} \tag{5.4}$$

Theorem 6. *Consider an economy $\mathfrak{E}_n$ described by the interactions matrix $\mathbf{P}$ and the parameter α. Let $\mathbf{v} \in \mathbb{R}^n$ be the vector of Katz-Bonacich centralities and $\boldsymbol{\Omega} \in \mathbb{S}_+$ is the covariance matrix of stochastic shocks.*

Then, the maximum of the problem (5.4) is

$$\alpha^2 \sum_{k=1}^{n} v_k^2, \tag{5.5}$$

[1] We recall that a matrix is said to be PSD if the associated quadratic form is nonnegative, i.e. $\mathbf{x}^\top \boldsymbol{\Omega}\, \mathbf{x} \geq 0$, $\forall \mathbf{x} \in \mathbb{R}^n$.

and is reached if and only if the covariance matrix $\mathbf{\Omega}^*$ *has the form*

$$\Omega^*_{ij} = \frac{v_i\, v_j}{\sum_{k=1}^{n} v_k^2}\, \forall\, i,j. \tag{5.6}$$

The theorem says that the squared Euclidean norm of *KB* centralities vector is the maximum reachable value when the sum of the variance of the shocks is fixed. The extremum point is given by the matrix $\mathbf{\Omega}^*$ of perfectly correlated shocks. Two shocks X, Y are perfectly correlated if $\mathrm{cov}(X,Y) = \sigma_X \sigma_Y$. Calling $\sigma = (\sigma_1, ..., \sigma_n)$ the vector of standard deviations, the optimal covariance matrix results $\mathbf{\Omega}^* = \sigma\sigma^\top$, where $\sigma = \mathbf{v}/\|\mathbf{v}\|$; the normalization is due to the constraint on the total amount of variance.

Proof. Given that the covariance matrix $\mathbf{\Omega}$ is real and symmetric it is diagonalizable by real orthonormal matrices. Hence, we rewrite $\mathbf{\Omega}$ as $\mathbf{\Omega} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^\top$, where $\mathbf{U}$ is an orthonormal matrix which columns are the eigenvector of $\mathbf{\Omega}$, and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is a diagonal matrix holding corresponding eigenvalues.

Then, we could reformulate the optimization problem as

$$\begin{aligned} &\max_{\mathbf{\Lambda},\mathbf{U}} \quad \mathbf{v}^\top \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^\top \mathbf{v} \\ &\text{s.t.} \quad \sum_{i=1}^{n} \lambda_i = 1. \end{aligned} \tag{5.7}$$

Calling $\mathbf{u}_{\cdot i}$ the i-th column of matrix $\mathbf{U}$, we could find an upper bound for the objective function as follow:

$$\begin{aligned} \Phi^*_A(\mathbf{P}) &= \max_{\mathbf{\Lambda},\mathbf{U}} \quad \lambda_1(\mathbf{v}^\top \mathbf{u}_{\cdot 1})^2 + ... + \lambda_n(\mathbf{v}^\top \mathbf{u}_{\cdot n})^2 \\ &\le \sum_{i=1}^{n} \lambda_j (\|\mathbf{v}\|\|\mathbf{u}_{\cdot i}\|)^2 \\ &= \|\mathbf{v}\|^2. \end{aligned} \tag{5.8}$$

(If) We have to prove that: if the matrix is defined as in (5.6) then the maximum $\|\mathbf{v}\|^2$ is reached. Defining the vector of standard deviations $\sigma = \mathbf{v}/\|\mathbf{v}\|$ we could obtain the optimal matrix as an outer product $\mathbf{\Omega}^* = \sigma\,\sigma^\top$. This matrix has rank 1 and there is only one eigenvalue, λ_j, different from 0 and equal to 1.

The eigenvector associated to eigenvalue 1 is equal to the vector of standard deviations. The above inequality holds true as an equality because the two vectors are proportional. As a conclusion we obtain that the maximum is

$$\Phi^*_A(\mathbf{P}) = \alpha^2 \sum_{k=1}^{n} v_k^2.$$

(Only if) We have to prove that: if the maximum $\|\mathbf{v}\|^2$ is reached then the optimal matrix is equal to (5.6). The equality $\Phi^*_A(\mathbf{P}) = \|\mathbf{v}\|^2$ holds if $\mathbf{v}$ is proportional to all the columns of $\mathbf{U}$ or if there is a column $\mathbf{u}_{\cdot max}$ associated with the eigenvalue $\lambda_{max} = 1$. Given that $\mathbf{U}$ is orthonormal, its columns are a basis in $\mathbb{R}^n$ and therefore if $\mathbf{v}$ is proportional to one of them is also orthogonal to all the others.

The resulting matrix $\mathbf{\Omega}^*$ that solves problem (5.4) has only an eigenvalue different from zero and equal to 1, whose associated eigenvector is $\mathbf{v}/\|\mathbf{v}\|$. Summarizing, the optimum matrix results

$$\mathbf{\Omega}^* = \frac{\mathbf{v}\,\mathbf{v}^\top}{\|\mathbf{v}\|^2}.$$

□

We now considering a particular nature of $\mathbf{\Omega}$. Consider the case of not correlated shocks. The covariance matrix becomes a diagonal matrix whose (i, i) entry represents the variance of the i-th shock. We could reformulate the problem (5.4) as

$$\begin{aligned} \max_{\sigma^2 \in \mathbb{R}^n} \quad & \alpha^2 \sum_{i=1}^{n} (v_i\, \sigma_i)^2 \\ \text{s.t.} \quad & \sum_{i=1}^{n} \sigma_i^2 = 1. \end{aligned} \tag{5.9}$$

When shocks have this special nature this proposition holds.

Proposition 8. *Assume that shocks that affect a network are all uncorrelated. Then, the maximum of problem (5.9) is*

$$\alpha^2\, v_{max}^2, \tag{5.10}$$

i.e. the highest values of the vector $\mathbf{v}$ *of KB centralities. The maximum is reached if the optimal vector of variances* $(\sigma^2)^*$ *distributes the total amount of variances only on the nodes that have the KB centrality equal to* v_{max}.

Proof. The objective function is a weighted sum of centralities, and the weights are the variances of shocks $0 \le \sigma_i^2 \le 1$, $\forall\, i$. Then:

$$\begin{aligned} \max_{\|\sigma\|=1} \sum_{i=1}^{n} v_i^2\, \sigma_i^2 &\le \max_{\|\sigma\|=1} v_{max}^2 \sum_{i=1}^{n} \sigma_i^2 \\ &= v_{max}^2. \end{aligned}$$

Satisfying the equality constraint in the first line means to have the diagonal of $\mathbf{\Omega}$ that distributes the sum of variances on the positions of agents with *KB* centrality equal to v_{max}. □

Summarizing, when shocks are not correlated, the highest value of μ_A is reached when the total effect of shocks is focused on the node with the highest centrality.

5.1.2 Planner's Intervention

Suppose that a *planner*, an external entity, wants that the network of n agents reaches a goal and has the power to intervene changing individuals' state. In the previous section we have talked about maximizing the expected value of y_A. Actually, the real goal of the planner is to minimize the aggregate volatility because we wish to have a stable network that varies slightly its expected macro state. Therefore, the planner of this section will minimize the maximum of the expected value of y_A acting indirectly on the variance of the shocks, using a finite budget. The new question is: how does the planner allocate the budget?

Compared to the previous problem, we add a vector of weights $\mathbf{q} \in \mathbb{R}^n_+$ which represents the intensity q_i of the intervention on the variance of shock i. Then, the optimization problem is formulated as follow:

$$\min_{\mathbf{q}\in\mathbb{R}_+:\sum_{i=1}^n q_i=1} \quad \max_{\boldsymbol{\Omega}} \quad \alpha^2\,\mathbf{v}^\top \boldsymbol{\Omega}\,\mathbf{v} \qquad \text{s.t.} \quad \sum_{i=1}^n \Omega_{ii}\, q_i = 1. \tag{5.11}$$

Given a vector of *KB* centralities, we will call the extremum value *best worst case* $\mathrm{bwc}(\mathbf{v})$ because the planner has to reduce the highest impact of shocks over μ_A.

Theorem 7. *Consider an economy $\mathfrak{E}_n$ described by the interactions matrix $\mathbf{P}$ and the parameter α. Let $\mathbf{v} \in \mathbb{R}^n_+$ be the vector of Katz-Bonacich centralities, and $\boldsymbol{\Omega} \in \mathbb{S}^n_+$ the covariance matrix of stochastic shocks.*

Then, the best worst case of problem (5.11) is

$$\left(\alpha \sum_{k=1}^n v_k\right)^2. \tag{5.12}$$

The optimum is achieved when the covariance matrix $\boldsymbol{\Omega}^*$ has the form

$$\Omega^*_{ij} = \phi\,\frac{v_i v_j}{q_i q_j}, \quad \phi = \left(\sum_{k=1}^n \frac{v_k^2}{q_k}\right)^{-1}, \quad \forall\, i,j, \tag{5.13}$$

and the asset of optimum weights is

$$\mathbf{q}^* = \frac{\mathbf{v}}{\sum_{k=1}^n v_k}. \tag{5.14}$$

The interesting fact about the proposition is that the bwc($\mathbf{v}$) is equal for all the networks independently from the topology of interactions. The amount of budget given to an agent has to be proportional to its *KB* centrality. The proposition states that the best worst case is founded when shocks are perfectly correlated and the planner allocates the budget proportionally to the magnitude of *KB* centralities. Using the asset of perfect weights the optimal matrix defined in (5.13) simply becomes $\mathbf{\Omega}^* = \mathbb{1}\,\mathbb{1}^\top$. In both cases, when there is an external planner or not, perfectly correlated shocks brings the expected value of y_A to reach its maximum value. Therefore, we have highlight that role of the planner is to keep this maximum value lower as possible.

Proof. Given that the covariance matrix is symmetric, it has real eigenvalues and orthonormal eigenvectors. We can therefore represent it as $\mathbf{\Omega} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^\top$, where $\mathbf{U}$ is an orthonormal matrix whose columns are eigenvectors of $\mathbf{\Omega}$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is a diagonal matrix holding corresponding eigenvalues. We will denote with $\mathbf{u}_{\cdot i}$ the i-th column of $\mathbf{U}$. For the sake of exposure, we call $\mathcal{V} \in \mathbb{R}^{n\times n}$ the outer product $\mathbf{v}\,\mathbf{v}^\top$ and $\mathcal{Q} = diag(\mathbf{q}) \in \mathbb{R}^{n\times n}$ the diagonal matrix of weights.

Firstly we have to find the optimal vector of shocks that maximize μ_A, considering the asset of weights fixed. Formally we define the optimization problem:

$$\begin{aligned} \Phi_A^*(\mathbf{P}, \mathbf{q}) = &\max_{\mathbf{U}, \mathbf{\Lambda}} \quad \sum_{i=1}^n \lambda_i (\mathbf{v}^\top \mathbf{u}_{\cdot i})^2, \\ &\text{s.t.} \quad \sum_{i=1}^n \lambda_i\, \mathbf{u}_{\cdot i}^\top \mathcal{Q}\, \mathbf{u}_{\cdot i} = 1. \end{aligned} \tag{5.15}$$

Defining the Lagrange function as

$$\mathcal{L}(\mathbf{\Lambda}, \mathbf{U}, \eta) = \sum_{i=1}^n \lambda_i (\mathbf{v}^\top \mathbf{u}_{\cdot i})^2 - \eta \left(\sum_{i=1}^n \lambda_i\, \mathbf{u}_{\cdot i}^\top \mathcal{Q}\, \mathbf{u}_{\cdot i} - 1 \right),$$

we want to find the extremum point using the method of Lagrange multipliers. Deriving the above equation with respect to λ_i and $\mathbf{u}_{\cdot i}$ we find that

$$\frac{\partial \mathcal{L}(\mathbf{\Lambda}, \mathbf{U}, \eta)}{\partial \lambda_i} = 0 \iff \mathbf{u}_{\cdot i}^\top \mathcal{V}\, \mathbf{u}_{\cdot i} = \eta\, \mathbf{u}_{\cdot i}^\top \mathcal{Q}\, \mathbf{u}_{\cdot i}, \; \forall i, \tag{5.16}$$

$$\frac{\partial \mathcal{L}(\mathbf{\Lambda}, \mathbf{U}, \eta)}{\partial \mathbf{u}_{\cdot i}} = 0 \iff \lambda_i \left(\mathcal{V}\, \mathbf{u}_{\cdot i} - \eta\, \mathcal{Q}\, \mathbf{u}_{\cdot i} \right) = \mathbf{0}, \; \forall i, \tag{5.17}$$

Re-elaborating the first equation we found that the objective function is

$$\sum_{i=1}^n \lambda_i (\mathbf{v}^\top \mathbf{u}_{\cdot i})^2 \; = \eta \sum_{i=1}^n \lambda_i\, \mathbf{u}_{\cdot i}^\top \mathcal{Q}\, \mathbf{u}_{\cdot i} \; = \eta,$$

that is, it is equal to the value of Lagrange multiplier.

Assuming $\mathcal{Q}$ non-singular we call $\mathbf{\Pi} = \mathcal{Q}^{-1}\mathcal{V}$. The term inside the brackets of equation (5.17) becomes the standard eigenvalues problem

$$\mathbf{\Pi}\,\mathbf{u}_{\cdot i} = \eta\,\mathbf{u}_{\cdot i}.$$

Therefore, the Lagrange multiplier is one of the eigenvalues of $\mathbf{\Pi}$. The matrix $\mathbf{\Pi}$ has rank, then the Lagrange multiplier is equal to

$$\eta^* = \sum_{i=1}^{n} \frac{v_i^2}{q_i},$$

a weighted sum of squared *KB* centralities.

Considering the new value of the Lagrange multiplier we have now to prove that equation (5.17) holds for all eigenvalues and associated eigenvector. We split the analysis in two cases.

- When the vector $\mathbf{u}_{\cdot i}$ is the eigenvector associated with eigenvalue η^* the term inside the brackets is equal to 0 and then the equation is satisfied. The resultant eigenvector associated to this eigenvalue is founded from the eigenvalue problem and is equal to

$$\mathbf{u}^* = \frac{\mathcal{Q}^{-1}\mathbf{v}}{\|\mathcal{Q}^{-1}\mathbf{v}\|}. \tag{5.18}$$

- When the vector $\mathbf{u}_{\cdot i}$ is another eigenvector we have to prove that the null term is λ_i. We start with the analysis of $\mathcal{V}\,\mathbf{u}_{\cdot i}$. The matrix $\mathcal{V}$ has rank one and rank-nullity theorem ensures that the null space has dimension $n-1$.

  Given that

$$\mathcal{V}\,\mathbf{u}^* = \eta^* \frac{\mathbf{v}}{\|\mathcal{Q}^{-1}\mathbf{v}\|} \tag{5.19}$$

  is different from zero, $\mathbf{u}^*$ does not belong to the kernel of $\mathcal{V}$ but belongs to the the row space of $\mathcal{V}$. It is possible to demonstrate that a vector $\mathbf{x}$ lies in the kernel of matrix $\mathbf{A}$ if and only if is orthogonal to every vector in the row space of $\mathbf{A}$. In this case the row space has dimension 1 and is composed by $\mathbf{u}^*$. Given that the column vectors of matrix $\mathbf{U}$ are orthogonal, all the other $n-1$ columns of $\mathbf{U}$ belongs to the kernel of $\mathcal{V}$; thus, we conclude that $\mathcal{V}\,\mathbf{u}_{\cdot i} = \mathbf{0}$ for all columns different from $\mathbf{u}^*$.

  We have proved that the first term in the brackets is equal to 0. Demonstrating that the latter is different from 0 is easier. Matrix $\mathcal{Q}$ is assumed non-singular and given that is diagonal each entries must be different from 0. The orthonormal vector $\mathbf{u}_{\cdot i}$ must have at least one entry different to zero because it has norm 1. Therefore, the term $\mathcal{Q}\,\mathbf{u}_{\cdot i}$ is always different from zero.

Summarizing, we have proved that the resultant vector in the brackets of equation (5.17) is different from 0 when the eigenvector is not $\mathbf{u}^*$. We conclude that $\lambda_i = 0$ for all the component different from the $*$ one.

Using the constraint, we find the value of the only eigenvalue different from 0

$$\lambda^* = \frac{\|\mathcal{Q}^{-1}\mathbf{v}\|^2}{\mathbf{v}^\top \mathcal{Q}^{-1}\mathbf{v}}. \tag{5.20}$$

Then, the covariance matrix of shocks that solves (5.15) is equal to

$$\mathbf{\Omega}^* = \lambda^* \left(\mathbf{u}^* \, \mathbf{u}^{*\top}\right). \tag{5.21}$$

Now that we have found the optimal matrix we have to solve the second problem

$$\begin{aligned} \Phi_A^*(\mathbf{P}) &= \min_{\mathbf{q}\in\mathbb{R}^n} \quad \Phi_A^*(\mathbf{P},\mathbf{q}), \\ &\text{s.t.} \quad \sum_{i=1}^n q_i = 1. \end{aligned} \tag{5.22}$$

Defining another Lagrange function

$$\mathcal{L}(\mathbf{q},\delta) = \sum_{i=1}^n \frac{v_i^2}{q_i} - \delta\left(\sum_{i=1}^n q_i - 1\right),$$

we equal its derivative, respect to vector $\mathbf{q}$, to 0 for finding the optimal weight vector

$$\frac{\partial \mathcal{L}(\mathbf{q},\delta)}{\partial q_i} = -\left(\frac{v_i}{q_i}\right)^2 - \delta = 0.$$

In this situation, the planner has to do not create unbalanced ratio. Suppose that there is a ratio strictly higher than the others and imagine that there is an *enemy* planner that decide where the shocks variance will be allocated. Obviously, the enemy planner will choose the higher ratio, because in that way the maximum feedback is reached.

To do not create a positive situation for the enemy planner, the good one has to make all the ratio equal. So the magnitude of planner intervention q_i has to be proportional to v_i and all the ratios have to be equal. The resulting vector of weight is

$$q_i^* = \frac{v_i}{\sum_{i=1}^n v_i}, \; \forall i,$$

and the best worst case results the square of the sum of *KB* centralities

$$\text{bwc}(\mathbf{v}) = \left(\alpha \sum_{i=1}^n v_i\right)^2.$$

□

We now consider the special case of uncorrelated shocks. When shocks are not correlated we recall that the covariance matrix is $\mathbf{\Omega} = \text{diag}(\sigma_1^2, ..., \sigma_n^2)$ and each element represents the variance of shock. With $\sigma \in \mathbb{R}^n$ we indicate the vector of standard deviation. The problem is formulated as follows:

$$\begin{aligned} \min_{\mathbf{q}\in\mathbb{R}_+ : \sum_{i=1}^n q_i = 1} \quad \max_{\sigma} \quad & \alpha^2 \sum_{i=1}^n \sigma_i^2 v_i^2 \\ \text{s.t.} \quad & \sum_{i=1}^n \sigma_i^2 q_i = 1. \end{aligned} \tag{5.23}$$

The following proposition defines the solution.

Proposition 9. *Assume that shocks that affect a network of n agents are all not correlated. Then, the bwc($\mathbf{v}$) of problem (5.23) is*

$$\alpha^2 \sum_{k=1}^n v_k^2, \tag{5.24}$$

and the asset of weights that solves the problem is

$$q_i = \frac{v_i^2}{\sum_{k=1}^n v_k^2}, \forall\, i. \tag{5.25}$$

Proof. First of all, we consider a fixed $\mathbf{q}$ and find the worst, maximum, case of μ_A. Defining the Lagrange function defined as

$$\mathcal{L}(\sigma, \eta) = \sum_{i=1}^n \sigma_i^2 v_i^2 - \eta \left(\sum_{i=1}^n \sigma_i^2 q_i - 1 \right),$$

we want to find the extremum point using the method of Lagrange multipliers.

Taking the derivative with respect to σ_i and setting the derivative equal to 0 we find that

$$\frac{\partial \mathcal{L}(\sigma, \eta)}{\partial \sigma_i} = 2v_i^2\sigma_i - 2\eta q_i \sigma_i = 0, \forall i \iff v_i^2 \sigma_i = \eta q_i \sigma_i, \forall i,$$

and hence, the maximum of the objective function becomes

$$\max_i \quad \frac{v_i^2}{q_i}. \tag{5.26}$$

The maximum implies that the worst case of the macro state is defined by the highest ratio of centralities and weights, and not only depends on the topology of the network. In fact, also where the budget is allocated is relevant.

Now, the planner has to decide how to distribute the weights to minimize the maximum ratio, that is,

$$\min_{\mathbf{q}\in\mathbb{R}_+, \sum_{i=1}^{n} q_i = 1} \quad \max_{j} \quad \frac{v_j^2}{q_j}$$

In this situation, the planner has to do not create unbalanced ratio. The magnitude of weight to allocate on agent i has to be proportional to the square of its *KB* centrality, and has to make the ratio equal for all the agents:

$$q_i \propto v_i^2, \text{ and } \frac{v_i^2}{q_i} = k, \, \forall i.$$

Using these devices and the constraint on the sum of the weights, we find that

$$q_i = \frac{v_i^2}{\sum_{i=1}^{n} v_i^2}, \, \forall i.$$

The resultant best worst case of μ_A is the squared Euclidean norm of *KB* centrality vector

$$\text{bwc}(\mathbf{v}) = \alpha^2 \sum_{i=1}^{n} v_i^2.$$

□

An interesting comparison between the two asset of weights could be done by inverting the nature of the shocks in the respective problems. Imagine that the planner prepares her/his network to receive not correlated shocks, i.e. she/he set the vector of weights to $q_i = v_i^2 / \left(\sum_{k=1}^{n} v_k^2\right) \forall i$; we indicate with q^0 this asset of weights. Unfortunately, once the network is prepared, a set of correlated shocks affects the economy. How much would the bwc($\mathbf{v}$) change from $\left(\sum_{k=1}^{n} v_k\right)^2$? Substituting the vector of weights in $\sum_{k=1}^{n}(v_k^2/q_k)$ we find that the difference between the old maximum (5.12) and the new value is

$$\begin{aligned} \Delta_{q^0,\rho=1} &= n\,\alpha^2 \sum_{k=1}^{n} v_k^2 - \left(\alpha \sum_{k=1}^{n} v_k\right)^2 \\ &= n\,\alpha^2 \left(\sum_{k=1}^{n} v_k^2 - \frac{n}{(1-\alpha)^2}\right) \\ &= (n\,\alpha)^2 \text{Var}(v_1, ..., v_n). \end{aligned} \tag{5.27}$$

The difference between the two maximum increases as the variance of centralities. A regular network that has no differences between agents does not show any changing in the best worst case; this result is obvious because $v_i / \sum_{k=1}^n v_k$ is equal to $v_i^2 / \sum_{k=1}^n v_k^2$ when *KB* centrality is $v_{regular} = 1/(1-\alpha)$. The divergence $\Delta_{\rho=1}$ is no more equal to 0 when hubs are present in the network. The existence of hubs and marginal agents increases the variance of centralities and therefore the $\Delta_{q^0,\rho=1}$. Disproportional networks show an higher decline of the maximum of μ_A when shocks are perfectly correlated and weights are not set as **??** says.

Moreover this difference is quadratic in the size of the network; if the number of agents $n \to \infty$ then the cost for having set the asset of weights wrong will tends to infinity too.

Consider now the opposite case. The planner prepares the network to receive perfectly correlated shocks, i.e. $q_i = v_i / \left(\sum_{k=1}^n v_k\right) \; \forall i$ but the ones that affect the network are not correlated; we indicate with q^1 this asset of weight. Substituting the vector of weights in the problem (5.26) we find that

$$\max_i \frac{v_i^2}{q_i} = \max_i v_i \sum_{k=1}^n v_k.$$

Hence, the difference between the old maximum (5.24) and the new one is

$$\Delta_{q^1,\rho=0} = \alpha^2 \left(v_{max} \sum_{k=1}^n v_k - \sum_{k=1}^n v_k^2 \right). \tag{5.28}$$

The three norm of *KB* centralities vector are related by the following inequality

$$v_{max} \|\mathbf{v}\|_1 \geq \|\mathbf{v}\|_2^2.$$

Then, we conclude that also in this case the deviance between the two term is not null. If we imagine that the difference between the two terms is a cost to be paid then a network with almost equal centralities results in no cost at all. Unfortunately it is not so clear the amount of deviance between the two term. The behavior of $\Delta_{q^1,\rho=0}$ suggests again that a disproportional network has higher decline when the weights are not set as Proposition 9 says.

5.2 Total Welfare

We recall that the macro state y_W is formulated as $\left(\mathbf{x}^\top \mathbf{x}\right)$. The performance measure is founded by taking the expected value of y_W:

$$\begin{aligned}
\mathbb{E}[y_W] &= \alpha^2 \, \mathbb{E}\left[\mathbf{S}^\top \left(\mathbf{L}^\top \mathbf{L}\right) \mathbf{S}\right] \\
&= \alpha^2 \, \mathbb{E}\left[\mathrm{Tr}\left(\mathbf{L}\,\mathbf{S}\,\mathbf{S}^\top \mathbf{L}^\top\right)\right] \\
&= \alpha^2 \, \mathrm{Tr}\left(\mathbf{L}\mathbb{E}\left[\mathbf{S}\mathbf{S}^\top\right] \mathbf{L}^\top\right).
\end{aligned}$$

In other words,

$$\mathbb{E}[y_W] = \alpha^2 \mathrm{Tr}\left(\mathbf{L}\,\boldsymbol{\Omega}\,\mathbf{L}^\top\right), \tag{5.29}$$

that is, the Leontief matrix is the most important instrument that defines how the shocks shape the macro state. From now on we will denote $\mathbb{E}[y_W]$ with $\mu_W = \Phi_W(\mathbf{P}, \mathbf{S})$, where $\Phi_W : \mathbb{R}^n \to \mathbb{R}$ expresses the dependence of the macro state on the nature of interactions matrix and on stochastic shocks.

Another way of describing μ_W is by the SVD of the Leontief matrix 2.4. Using the SVD, the output of the economy could be rewritten as

$$\begin{aligned}\mu_W &= \alpha^2 \mathbb{E}\left[\mathbf{S}^\top\left(\mathbf{U}\,\boldsymbol{\Gamma}^\top\boldsymbol{\Gamma}\,\mathbf{U}^\top\right)\mathbf{S}\right] \\ &= \alpha^2 \mathbb{E}\left[\mathrm{Tr}\left(\boldsymbol{\Gamma}\,\mathbf{U}^\top\mathbf{S}\,\mathbf{S}^\top\mathbf{U}\,\boldsymbol{\Gamma}^\top\right)\right],\end{aligned}$$

that using the SVD basis $\underline{\boldsymbol{\Omega}} = \mathbf{U}^\top\boldsymbol{\Omega}\,\mathbf{U}$ becomes

$$\mu_W = \mathrm{Tr}\left(\boldsymbol{\Gamma}\,\underline{\boldsymbol{\Omega}}\,\boldsymbol{\Gamma}^\top\right). \tag{5.30}$$

We will use both equations interchangeably; in fact, even if equation (5.29) is easier to interpret, the above one is useful when we have to describe the optimization problems in a compact form.

As for μ_V, we now start from idiosyncratic shocks. When $\mathbf{S}$ describes a vector of i.i.d. shocks with null mean and variance σ^2, $\boldsymbol{\Omega}$ is equal to $\sigma^2\mathbb{I}$. The following result resumes how μ_W is conditioned by the Leontief matrix:

Proposition 10. *Let $\mathfrak{E}_n$ be an economy defined by the interactions matrix $\mathbf{P}$, the parameter α and the vector of stochastic shocks $\mathbf{S}$. Suppose that shocks are independent and identically distributed with expected value 0 and finite variance σ^2. We denote with γ_i the i-th singular value of Leontief matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$.*

Then, the ex-ante performance of the macro state y_W is

$$\mu_V = (\alpha\sigma)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} L_{ji}^2 \tag{5.31}$$

or equally

$$\mu_V = (\alpha\sigma)^2 \sum_{i=1}^{n} \gamma_i. \tag{5.32}$$

The proposition says that the performance measure μ_W is increasing in the norm of the columns of $\mathbf{L}$, or equally, in the sum of the singular values. In fact, the trace of $\mathbf{L}\,\mathbf{L}^\top$ is equal to the sum of its eigenvalues, that in this case corresponds to the sum of singular values of $\mathbf{L}$.

From en *ex-ante* point of view it is not important which agent is affected when idiosyncratic shocks affect the network. Assuming a finite variance of the shocks, values of the performance metric depends on the trace of $\mathbf{L}\,\mathbf{L}^\top$. Hence, classifying the expected macro state of different networks means understanding how much the sum of singular values is high. Here we want to express μ_W as a function of the number of agents and understand its behavior when n tends to infinity.

When the network is regular the Leontief matrix is symmetric and the square root of singular values are equal to eigenvalues of $\mathbf{L}$. Calling λ the eigenvalues of matrix $\mathbf{P}$ associated to a regular network, equation (5.32) becomes

$$\mu_W = (\alpha\sigma)^2 \sum_{i=1}^{n} \frac{1}{(1-\alpha\lambda_i)^2}. \tag{5.33}$$

When the network is not regular the only way to calculate the sum of singular values is through the direct calculus of Leontief matrix. Once that the $\mathbf{L}$ is found, it is possible to find $\mathbf{L}\,\mathbf{L}^\top$ and after calculate the trace as the sum of diagonal elements. What follows are example of the behavior of μ_W in academic networks, also considered in previous chapters. We refer to Section 2.2.1 where the calculus of eigenvalues of regular graph was already done.

Complete graph K_n. Consider a complete network with n agents. The complete graph is $(n-1)$-regular, hence the spectrum of $\mathbf{P} = \frac{1}{n-1}\mathbf{W}$ is

$$\sigma(\mathbf{P}) = \left\{1, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right\}.$$

Given the equation (5.33), the trace of $\mathbf{L}^2$ results

$$\mathrm{Tr}\left(\mathbf{L}^2\right) = \frac{1}{(1-\alpha)^2} + \frac{(n-1)^3}{[(n-1)+\alpha]^2}.$$

Considering the *large scale* limit, i.e. when $n \to \infty$, we want to express the above equation as function of n and of $\mathcal{C}_\alpha$, a parameter that depends on a fixed α. Then, we find that

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathbf{L^2}\right) = n\,\mathcal{C}_\alpha, \text{ where } \mathcal{C}_\alpha = 1, \tag{5.34}$$

namely, given n tending towards ∞ the dependence of the trace is simply linear in n. □

Barbell graph $B_{n/2}$. Consider now the Barbell graph $B_{n/2}$ composed by two independent complete graphs each of which has $n/2$ agents. For the sake of simplicity consider an even number of agents; this simplification becomes irrelevant when $n \to \infty$. Resulting adjacency matrix is similar to a block diagonal matrix with exception of two values, that represent the link between the two components.

The barbell graph is not regular and so we can not calculate eigenvalues as the sum of characters. Instead of using properties of Cayley graphs we have to explicitly calculate the Leontief matrix using the recursive definition.

We have already demonstrated that $|\alpha\,\lambda_i| < 1$ for all the eigenvalues. Given that this assumption holds, matrix $\alpha\,\mathbf{P} \to 0$ when the exponent of the Neumann series tends to infinity. Therefore we could approximate the Leontief matrix with

$$\mathbf{L} \simeq \mathbb{I} + \alpha\,\mathbf{P} + \alpha^2\mathbf{P}^2 \ \ [19].$$

For the sake of exposition we also use another assumption; it is not a restriction to consider all the element of $\mathbf{P}$ equal, in fact $1/(n-1) \simeq 1/n$, when $n \to \infty$.

The explicit formula is not here reported due to its complicate algebraic form. On the other hand, the result for $n \to \infty$ is equal to the one of the complete graph:

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathbf{L}\,\mathbf{L}^\top\right) = n\mathcal{C}_\alpha, \text{ where } \mathcal{C}_\alpha = 1. \tag{5.35}$$

Once again it seems that the bridge between the two components does not give any noticeable variation on the performance of the macro state. □

Cycle graph C_n. Considering that the cycle is a 2-regular graph, the spectrum of $\mathbf{P}$ associated to the network is

$$\sigma(\mathbf{P}) = \left\{\lambda_k = \cos\left(\frac{2\pi}{n}k\right), k = 0,1,...,n-1\right\}.$$

Therefore, the trace of $\mathbf{L}^2$ results

$$\mathrm{Tr}\left(\mathbf{L}^2\right) = \sum_{k=0}^{n-1}\left[1 - \alpha\cos\left(\frac{2\pi}{n}k\right)\right]^{-2}.$$

Since $\alpha < 1$, the function $f(x) = (1 - \alpha\,\cos(x))^{-2}$ is a continuous function over the domain $[0,2\pi]$. The definite integral of a continuous function f over the interval $[a, b]$ is the limit of a Riemann sum as the number of subdivisions approaches infinity. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x f(x_i), \ \text{ where } \ \Delta x = \frac{b-a}{n},\ x_i = a + \Delta x\,i.$$

Using the above definition the summation of $\mathbf{L}^2$ becomes

$$\lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=0}^{n-1}\left[1 - \alpha\cos\left(\frac{2\pi}{n}k\right)\right]^{-2} \simeq \int_0^{2\pi}\left[1 - \alpha\cos(x)\right]^{-2}dx,$$

and the resulting value of the trace when the number of agents tends to infinity results

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathbf{L}^2\right) = n\,\mathcal{C}_\alpha, \text{ where } \mathcal{C}_\alpha = \frac{1}{2\pi}\int_0^{2\pi} \left[1-\alpha\cos(x)\right]^{-2} dx. \tag{5.36}$$

The trace results again linear in n but the role of parameter α is also relevant. An upper bound for $\mathcal{C}_\alpha$ is $1/(1-\alpha)^2$; hence, differently from the complete graph, when α is near to 1 the value of the trace shift the straight line to an higher intercept.

We have previously seen that a product of $d > 1, \in \mathbb{N}$ cycle graphs generates a toroidal graph. Recalling that the spectrum of $\mathbf{P}$ of a toroidal graph is

$$\sigma(\mathbf{P}_{tor}) = \left\{\lambda_{k_1,\dots,k_d} = 2\sum_{i=1}^{d}\cos\left(\frac{2\pi}{n}k_i\right), (k_1, ..., k_d) \in \{0,1, ..., n-1\}^d\right\},$$

the trace of $\mathbf{L}^2$ results

$$\mathrm{Tr}\left(\mathbf{L}^2_{tor}\right) = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\cdots\sum_{k_d=0}^{n-1}\left(1-\frac{2\alpha}{d}\sum_{i=0}^{d}\cos\left(\frac{2\pi}{n}k_i\right)\right)^{-2}.$$

When $n \to \infty$ we could replace summations with Riemann integrals doing same substitutions as in the cycle graph. The behavior of $\mathrm{Tr}\left(\mathbf{L}^2\right)$ when $n \to \infty$ becomes

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathbf{L}^2_{tor}\right) \simeq n^d \mathcal{C}_\alpha \tag{5.37}$$

where

$$\mathcal{C}_\alpha = \frac{1}{(2\pi)^d}\iint\cdots\int_0^{2\pi}\left[1-\frac{2\alpha}{d}\sum_{i=0}^{d}\cos(x_i)\right]^{-2} dx_1 dx_2 ... dx_d.$$

Even if the parameter $\mathcal{C}_\alpha$ is not informative, we could state that the trace increases again proportionally to the dimension of graph, i.e. to the number of nodes, as in previous cases. □

Star graph S_n. Consider now the star graph S_n with n nodes. Instead of considering the inverse of $(\mathbb{I}-\alpha\mathbf{P})$, the Leontief matrix is founded using $\sum_{i=0}^{\infty}\alpha^i\mathbf{P}^i$. As considered in the second chapter, matrices with odd powers are all equal to $\mathbf{P}_{odd}$ and ones with even powers are all equal to $\mathbf{P}_{even}$.

Given that, Leontief matrix is

$$\begin{aligned}\mathbf{L} &= \mathbb{I} + (\alpha+\alpha^3+\dots)\mathbf{P}_{odd} + (\alpha^2+\alpha^4+\dots)\mathbf{P}_{even}\\ &= \mathbb{I} + \frac{\alpha}{1-\alpha^2}\mathbf{P}_{odd} + \frac{\alpha^2}{1-\alpha^2}\mathbf{P}_{even},\end{aligned}$$

the trace of $\mathbf{L}\,\mathbf{L}^\top$ results

$$\mathrm{Tr}\left(\mathbf{L}\,\mathbf{L}^\top\right) = \frac{n^2\alpha^2 + (n-1)(1-\alpha^2)\left[n(1-\alpha^2)+2\alpha^2\right]}{(n-1)(1-\alpha^2)^2}.$$

Considering the large scale limit, i.e. when n grows and α is fixed, the trace of $\mathbf{L}\,\mathbf{L}^\top$ results

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathbf{L}\,\mathbf{L}^\top\right) = n\mathcal{C}_\alpha, \text{ where } \mathcal{C}_\alpha = \frac{1}{(1-\alpha^2)^2}. \tag{5.38}$$

Like in the others graphs the dependence of $\mathrm{Tr}\left(\mathbf{L}\,\mathbf{L}^\top\right)$ is linear in n but the parameter $\mathcal{C}_\alpha$ makes great differences with others topologies. It is clear that when α takes higher values, parameter $\mathcal{C}_\alpha$ explodes. □

We now return on the comparison between μ_W in different topologies. We discuss which network outperforms others and which one has worst performance, classifying different topologies that we have considered till now. We will compare only one-dimensional networks, i.e. we do not consider toroidal graph. Given the results of $\mathrm{Tr}\left(\mathbf{L}\,\mathbf{L}^\top\right)$, these are the expected values of y_W when $n \to \infty$:

$$\begin{aligned}
&\text{Complete graph} && \mu_W(K_n) = n\,(\alpha\,\sigma)^2 \\
&\text{Barbell graph} && \mu_W(B_k) = n\,(\alpha\,\sigma)^2 \\
&\text{Cycle graph} && \mu_W(C_n) = n\,\frac{(\alpha\,\sigma)^2}{2\pi}\int_0^{2\pi}\left[1-\alpha\cos(x)\right]^{-2}dx \\
&\text{Star graph} && \mu_W(S_n) = n\left(\frac{\alpha\,\sigma}{1-\alpha^2}\right)^2.
\end{aligned} \tag{5.39}$$

In the performance metric μ_W it is $\mathcal{C}_\alpha$ that makes an economy better than the others. We recall that the parameter α represents the state of the world known to all the agents; formally, it is considered as a parameter that weights direct and indirect interconnections between agents.

Figure 5.2 shows an exponential growth for $\mathcal{C}_\alpha$ of star and cycle network when α varies in its domain. This figure explains that when α approaches 1 the order of difference between μ_W in a complete network and in a star or cycle network is approximately of one order of magnitude. Notable difference in $\mathcal{C}_\alpha$ starts approximately when $\alpha = 0.8$.

Again, the not intuitive result concerns the cycle network. When longer paths are mainly taken into account the effects of shocks are spread all over the ring, reaching also opposite agents. In complete and barbell networks, where every node is highly interlinked, shock that affects a particular agent is immediately shared with almost all the other agents, which in turn will again share the intensity of the shock. This effect explains why these networks has lower values of μ_W.

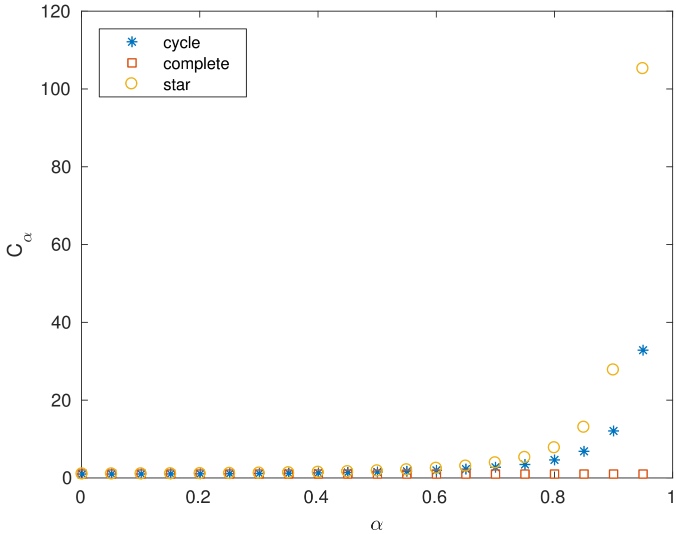

Figure 5.2. How parameter $\mathcal{C}_\alpha$ changes in different networks when α grows in its domain.

Consider i.i.d. shocks with variance $\sigma^2 = 1$. How the performance metric μ_W changes when $n \to \infty$ is depicted in Figure 5.3. In light of the previous figure we have set $\alpha = 0.8$ just for noticing differences n μ_W.

As a conclusion we state that when the number of agents grows to infinity the star network outperforms ring and complete network. Once again, if the goal is to reach higher values of the expected value of the macro state, the presence of an hub makes the star network be the topology to prefer.

5.2.1 Non-idiosyncratic shocks

The previous analysis focus the attention on the role of the trace of $\mathbf{L}\,\mathbf{L}^\top$ because the covariance matrix has a simple nature $\sigma^2\mathbb{I}$. Suppose now that the n shocks are not i.i.d.; in this case, $\boldsymbol{\Omega}$ gives a more intricate contribute to μ_W.

We are interested in finding which is the best covariance matrix that maximize the expected value of y_W when it is known the Leontief matrix of the system. Under the assumption of not idiosyncratic shocks, we will study the maximization

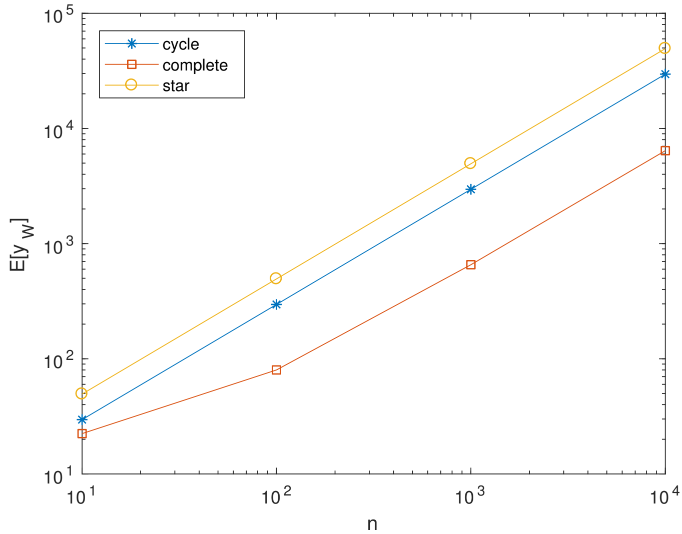

Figure 5.3. How the performance metric μ_W changes in different networks when $n \to \infty$.

problem defined as follows:

$$\begin{aligned} \Phi_W^*(\mathbf{P}) = \max_{\boldsymbol{\Omega} \in \mathbb{S}_+^{n \times n}} \quad & \operatorname{Tr}\left(\mathbf{L}\,\boldsymbol{\Omega}\,\mathbf{L}^\top\right) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \Omega_{ii} = 1. \end{aligned} \tag{5.40}$$

Theorem 8. *Consider an economy* $\mathfrak{E}_n$ *described by the its interactions matrix* $\mathbf{P}$ *and the parameter* α*. Let* $\mathbf{L} \in \mathbb{R}^{n \times n}$ *be the Leontief matrix of the economy,* γ_i *its singular values, and* $\boldsymbol{\Omega} \in \mathbb{S}_+$ *the covariance matrix of stochastic shocks.*

Then, the maximum of the problem (5.40) *is*

$$\alpha^2 \gamma_{max}, \tag{5.41}$$

and is achieved if the matrix is $\boldsymbol{\Omega}^* = \sigma^*(\sigma^*)^\top$*, where* σ^* *is a column vector proportional to the first principal component of* $\mathbf{L}$*.*

As in the previous chapter, we have that the solution is not explicative considering the measures taken into account up to now. The highest singular value and the first principal component associated are good summaries just because the singular value decomposition is in itself a good algebraic procedure. Unfortunately, the efficiency problem of calculating them when the number of agents is large is a well known numeric problem.

As in the previous chapter, this theorem states that the highest singular value of Leontief matrix of an economy resumes the highest possible value reached from μ_W. Moreover, it states that this value is reached when the vector of standard deviations is proportional to the first principal component, that is, in special cases, something similar to the eigenvector Bonacich centrality.

Proof. First of all, we have to find the maximum of problem (5.40). Instead of considering the standard problem we prefer to use the SVD form (5.30). In particular, we focus on the sum of diagonal element of $\underline{\boldsymbol{\Omega}} = \mathbf{U}^\top \boldsymbol{\Omega}\, \mathbf{U}$, that is

$$\begin{aligned}\sum_{i=1}^{n} \underline{\Omega}_{ii} &= \sum_{i=1}^{n}\left[\sum_{k=1}^{n} u_{ki}\left(\sum_{j=1}^{n} \Omega_{jk} u_{ji}\right)\right]\\ &= \sum_{k=1}^{n}\left[\sum_{j=1}^{n} \Omega_{jk}\left(\sum_{i=1}^{n} u_{ki} u_{ji}\right)\right]\\ &= \sum_{k=1}^{n} \Omega_{kk},\end{aligned}$$

where the last equation derives from the orthonormality of rows of matrix $\mathbf{U}$. Therefore, also the sum of diagonal elements of $\underline{\boldsymbol{\Omega}}$ is equal to 1.

Now we move back to the optimization function; we could rewrite y_W as

$$y_W = \sum_{i=1}^{n} \underline{\Omega}_{ii}\gamma_i.$$

We have already encountered a similar form. To find the optimum we could observe that

$$\begin{aligned}\Phi^*_W(\mathbf{P}) &= \max_{\sum_{i=1}^{n} \underline{\Omega}_{ii}=1} \sum_{i=1}^{n} \underline{\Omega}_{ii}\gamma_i\\ &\leq \gamma_{max}\sum_{i=1}^{n} \underline{\Omega}_{ii}\\ &= \gamma_{max}.\end{aligned} \tag{5.42}$$

(If) We have to prove that if the matrix is $\underline{\boldsymbol{\Omega}}^*$ then the maximum γ_{max} is reached. The theorem states that this matrix is formed by a vector σ^* proportional to the first principal component, that we indicate with $\mathbf{u}_{\cdot 1}$. Consider the element $\underline{\Omega}^*_{11}$:

$$\underline{\Omega}^*_{11} = \sum_{k=1}^{n} u_{k1}\sigma^*_k\left(\sum_{j=1}^{n} u_{j1}\sigma^*_j\right).$$

Given that the vector σ is proportional to the first principal component we find that $\underline{\Omega}^*_{11} = 1$. Therefore all the others diagonal elements of $\underline{\mathbf{\Omega}}^*$ are null. As a consequence, the above inequality (5.42) becomes an equality and results that $\Phi^*_W(\mathbf{P}) = \gamma_{max}$.

The only if condition is true only when the matrix $\mathbf{P}$ is symmetric. In fact, we have to prove that if the maximum γ_{max} is reached then the optimal matrix is $\mathbf{\Omega}^*$. That is, we have to prove that γ_{max} has multiplicity 1 and that $\Omega_{11} = 1$ while all the others elements of $\mathbf{\Omega}$ are null.

Proposition 1 ensures that if the graph is strongly connected, the highest eigenvalue of $\mathbf{P}$ has multiplicity 1. As a consequence, also the highest eigenvalue of $\mathbf{L}$ has multiplicity 1. If the matrix $\mathbf{L}$ is symmetric, for example for regular networks, also γ_{max} has multiplicity 1 because the highest singular value is equal to the square of the highest eigenvalue of $\mathbf{L}$. In this case γ_{max} is reached only when the diagonal element $\underline{\Omega}_{11} = 1$ and all the others are null. This happens when the covariance matrix is defined by the outer product of the first principal component, that is, the matrix is defined as $\mathbf{\Omega}^* = \sigma^* (\sigma^*)^\top$, with σ^* a vector of standard deviations proportional to $\mathbf{u}_{\cdot 1}$.

If the matrix $\mathbf{L}$ is not symmetric we could not conclude that the multiplicity of γ_{max} is 1. As a consequence, $\mathbf{\Omega}^* = \sigma^* (\sigma^*)^\top$, with σ^* a vector of standard deviations proportional to the first principal component of $\mathbf{L}$, could not be the unique solution for the problem. □

As for the performance metric μ_V, we now start the analysis of two particular nature of $\mathbf{\Omega}$: one that represents perfectly correlated shocks, and the other that describes not correlated shocks.

Consider the case of uncorrelated shocks. The covariance matrix becomes a diagonal matrix whose (i, i) entry represents the variance of the i-th shock. We could reformulate the problem (5.40) as

$$\begin{aligned} \max_{\sigma^2 \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ji}^2 \sigma_i^2 \\ \text{s.t.} \quad & \sum_{i=1}^{n} \sigma_i^2 = 1. \end{aligned} \tag{5.43}$$

When shocks are not correlated this proposition holds.

Proposition 11. *Assume that shocks that affect a network of n agents are all not correlated. Then, the maximum of problem (5.43) is*

$$\alpha^2 \sum_{j=1}^{n} L_{jmax}^2. \tag{5.44}$$

The maximum is reached if the optimal vector of variances $(\sigma^2)^*$ *distributes the total amount of variances only on the nodes that have* $\sum_{j=1}^{n} L_{j\cdot}^2$ *equal to* $\sum_{j=1}^{n} L_{jmax}^2$.

As for Proposition 8 all the variance of shocks has to be addressed to the agent i that has higher value of $\sum_{j=1}^{n} L_{ji}$; this value represents the variation of overall influence of agent i and we have previously used it in the *ex-post* analysis. Therefore, we conclude that the variance is allocated to the most systemically important agent, as defined in the previous chapter. Figure **??** clearly indicates that the star graph outperforms cycle, complete, and barbell graph.

Proof. The objective function of problem (5.43) is a weighted sum of z where weights are the variances of shocks. Then:

$$\begin{aligned}\max_{\|\sigma\|=1} \sum_{i=1}^{n}\sum_{j=1}^{n} L_{ji}^2\,\sigma_i^2 &\leq \sum_{j=1}^{n} L_{jmax}^2 \sum_{i=1}^{n} \sigma_i^2 \\ &= \sum_{j=1}^{n} L_{jmax}^2.\end{aligned}$$

Satisfying the equality constraint in the first line means to have the diagonal of $\mathbf{\Omega}$ that distributes the sum of variances on the positions of agents with $\sum_{j=1}^{n} L_{ji}^2$ equal to $\sum_{j=1}^{n} L_{jmax}^2$. □

Chapter 6

Conclusion

The thesis presented the role of network's structure within economic models with a particular attention spent on Katz-Bonacich centrality and on the Leontief matrix. Our main contribute on the general analysis made in the article [2] was to gave direct explanation of macroeconomic outcomes as function of these network properties. Starting from the assumption of idiosyncratic shocks we have developed the study considering even more general forms, highlighting the role of different type of shocks in shaping the macro state of the system.

As expected, the quantities that come out as being important have connections with ones studied in the economic network literature. For instance, we have found that, in our squared models, the Katz-Bonacich centralities vector and singular values of Leontief matrix are simple measures that could summarize network's structure properties. The problems of optimization on one of the macro states have highlighted that the euclidean and the maximum norm of Katz-Bonacich centralities vector are the fundamental quantities that explain reachable maximum of the aggregate level.

This conclusion makes us think that in real networks, the importance/power of a system, considered as a macro economic force, is concentrated in the hands of few agents. Once again, the power-law distribution explain the behavior of real network; the existence of nodes much larger than the average are the heart of the network. This concept reflects very well what has been found; in economic networks, if there is an agent much more central than the others, will make the difference at the aggregate level.

Clearly these results should be tested with real data; in the thesis, only academic cases were considered. Even if these examples could reflect some patterns of real networks a more truthful simulation could demonstrate the obtained results. A first analysis that could be done in a real simulation, in direct continuity with what has been found, is to understand how many hubs are necessary to weaken their power. In other words, it could be interesting to understand when an hub stop being such.

Given that the thesis aimed to understand the functionalities of networks structure, our model did not describe shocks dynamic. The first direction will be to modify the starting model for the diffusion of shocks in the economic context. For example, the SIS or SIR epidemic models could describe shocks as infections and study the fluctuations of the macroeconomic outcome, that is, a macro measure that represents the behavior of the collectivity. As a result, this will allow us to study how shocks to financial or production institutions can propagate throughout the whole system, potentially generating a systemic crisis.

Agents in networks are alive, that is, they constantly change their way of acting. Consider economic systems means to models agents' behavior with games theory. For example, the point of view of the article '"Networks,Shocks, and Systemic Risk" [2] is mainly focused on analyzing a system in which agents' interactions are simple, that is, in which agents are strategic complements and information is complete. Formally, information is complete when knowledge about agents' characteristics is available to all members, and agents are strategic complements when players' utilities are positively correlated. Another interesting approach could be to consider games with incomplete information, in which agents are not aware of the general state, but have their own global vision.

A second direction will be based on the dynamic of networks topology. The structure of the network plays a fundamental role in translating microeconomic shocks into macroeconomic outcomes. The starting point for analyze real systems is to consider networks that evolve on time. Within a system, each agent continuously changes its interactions, adding some and eliminating others: these transformations lead to vary the structure of the network. Suppose now that a shock damages a particular agent, what will be the most probably structure adopted by agents to make the shock "innocuous"?

A very interesting and curious approach on these problems is the one adopted in the section of planner intervention. Assuming that outside the network exists a planner that can shape incentives in order to optimize the macro state of the system, being limited by constraints, a lot of empirical situation can be modeled. Using this vision, we could analyze the two cases mentioned above: assuming that there is a planner who has a finite number of edges to place to maximize the macro state, where should she/he put them? If the planner has to choose the agent to give a positive shock, who should she/he target to get the maximum macro status?

As a final conclusion, we state that this thesis was an introduction to the theory of shocks on economic networks; many questions arise to better understand the reality of these systems, that, even if created by human beings, are not predictable.

Bibliography

[1] Acemoglu, D., Carvalho, V. M., Ozdaglar, A., and Tahbaz-Salehi, A. The network origins of aggregate fluctuations. *Econometrica 80*, 5 (2012), 1977–2016.

[2] Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. Networks, shocks, and systemic risk. Tech. rep., National Bureau of Economic Research, 2015.

[3] Ballester, C., Calvó-Armengol, A., and Zenou, Y. Who's who in networks. wanted: The key player. *Econometrica 74*, 5 (2006), 1403–1417.

[4] Barabási, A.-L., et al. *Network science.* Cambridge university press, 2016.

[5] Bonacich, P. Power and centrality: A family of measures. *American journal of sociology 92*, 5 (1987), 1170–1182.

[6] Bramoullé, Y., and Kranton, R. Games played on networks.

[7] Bramoullé, Y., Kranton, R., and D'amours, M. Strategic interaction and networks. *American Economic Review 104*, 3 (2014), 898–930.

[8] Carli, R., Fagnani, F., Speranzon, A., and Zampieri, S. Communication constraints in the average consensus problem. *Automatica 44*, 3 (2008), 671–684.

[9] Carvalho, V. M. From micro to macro via production networks. *Journal of Economic Perspectives 28*, 4 (2014), 23–48.

[10] Dequiedt, V., and Zenou, Y. Local and consistent centrality measures in networks.

[11] Duchin, F., Steenge, A. E., et al. Mathematical models in input-output economics. *Rensselaer Polytechnic Institute, Troy, NY* (2007).

[12] Easley, D., and Kleinberg, J. *Networks, crowds, and markets: Reasoning about a highly connected world.* Cambridge University Press, 2010.

[13] Fagnani, F. Lecture notes on dynamics over networks, 2017.

[14] Galeotti, A., Golub, B., and Goyal, S. Targeting interventions in networks. *arXiv preprint arXiv:1710.06026* (2017).

[15] Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. Network games. *The review of economic studies 77*, 1 (2010),

218–244.

[16] GRIGORYAN, A. Lecture notes on analysis on graphs, 2009.

[17] JACKSON, M. O., AND ZENOU, Y. Games on networks. In *Handbook of game theory with economic applications*, vol. 4. Elsevier, 2015, pp. 95–163.

[18] KATZ, L. A new status index derived from sociometric analysis. *Psychometrika 18*, 1 (1953), 39–43.

[19] PETERSEN, K. B., PEDERSEN, M. S., ET AL. The matrix cookbook. *Technical University of Denmark 7*, 15 (2008), 510.

[20] SCHWEITZER, F., FAGIOLO, G., SORNETTE, D., VEGA-REDONDO, F., VESPIGNANI, A., AND WHITE, D. R. Economic networks: The new challenges. *science 325*, 5939 (2009), 422–425.

[21] TERRAS, A. *Fourier analysis on finite groups and applications*, vol. 43. Cambridge University Press, 1999.

[22] TREVISAN, L. Lecture notes on expansion, sparsest cut, and spectral graph theory, 2013.

[23] ZENOU, Y. Key players.