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Hybrid Antenna Measurement and Simulations

Relatore
prof. Giuseppe Vecchi

Correlatori:
dott. Giorgio Giordanengo
dott. Marco Righero

Lorenzo Ciorba
matricola: 233348

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A mamma, papà
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Abstract

The thesis analyzes a method [1] to determine the electric far-field radiated by electrically large antennas (or antennas placed over structures) using few measured near field samples of the electric field and numerically constructed expansion functions. Full test of an antenna can be complex and time consuming. When working with classical expansion functions, namely vector spherical harmonics for classical spherical range measurement, an estimate on the number N of sampling points one needs to acquire is given by

$$N = \frac{4\pi r^2}{(\frac{\lambda}{2})^2} \tag{1}$$

where r is the radius of the minimum sphere enclosing the radiating structure and λ the wavelength. When structures which are large in terms of wavelength are considered, as antennas placed on satellite or other platforms, this number is so large that makes the measurement easily impractical. The aim of this method is to use information about the antenna, the scattering structure and the far-field of the antenna in isolation to drastically reduce the number of sampling points needed to determine the radiated far-field of the antenna mounted on the platform. The classic reconstruction method consists in measuring spherical near field samples and then expand them through vector spherical wave functions [6], i.e. a set of basis functions that is general and can be used to reconstruct the electric field of any radiating structure. This method, instead, employs a numerical basis adapted to the structure under test. More precisely, the antenna is enclosed in a virtual surface B, where unknown electric and magnetic currents are placed. We determine these currents imposing matching between the field radiated by the currents and the near-field samples acquired, and then use them to evaluate the far-field. The radiation is computed with a Green's function that takes into account the presence of the platform where the antenna is mounted. From a computational viewpoint, we construct triangular meshes of the surface B and of the platform; we express the unknown electric and magnetic currents defined on B as a linear combination of piecewise linear basis functions with limited support; we numerically compute the field radiated by each of these elementary basis function; we enforce matching between the measured samples of the field and the field due to a linear combination of the fields radiated by each basis function. Once appropriate coefficients are determined, we know the currents defined over B and are then able to evaluate the field radiated by these currents in any point of the space outside B. In particular, we are able to evaluate the far-field. The method requires a large computational effort to build the numerical basis, but there are two main advantages in the method. The economic cost of the simulations is much lower than the cost of acquiring many samples of the field radiated by the antenna; once computed, we can reuse the fields due to basis functions to express the field radiated by any antenna contained in the surface B working at the considered frequency, thus tests of different antennas placed over the same structure become particularly fast.
We show results of reconstructions of simulated electric fields of

- a dipole over a plane mock-up at frequency 3 GHz,

- a reflector antenna at frequency 8 GHz,

while in the last chapter we show results of a reconstruction with measured near field samples of the same reflector antenna.

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Contents

Chapter 1

A brief introduction

In this first chapter we want to introduce relations that involve electric and magnetic currents defined over a surface and electric and magnetic fields they produce.

1.1 Maxwell's equations

We recall Maxwell's equations in vacuum and in free space without sources (electric/magnetic currents) for electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$

$$\text{curl } \mathcal{E}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t} \mathcal{H}(\mathbf{r}, t) \tag{1.1}$$

$$\text{curl } \mathcal{H}(\mathbf{r}, t) = \epsilon \frac{\partial}{\partial t} \mathcal{E}(\mathbf{r}, t) \tag{1.2}$$

We list the main electromagnetic constants that we will use:

- $\epsilon \simeq 8.85410^{-12} F/m$ is the permittivity of the vacuum;
- $\mu \simeq 4\pi 10^{-7} H/m$ is the permeability of the vacuum;
- $k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$ is the wavenumber; $[k] = \frac{1}{m}$
- $\lambda = \frac{c}{f}$ is the wavelength and c is the speed of light $c \simeq 3 \cdot 10^8 \frac{m}{s}$; $[\lambda] = m$
- $f = \frac{\omega}{2\pi}$ is the frequency; $[f]$ =Hz=$\frac{1}{s}$
- $\eta = \sqrt{\mu/\epsilon}$ is the impedance.

We are interested to find a stationary solution of Maxwell's equations so we will assume that each quantity is a periodic function of time and that it can be expressed as

$$\mathcal{E}(\mathbf{r}, t) = \Re(\mathbf{E}(\mathbf{r})e^{j\omega t}) = \Re(\mathbf{E}(\mathbf{r}))\cos(\omega t) - \Im(\mathbf{E}(\mathbf{r}))\sin(\omega t) \tag{1.3}$$

where $j = \sqrt{-1}$ is the imaginary unit and $\mathbf{E} : \mathbb{R}^3 \longrightarrow \mathbb{C}^3$ a time independent function. We note that fixed an $\mathbf{r} \in \mathbb{R}^3$ the function $t \longmapsto \mathcal{E}(\mathbf{r}, t)$ is a curve: if $\Re(\mathbf{E}(\mathbf{r})) \times \Im(\mathbf{E}(\mathbf{r})) = 0$, i.e. one of the vectors is zero or vectors are linearly dependent, then the curve becomes a line and the electric field is said to be linear polarized; instead if $||\Re(\mathbf{E}(\mathbf{r}))|| = ||\Im(\mathbf{E}(\mathbf{r}))||$ and $\Re(\mathbf{E}(\mathbf{r}))\cdot\Im(\mathbf{E}(\mathbf{r})) = 0$ then the field is said to be circularly polarized; in the other cases the field is said to be elliptically polarized. Through this hypotesis Maxwell's equations become time independent relations over space

$$\text{curl }\mathbf{E} = -j\omega\mu\mathbf{H} \tag{1.4}$$

$$\text{curl }\mathbf{H} = j\omega\epsilon\mathbf{E} \tag{1.5}$$

with the little disadvantage that new fields are complex valued functions. We will use these equations in the exterior space of a body that we will model as a PEC (perfect electric conductor), i.e. a material with infinite conductivity or equivalently zero resistivity. It is known that in the region D enclosed by the PEC, the eletric and magnetic fields vanish identically. In this first chapter we will assume that $D \subseteq \mathbb{R}^3$ is a bounded, connected open set with C^2-smooth boundary ∂D. Given $\mathbf{x} \in \mathbb{R}^3$ we denote by $|\mathbf{x}|$ its norm $|\mathbf{x}| = x_1^2 + x_2^2 + x_3^2$. In electromagnetism, given $k > 0$, the function $G : \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \text{ s.t. } \mathbf{x} \neq \mathbf{y}\} \longrightarrow \mathbb{C}$

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{-jk|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \quad \mathbf{x} \neq \mathbf{y} \tag{1.6}$$

plays a fundamental role and is called Green's function. It can be checked that

$$\nabla_x G(\mathbf{x}, \mathbf{y}) = -\frac{e^{-jk|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|^3}(1 + jk|\mathbf{x}-\mathbf{y}|)\Big(\mathbf{x}-\mathbf{y}\Big) \quad \mathbf{x} \neq \mathbf{y} \tag{1.7}$$

The function G appears in several integral operators that we will use, so now we recall a lemma about the integrability of $G(\mathbf{x}, \cdot)$.

Lemma 1.1.1. *[2] Let $G : \{(\mathbf{x}, \mathbf{y}) \in \partial D \times \partial D \text{ s.t. } \mathbf{x} \neq \mathbf{y}\} \longrightarrow \mathbb{C}$ the Green's function defined in 1.6. Then $\int_{\partial D} G(\mathbf{x}, \mathbf{y}) ds(\mathbf{y})$ exists $\forall \mathbf{x} \in \partial D$ and there exists $c > 0$ such that*

$$\int_{\partial D \setminus B(x,\tau)} |G(\mathbf{x}, \mathbf{y})| ds(\mathbf{y}) \leq c \quad \forall \mathbf{x} \in \partial D \quad \forall \tau > 0 \tag{1.8}$$

and

$$\Big| \int_{\partial D \setminus B(x,\tau)} \nabla_x G(\mathbf{x}, \mathbf{y}) ds(\mathbf{y}) \Big| \leq c \quad \forall \mathbf{x} \in \partial D \quad \forall \tau > 0 \tag{1.9}$$

We denote by $\mathbf{n}$ the unit normal vector to the surface ∂D, by $C^1(D)$ the set of functions $u : D \longrightarrow \mathbb{C}$ such that u is differentiable in D, by $C^1(D, \mathbb{C}^3)$ the set

$$C^1(D, \mathbb{C}^3) = \{u : D \longrightarrow \mathbb{C}^3 : \ u_j \in C^1(D) \text{ for } j = 1,2,3\} \tag{1.10}$$

Theorem 1.1.1 (Stratton-Chu formula). *[2] Let $\epsilon, \mu, \omega > 0$ real constants and $\mathbf{E}^+, \mathbf{H}^+ \in C^1(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3) \cap C(\mathbb{R}^3 \setminus D, \mathbb{C}^3)$ solutions of Maxwell's equations*

$$\textit{curl } \mathbf{E}^+ + j\omega\mu\mathbf{H}^+ = 0 \quad \textit{curl } \mathbf{H}^+ - j\omega\epsilon\mathbf{E}^+ = 0 \quad \textit{in } \mathbb{R}^3 \setminus \overline{D} \tag{1.11}$$

which satisfy also the Silver Muller radiation condition

$$\sqrt{\epsilon}\mathbf{E}^+(\mathbf{x}) - \sqrt{\mu}\mathbf{H}^+(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} = O(\frac{1}{|\mathbf{x}|^2}) \tag{1.12}$$

uniformly with respect to $\mathbf{x}/|\mathbf{x}| \in S^2$. *Let* $\mathbf{E}^-, \mathbf{H}^- \in C^1(D, \mathbb{C}^3) \cap C(\overline{D}, \mathbb{C}^3)$ *solutions of Maxwell's equations*

$$curl\, \mathbf{E}^- + j\omega\mu\mathbf{H}^- = 0 \quad curl\, \mathbf{H}^- - j\omega\epsilon\mathbf{E}^- = 0 \quad in\ D \tag{1.13}$$

Then

$$\begin{aligned} curl \int_{\partial D} G(\mathbf{x},\mathbf{y})\mathbf{n}(\mathbf{y}) \times [\mathbf{E}^+(\mathbf{y}) - \mathbf{E}^-(\mathbf{y})]ds(\mathbf{y}) + \\ + \frac{j}{\omega\epsilon}\nabla \int_{\partial D} G(\mathbf{x},\mathbf{y})\mathrm{Div}[\mathbf{n} \times (\mathbf{H}^- - \mathbf{H}^+)](\mathbf{y})ds(\mathbf{y}) + \\ + j\omega\mu \int_{\partial D} G(\mathbf{x},\mathbf{y})\mathbf{n}(\mathbf{y}) \times [\mathbf{H}^-(\mathbf{y}) - \mathbf{H}^+(\mathbf{y})]ds(\mathbf{y}) = \\ = \begin{cases} \mathbf{E}^+(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D} \\ \mathbf{E}^-(\mathbf{x}) & \mathbf{x} \in D \end{cases} \end{aligned}$$

where Div *denotes the surface divergence and*

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}e^{-jk|\mathbf{x}-\mathbf{y}|} \quad \mathbf{x} \neq \mathbf{y} \tag{1.14}$$

is the Green's function.

For an $\mathbf{x} \in \partial D$ we denote by

$$\lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{E}(\mathbf{x}^+) \tag{1.15}$$

the one-sided limit of $\mathbf{E}$ as $\mathbf{x}^+ \in \mathbb{R}^3 \setminus \overline{D}$ approaches $\mathbf{x} \in \partial D$ from the exterior region. Similarly we denote by $\lim_{\mathbf{x}^- \to \mathbf{x}} \mathbf{E}(\mathbf{x}^-)$ the one-sided limit of $\mathbf{E}$ as $\mathbf{x}^- \in D$ approaches $\mathbf{x} \in \partial D$ from the interior region. Given fields $\mathbf{E}, \mathbf{H}$ solutions of Maxwell's equations in $\mathbb{R}^3 \setminus \partial D$ the electric and magnetic currents $\mathbf{J}, \mathbf{M}$ on ∂D are defined as

$$\mathbf{J}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times \left[\lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{H}(\mathbf{x}^+) - \lim_{\mathbf{x}^- \to \mathbf{x}} \mathbf{H}(\mathbf{x}^-)\right] \ \mathbf{x} \in \partial D \tag{1.16}$$

$$\mathbf{M}(\mathbf{x}) = \left[\lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{E}(\mathbf{x}^+) - \lim_{\mathbf{x}^- \to \mathbf{x}} \mathbf{E}(\mathbf{x}^-)\right] \times \mathbf{n}(\mathbf{x}) \ \ \mathbf{x} \in \partial D \tag{1.17}$$

We can rewrite Stratton-Chu formula in the following form:

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi - \frac{1}{\epsilon}\nabla \times \mathbf{F} \tag{1.18}$$

where we defined for $\mathbf{r} \in \mathbb{R}^3 \setminus \partial D$

$$\mathbf{A}(\mathbf{r}) = \mu \int_{\partial D} \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \tag{1.19}$$

$$\mathbf{F}(\mathbf{r}) = \epsilon \int_{\partial D} \mathbf{M}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \tag{1.20}$$

$$\phi(\mathbf{r}) = \frac{j}{\epsilon\omega} \int_{\partial D} \mathrm{Div}\ \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \tag{1.21}$$

1.2 The scattering problem

We now analyze a problem that will arise in next chapters: the scattering problem for a PEC. Given incident fields $\mathbf{E}^{inc}, \mathbf{H}^{inc}$ that satisfy Maxwell's equations

$$curl\mathbf{E}^{inc} + j\omega\mu\mathbf{H}^{inc} = 0 \quad curl\mathbf{H}^{inc} - j\omega\epsilon\mathbf{E}^{inc} = 0 \quad \text{in } \mathbb{R}^3 \tag{1.22}$$

find the total fields $\mathbf{E}^{tot}, \mathbf{H}^{tot} \in C^1(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3) \cap C(\mathbb{R}^3 \setminus D, \mathbb{C}^3)$ that satisfy

$$curl\mathbf{E}^{tot} + j\omega\mu\mathbf{H}^{tot} = 0 \;\; curl\mathbf{H}^{tot} - j\omega\epsilon\mathbf{E}^{tot} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{1.23}$$

$$\mathbf{H}^{tot} = \mathbf{E}^{tot} = 0 \quad \text{in } D \tag{1.24}$$

with boundary condition for $\mathbf{E}^{tot}$

$$\mathbf{n}(\mathbf{x}) \times \lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{E}^{tot}(\mathbf{x}^+) = 0 \;\; \forall \mathbf{x} \in \partial D \tag{1.25}$$

and the scattered fields $\mathbf{E}^s := \mathbf{E}^{tot} - \mathbf{E}^{inc}$, $\mathbf{H}^s := \mathbf{H}^{tot} - \mathbf{H}^{inc}$ satisfy the Silver Muller radiation condition

$$\sqrt{\epsilon}\mathbf{E}^s(\mathbf{x}) - \sqrt{\mu}\mathbf{H}^s(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} = O(\frac{1}{|\mathbf{x}|^2}) \tag{1.26}$$

uniformly with respect to $\mathbf{x}/|\mathbf{x}| \in S^2$.

Because total and incident fields satisfy Maxwell's equations in $\mathbb{R}^3 \setminus \partial D$ then also scattered fields $\mathbf{E}^s, \mathbf{H}^s$ are solutions of them. It is clear that the solution in D for scattered fields is $\mathbf{E}^s = -\mathbf{E}^{inc}$ and $\mathbf{H}^s = -\mathbf{H}^{inc}$. The scattering problem is part of the following exterior boundary value problem.

Exterior boundary value problem

Given a vector valued function such that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0 \; \forall \mathbf{x} \in \partial D$, i.e. a tangential field, find $\mathbf{E}, \mathbf{H} \in C^1(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3) \cap C(\mathbb{R}^3 \setminus D, \mathbb{C}^3)$ that satisfy

$$curl\mathbf{E} + j\omega\mu\mathbf{H} = 0 \;\; curl\mathbf{H} - j\omega\epsilon\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{1.27}$$

$$\mathbf{n}(\mathbf{x}) \times \lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{E}(\mathbf{x}^+) = \mathbf{f}(\mathbf{x}) \;\; \forall \mathbf{x} \in \partial D \tag{1.28}$$

$$\sqrt{\epsilon}\mathbf{E}(\mathbf{x}) - \sqrt{\mu}\mathbf{H}(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} = O(\frac{1}{|\mathbf{x}|^2}) \tag{1.29}$$

uniformly with respect to $\mathbf{x}/|\mathbf{x}| \in S^2$. Last condition is called Silver Muller radiation condition.

Scattering problem

We will consider the Exterior boundary value problem for scattered fields $\mathbf{E}^s, \mathbf{H}^s$ with $\mathbf{f} = -\mathbf{n} \times \mathbf{E}^{inc}$. We can apply Stratton-Chu formula to $\mathbf{E}^s$ and define currents $\mathbf{J}$ and $\mathbf{M}$ with respet to the tangential jump of $\mathbf{E}^s$ and $\mathbf{H}^s$ across the surface. Because $\mathbf{E}^{inc}$ is continuous then $\mathbf{E}^{tot}$ and $\mathbf{E}^s$ have the same jump of the tangential component across the surface, thus

$$\mathbf{M}(\mathbf{x}) = \left[\lim_{\mathbf{x}^+ \to \mathbf{x}} \mathbf{E}^s(\mathbf{x}^+) - \lim_{\mathbf{x}^- \to \mathbf{x}} \mathbf{E}^s(\mathbf{x}^-) \right] \times \mathbf{n}(\mathbf{x}) = \mathbf{E}^{tot}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial D \quad (1.30)$$

The electric scattered field $\mathbf{E}^s$ is linked with the current $\mathbf{J}$ through

$$\mathbf{n} \times \mathbf{E}^s = -j\omega \mathbf{n} \times \mathbf{A} - \mathbf{n} \times \nabla \phi \quad (1.31)$$

now applying the boundary condition $\mathbf{n} \times (\mathbf{E}^{inc} + \mathbf{E}^s) = 0$ we obtain that

$$-\mathbf{n}(\mathbf{r}) \times \mathbf{E}^{inc}(\mathbf{r}) = -j\omega \mathbf{n}(\mathbf{r}) \times \mathbf{A}(\mathbf{r}) - \mathbf{n}(\mathbf{r}) \times \nabla \phi(\mathbf{r}) \quad \forall \mathbf{r} \in S \quad (1.32)$$

We define the operator $\mathcal{L}$ as

$$\mathcal{L}(\mathbf{J}) = j\omega \mathbf{n} \times \mathbf{A} + \mathbf{n} \times \nabla \phi \quad (1.33)$$

or explicitly $\forall \mathbf{r} \in S$

$$\mathcal{L}(\mathbf{J})(\mathbf{r}) = j\omega\mu \mathbf{n}(\mathbf{r}) \times \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') ds(\mathbf{r}') + \frac{j}{\epsilon\omega} \mathbf{n}(\mathbf{r}) \times \nabla \int_S G(\mathbf{r}, \mathbf{r}') \text{ Div } \mathbf{J}(\mathbf{r}') ds(\mathbf{r}') \quad (1.34)$$

we obtain the following integral equation $\forall \mathbf{r} \in S$

$$(\mathbf{n} \times \mathbf{E}^{inc})(\mathbf{r}) = j\omega\mu \mathbf{n}(\mathbf{r}) \times \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') ds(\mathbf{r}') + \frac{j}{\epsilon\omega} \mathbf{n}(\mathbf{r}) \times \nabla \int_S G(\mathbf{r}, \mathbf{r}') \text{ Div } \mathbf{J}(\mathbf{r}') ds(\mathbf{r}') \quad (1.35)$$

We conclude this section with a little remark. There are two possible boundary conditions for EFIE [19]. The first is the one that we use and is $\mathbf{n} \times \mathbf{E}^{tot} = 0$ and is called N-EFIE while the second one is $\mathbf{n} \times \mathbf{n} \times \mathbf{E}^{tot} = 0$ and is called T-EFIE. These two formulations are mathematically equivalent but obviusly their implementations are different and one method, if it produces an ill-conditioned matrix, could be worse then the other one.

1.3 Far field

It is known that the electric field $\mathbf{E}$ generated by an electric current $\mathbf{J}$ and/or a magnetic current $\mathbf{M}$ takes a simpler form in the case we compute $\mathbf{E}$ in a region far from the sources $\mathbf{J}$ and $\mathbf{M}$. We note that multiplying the Silver Muller radiation condition

$$\sqrt{\epsilon}\mathbf{E}(\mathbf{x}) - \sqrt{\mu}\mathbf{H}(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} = O(\frac{1}{|\mathbf{x}|^2}) \quad (1.36)$$

for the unit normal vector to S^2 we have that the radial part of $\mathbf{E}(\mathbf{x})$ is

$$\mathbf{E}(\mathbf{x}) \cdot \frac{\mathbf{x}}{|\mathbf{x}|} = O(\frac{1}{|\mathbf{x}|^2}) \quad (1.37)$$

uniformly with respect to $\mathbf{x}/|\mathbf{x}| \in S^2$, while the tangential components have a slower decay. More precisely we consider a surface S and an electric current $\mathbf{J}$ and a magnetic current

$\mathbf{M}$ defined over S. We denote with D the diameter of the smallest sphere that encloses S. Using spherical coordinates (r, θ, ϕ), the electric field can be decomposed as

$$\mathbf{E}(r, \theta, \phi) = \mathbf{E}^{\infty}(r, \theta, \phi) + o\left(\frac{1}{r}\right) \quad r \to +\infty \tag{1.38}$$

If $r \gg \lambda$ and $r \geq 2D^2/\lambda$ we can approximate the electric field $\mathbf{E}(r, \theta, \phi) \simeq \mathbf{E}^{\infty}(r, \theta, \phi)$ with the far field (for simplicity of notation we will still use the symbol "=" instead of "$\simeq$"):

$$\begin{aligned}\mathbf{E}(r, \theta, \phi) = -j\omega\Big[\hat{\theta}(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi)\Big]\hat{\theta}(\theta, \phi) - j\omega\Big[\hat{\phi}(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi)\Big]\hat{\phi}(\theta, \phi) + \\ + j\omega\eta\Big[\hat{\theta}(\theta, \phi) \cdot \mathbf{F}(r, \theta, \phi)\Big]\hat{\phi}(\theta, \phi) - j\omega\eta\Big[\hat{\phi}(\theta, \phi) \cdot \mathbf{F}(r, \theta, \phi)\Big]\hat{\theta}(\theta, \phi)\end{aligned}$$

or briefly

$$\mathbf{E} = -j\omega(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}) \cdot \mathbf{A} + j\omega\eta(\hat{\phi}\hat{\theta} - \hat{\theta}\hat{\phi}) \cdot \mathbf{F} \tag{1.39}$$

where $\eta = \sqrt{\mu/\epsilon}$ and

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{3} x_i y_i \tag{1.40}$$

and

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi r} e^{-jkr} \int_S \mathbf{J}(\mathbf{r'}) e^{jk\hat{r}(\theta,\phi)\cdot\mathbf{r'}} ds(\mathbf{r'}) \tag{1.41}$$

$$\mathbf{F}(r, \theta, \phi) = \frac{\epsilon}{4\pi r} e^{-jkr} \int_S \mathbf{M}(\mathbf{r'}) e^{jk\hat{r}(\theta,\phi)\cdot\mathbf{r'}} ds(\mathbf{r'}) \tag{1.42}$$

where

$$\hat{r}(\theta, \phi) = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)) \tag{1.43}$$

is the normal unit vector of the sphere and

$$\hat{\theta}(\theta, \phi) = (\cos(\phi)\cos(\theta), \sin(\phi)\cos(\theta), -\sin(\theta)) \tag{1.44}$$

$$\hat{\phi}(\theta, \phi) = (-\sin(\phi), \cos(\phi), 0) \tag{1.45}$$

are the tangential vectors to the sphere.

Chapter 2

The scattering problem

In this chapter we introduce the ambient space for the solution of the scattering problem and then we report a theorem on the existence and uniqueness of the solution [2]. We don't introduce exhaustively notions and instruments needed for the theorem but we recall only the fundamental results. For this chapter we decompose the electric field $\mathcal{E}$ as

$$\mathcal{E}(\mathbf{r}, t) = \Re(\mathbf{E}(\mathbf{r})e^{-j\omega t}) \tag{2.1}$$

instead of

$$\mathcal{E}(\mathbf{r}, t) = \Re(\mathbf{E}(\mathbf{r})e^{j\omega t}) \tag{2.2}$$

We note that if

$$\mathcal{E}(\mathbf{r}, t) = \Re(\mathbf{E}_2(\mathbf{r})e^{-j\omega t}) = \Re(\mathbf{E}(\mathbf{r}))\cos(\omega t) + \Im(\mathbf{E}(\mathbf{r}))\sin(\omega t) \tag{2.3}$$

$$\mathcal{E}(\mathbf{r}, t) = \Re(\mathbf{E}_1(\mathbf{r})e^{j\omega t}) = \Re(\mathbf{E}(\mathbf{r}))\cos(\omega t) - \Im(\mathbf{E}(\mathbf{r}))\sin(\omega t) \tag{2.4}$$

then the two different solutions are linked through

$$\mathbf{E}_2 = \overline{\mathbf{E}}_1 \tag{2.5}$$

where $\overline{\mathbf{E}}_1$ is the complex coniugate of $\mathbf{E}_1$. Green function G is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}e^{jk|\mathbf{r}-\mathbf{r}'|} \tag{2.6}$$

with $k = \omega\sqrt{\epsilon\mu}$. The scattering equation is $\forall \mathbf{r} \in S$

$$j\omega\mu\left[\mathbf{n}(\mathbf{r})\times\int_{\partial D}\mathbf{J}(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')ds(\mathbf{r}')\right] + \frac{j}{\epsilon\omega}\left[\mathbf{n}(\mathbf{r})\times\nabla\int_{\partial D}\mathrm{Div}\ \mathbf{J}(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')ds(\mathbf{r}')\right] = -\mathbf{n}(\mathbf{r})\times\mathbf{E}^{inc}(\mathbf{r}) \tag{2.7}$$

where $\mathbf{n}$ is the normal unit vector to the surface S. Now defining the vector function

$$\mathbf{a} := \frac{j}{\epsilon\omega}\mathbf{J} \tag{2.8}$$

the scattering equation becomes

$$k^2\left[\mathbf{n}(\mathbf{r})\times\int_{\partial D}\mathbf{a}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}')\right]+\left[\mathbf{n}(\mathbf{r})\times\nabla\int_{\partial D}\text{ Div }\mathbf{a}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}')\right]=-\mathbf{n}(\mathbf{r})\times\mathbf{E}^{inc}(\mathbf{r}) \quad (2.9)$$

Now defining the trace operator γ_t such that $\gamma_t(\mathbf{a}) = \mathbf{n}\times\mathbf{a}$ and introducing the symbolic notation

$$< a, b >_{\partial D}= \int_{\partial D} b(y)a(y)ds(y) \quad (2.10)$$

where a is a scalar or vectorial function and b is a scalar function; the equation becomes

$$\mathcal{L}\mathbf{a} = -\gamma_t\mathbf{E}^{inc} \quad (2.11)$$

where $\tilde{\mathcal{L}}$ is

$$\tilde{\mathcal{L}}\mathbf{a}(\mathbf{r}) = k^2 < \mathbf{a}, G(\mathbf{r},\cdot) >_{\partial D} +\nabla < Div\ \mathbf{a}, G(\mathbf{r},\cdot) >_{\partial D} \quad (2.12)$$

and $\mathcal{L} := \gamma_t\tilde{\mathcal{L}}$. Our aim is to study the existence and uniqueness of the equation $\mathcal{L}\mathbf{a} = \mathbf{f}$ for some vectorial function $\mathbf{f}$ which is tangential to the surface ∂D. In the following we will denote vectors without bold text.

2.1 Preliminaries

We don't give the definition of fractional Sobolev space, we will read it as image space of a trace operator. For proofs, definitions and details we refer to [2]. In this chapter we suppose that $D \subset \mathbb{R}^3$ is a bounded Lipschitz domain. We denote with $C^k(\overline{D})$ the set of functions $u : \overline{D} \longrightarrow \mathbb{C}$ such that u is k times differentiable in D and all derivatives can be continuosly extended to $\overline{D}$. We also define

$$C^k(\overline{D},\mathbb{C}^3) = \{u : \overline{D} \longrightarrow \mathbb{C}^3 \mid u_j \in C^k(\overline{D}) \text{ for } j = 1,2,3\} \quad (2.13)$$

$$C^k_0(\overline{D}) = \{u \in C^k(\overline{D}) : \text{ supp}(u) \text{ is compact and supp}(u) \subseteq D\} \quad (2.14)$$

$$C^k_0(D,\mathbb{C}^3) = \{u \in C^k(\overline{D},\mathbb{C}^3) : u_j \in C^k_0(D) \text{ for } j = 1,2,3\} \quad (2.15)$$

$$L^p(D) = \{u : D \longrightarrow \mathbb{C} \ : u \text{ is Lebesgue measurable and } \int_D |u|^p dx < \infty\} \quad (2.16)$$

$$L^p(D,\mathbb{C}^3) = \{u : D \longrightarrow \mathbb{C}^3 : \ u_j \in L^p(D) \text{ for } j = 1,2,3\} \quad (2.17)$$

where the support of a measurable function u is defined as

$$\text{supp}(u) = \cap\{K \subseteq \overline{D} : K \text{ is closed and } u(x) = 0 \text{ a.e. on } D \setminus K\} \quad (2.18)$$

Definition 2.1. *A function $v \in L^2(D,\mathbb{C}^3)$ has a variational gradient if there exists $w \in L^2(D,\mathbb{C}^3)$ such that*

$$\int_D v\cdot\nabla\psi dx = -\int_D w\cdot\psi dx \quad \forall\psi \in C^\infty_0(D) \quad (2.19)$$

In this case we define $\nabla v := w$

Definition 2.2. *A function $v \in L^2(D, \mathbb{C}^3)$ has a variational curl if there exists $w \in L^2(D, \mathbb{C}^3)$ such that*

$$\int_D v \cdot \; curl\, \psi dx = \int_D w \cdot \psi dx \quad \forall \psi \in C_0^\infty(D, \mathbb{C}^3) \tag{2.20}$$

In this case we define curl $v := w$

$$H^1(D) := \{u \in L^2(D) : u \text{ has a variational gradient } \nabla u \in L^2(D, \mathbb{C}^3)\} \tag{2.21}$$

$$H(curl, D) := \{u \in L^2(D) : u \text{ has a variational curl in } L^2(D, \mathbb{C}^3)\} \tag{2.22}$$

The scalar product on $H^1(D)$ is defined as

$$(u, v)_{H^1(D)} := (u, v)_{L^2(D)} + (\nabla u, \nabla v)_{L^2(D)} = \int_D [u\overline{v} + \nabla u \nabla \overline{v}] dx \tag{2.23}$$

The scalar product on $H(curl, D)$ is defined as

$$(u, v)_{H(curl,D)} := (u, v)_{L^2(D)} + (\text{ curl } u, \text{ curl } v)_{L^2(D)} \tag{2.24}$$

It can be proved that $H^1(D)$ and $H(curl, D)$ with their inner products are Hilbert spaces [2]. We define the space $H_0(curl, D)$ as the closure of the space $C_0^\infty(D, \mathbb{C}^3)$ in $H(curl, D)$ with respect to the scalar product previously defined.

Proposition 2.1.1. *[2] The trace operators $\gamma_0 : H^1(D) \longrightarrow H^{1/2}(\partial D)$ such that $\gamma_0 u = u|_{\partial D}$ and $\gamma_t : H(curl, D) \longrightarrow H^{-1/2}(Div, \partial D)$ such that $\gamma_t u = n \times u|_{\partial D}$ and $\gamma_T : H(curl, D) \longrightarrow H^{-1/2}(Curl, \partial D)$ such that $\gamma_T u = (n \times u|_{\partial D}) \times n$ are well defined and bounded. Furthermore it holds that $Ker(\gamma_t) = Ker(\gamma_T) = H_0(curl, D)$.*

We recall that the normal vector $n(x)$ exists for almost all $x \in \partial D$ because D is a Lipschitz domain.

Now we have to give a sense to the bilinear form $< \cdot, \cdot >_{\partial D}$.

2.1.1 Definition of $< \cdot, \cdot >_{\partial D}$

The space $H^{-1/2}(Div, \partial D)$ can be read as the space image of the trace operator γ_t or as a subspace of $H^{-1/2}(\partial D, \mathbb{C}^3) = \left(H^{1/2}(\partial D, \mathbb{C}^3)\right)'$. Similarly $H^{-1/2}(Curl, \partial D)$ is the image space of the trace operator γ_T or a subspace of $H^{-1/2}(\partial D, \mathbb{C}^3) = \left(H^{1/2}(\partial D, \mathbb{C}^3)\right)'$.
We define for $\psi \in H^{-1/2}(Curl, \partial D)$

$$< a, \psi >_* = \int_D \tilde{\psi} \cdot curl \tilde{a} - \tilde{a} \cdot curl \tilde{\psi} dx \tag{2.25}$$

where $\tilde{\psi}, \tilde{a} \in H(Curl, D)$ are any functions such that $\psi = \gamma_T \tilde{\psi}$ and $a = \gamma_t \tilde{a}$; the right hand side does not depend on the choice of extensions $\tilde{\psi}, \tilde{a}$.

We recall that $\gamma_0 : H^1(D) \longrightarrow H^{1/2}(\partial D)$, with abuse of notation we denote the vector trace operator still with $\gamma_0 : H^1(D, \mathbb{C}^3) \longrightarrow H^{1/2}(\partial D, \mathbb{C}^3)$ such that $f \overset{\gamma_0}{\longmapsto} f|_{\partial D}$. It can be proved that $H^1(D, \mathbb{C}^3)$ is boundedly embedded in $H(curl, D)$, thus γ_T is well defined and continuous on $H^1(D, \mathbb{C}^3)$.

Theorem 2.1.1. *[2] The space $H^{-1/2}(Div, \partial D)$ is contained in $H^{-1/2}(\partial D, \mathbb{C}^3)$. The identification is given by the application $a \longmapsto l_a$ with $a \in H^{-1/2}(Div, \partial D)$ and $l_a \in H^{-1/2}(\partial D, \mathbb{C}^3)$ given by*

$$< l_a, \psi >=< a, \gamma_T \tilde{\psi} >_* \quad \psi \in H^{1/2}(\partial D, \mathbb{C}^3) \tag{2.26}$$

with $< \cdot, \cdot >_*$ as in 2.25 and $\tilde{\psi} \in H^1(D, \mathbb{C}^3)$ is any extension of ψ such that $\gamma_0 \tilde{\psi} = \psi$.

In the next step we define $< a, \psi >_{\partial D}$ for a scalar function ψ. We define componentwise $< \cdot, \cdot >_{\partial D}$: $H^{-1/2}(Div, \partial D) \times H^{1/2}(\partial D) \longrightarrow \mathbb{C}^3$ as

$$(< a, \psi >_{\partial D})_k =< l_a, \psi e_k >=< a, \gamma_T(\tilde{\psi} e_k) >_* \quad k = 1,2,3 \tag{2.27}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis $e_1 = (1,0,0)'$ $e_2 = (0,1,0)'$ $e_3 = (0,0,1)'$ of $\mathbb{C}^3$ and $\tilde{\psi} \in H^1(D)$ is any extension of ψ such that $\gamma_0 \tilde{\psi} = \psi$. For smooth functions it holds that

$$< a, \psi >_{\partial D}= \int_{\partial D} a(y)\psi(y)ds(y) \tag{2.28}$$

It can be showed using the following theorem with $A = \tilde{a}$ and $B = e_j \tilde{\psi}$ for $j = 1,2,3$.

Theorem 2.1.2. *[2] Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $A, B \in C^1(D, \mathbb{C}^3) \cap C(\overline{D}, \mathbb{C}^3)$. Then*

$$\int_D (B \cdot curlA - A \cdot curlB)dx = \int_{\partial D} (n \times A) \cdot Bds \tag{2.29}$$

We consider an extension $\tilde{\psi}$ of ψ such that $\gamma_0 \tilde{\psi} = \psi$ and a function $\tilde{a}$ defined in $\overline{D}$ such that $\gamma_t \tilde{a} = a$, for $j = 1,2,3$ we have

$$(< a, \psi >_{\partial D})_j =< a, \gamma_T(e_j \tilde{\psi}) >_*= \int_D \tilde{\psi} e_j \cdot curl\tilde{a} - \tilde{a} \cdot curl(\tilde{\psi} e_j) =$$
$$= \int_{\partial D} \psi(n \times \tilde{a}|_{\partial D}) \cdot e_j ds = \int_{\partial D} \psi a \cdot e_j ds = e_j \cdot \int_{\partial D} \psi a \, ds$$

2.1.2 Surface divergence

We recall that the projection onto the tangent plane with unit normal vector n of a vector $f \in \mathbb{R}^3$ is $f - (f \cdot n)n = n \times (f \times n) = (n \times f) \times n$. Let $D \subset \mathbb{R}^3$ a C^2-smooth domain with boundary ∂D with parametrizations $\{\Psi_j,\ j = 1, ..., m\}$ and partition of unity $\{\phi_j, j = 1, ..., m\}$. We define

$$C^1(\partial D) = \{f \in C(\partial D) : (\phi_j f) \circ \Psi \in C^1 \text{ for } j = 1, ..., m\} \tag{2.30}$$

$$C^1(\partial D, \mathbb{C}^3) = \{F \in C(\partial D, \mathbb{C}^3) : F_j \in C^1(\partial D) \text{ for } j = 1,2,3\} \tag{2.31}$$

$$C_t(\partial D) = \{F \in C(\partial D, \mathbb{C}^3) : F \cdot n = 0\} \tag{2.32}$$

$$C_t^1(\partial D) = C_t(\partial D) \cap C^1(\partial D, \mathbb{C}^3) \tag{2.33}$$

Definition 2.3. *[2] Let $D \subset \mathbb{R}^3$ a C^2-smooth domain with boundary ∂D. Let $f \in C^1(\partial D)$, $F \in C^1_t(\partial D)$, U an open subset of $\mathbb{R}^3$ s.t. $\partial D \subset U$ and $\tilde{f} \in C^1(U)$ and $\tilde{F} \in C^1(U, \mathbb{C}^3)$ extensions of f and F respectively. The surface gradient of f is*

$$\text{Grad } f := n \times (\nabla \tilde{f} \times n) \quad \text{on } \partial D \tag{2.34}$$

The surface divergence of F is

$$\text{Div } F := div\ \tilde{F} - n \times (\tilde{F}'n) \quad \text{on } \partial D \tag{2.35}$$

where $n(x)$ is the exterior unit normal vector at $x \in \partial D$ and $\tilde{F}'(x) \in \mathbb{C}^{3\times 3}$ is the Jacobian matrix of $\tilde{F}$ at $x \in U$.

With the choice of $U = \mathbb{R}^3$ it can be proved that such extensions exist and definitions of surface gradient and divergence are independent of the extension [2].

Lemma 2.1.1. *[2] Let $D \subset \mathbb{R}^3$ a C^2-smooth domain with boundary ∂D and $f \in C^1(\partial D)$, $F \in C^1_t(\partial D)$ with extensions $\tilde{f} \in C^1(\mathbb{R}^3)$ and $\tilde{F} \in C^1(\mathbb{R}^3, \mathbb{C}^3)$ respectively. Then*

$$\int_{\partial D} f \text{ Div } F ds = -\int_{\partial D} F \cdot \text{ Grad } f ds \tag{2.36}$$

and

$$\int_{\partial D} \text{Div } F ds = 0 \tag{2.37}$$

Now we will give the definition of the variational form of the surface divergence inspired from the previous integral equality. It can be proved that for $\tilde{\psi} \in H^1(D)$ it holds that $\nabla\tilde{\psi} \in H(curl, D)$.

Definition 2.4. *[2] Let $a \in H^{-1/2}(Div, \partial D)$. The surface divergence* Div $a \in H^{-1/2}(\partial D)$ *is defined as the linear bounded functional*

$$< \text{ Div } a, \psi >_{\partial D} = - < a, \gamma_T \nabla \tilde{\psi} >_* \quad \psi \in H^{1/2}(\partial D) \tag{2.38}$$

where $\tilde{\psi} \in H^1(D)$ is an extension of ψ such that $\gamma_0 \tilde{\psi} = \psi$.

It can be showed that this definition is independent of the choice of the extension $\tilde{\psi}$. For smooth functions a, ψ we have that

$$< \text{ Div } a, \psi >_{\partial D} = \int_{\partial D} \psi \text{ Div } a\ ds \tag{2.39}$$

2.2 Existence and uniqueness

It is known that exist a set of values of k such that the scattering problem does not have unique solution, fortunately these values are only countable and do not form a continuous set.

Definition 2.5. *We say that k^2 is not an eigenvalue of $curl^2$ in D with respect to the boundary condition $n \times E = 0$ if the following problem*

$$\int_D [curlE \cdot curl\psi - k^2 E \cdot \psi] dx = 0 \quad \forall \psi \in H_0(curl, D) \qquad E \in H_0(curl, D) \tag{2.40}$$

has only the trivial solution $E = 0$.

Theorem 2.2.1. *[18] The problem: find $\lambda \neq 0$ such that exists a non-zero $E \in H_0(curl, D)$ such that*

$$\int_D [curlE \cdot curl\psi - \lambda E \cdot \psi] dx = 0 \quad \forall \psi \in H_0(curl, D) \tag{2.41}$$

has a countable set of real and positive eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$. Furthermore each eigenspace is finite dimesnional and all the eigenfunctions can be chosen to be real.

We now recall the exterior boundary value problem and then we state it in a variational formulation.

Exterior boundary value problem

Given a vector valued function such that $f(x) \cdot n(x) = 0 \ \forall x \in \partial D$ i.e. a tangential field, find E, H that satisfy

$$curlE - j\omega\mu H = 0 \qquad curlH + j\omega\epsilon E = 0 \tag{2.42}$$

$$n \times E = f \tag{2.43}$$

$$\sqrt{\epsilon}E(x) - \sqrt{\mu}H(x) \times \frac{x}{|x|} = O(\frac{1}{|x|^2}) \tag{2.44}$$

uniformly with respect to $x/|x| \in S^2$. Last condition is the Silver Muller condition. We define

$$H_{loc}(curl, \mathbb{R}^3 \setminus \overline{D}) := \{u : \mathbb{R}^3 \setminus \overline{D} \longrightarrow \mathbb{C}^3 : u|_B \in H(curl, B) \text{ for all balls } B\} \tag{2.45}$$

In the variational form for the field E the exterior boundary value problem becomes: given $f \in H^{-1/2}(Div, \partial D)$, find $E \in H_{loc}(curl, \mathbb{R}^3 \setminus \overline{D})$ such that

$$\int_{\mathbb{R}^3 \setminus \overline{D}} [curlE \cdot curl\psi - k^2 E \cdot \psi] dx = 0 \quad \forall \psi \in H_0(curl, \mathbb{R}^3 \setminus \overline{D}) \tag{2.46}$$

$$\gamma_t E = f \tag{2.47}$$

and E and H satisfy the S.M. condition.

The theorem

Let $Q \subseteq \mathbb{R}^3$ a bounded domain such that $\partial D \subseteq Q$. We define the operator $\tilde{\mathcal{L}}$ by $\forall x \in Q$

$$\tilde{\mathcal{L}}a(x) = \nabla < Div\ a, G(x,\cdot) >_{\partial D} + k^2 < a, G(x,\cdot) >_{\partial D} \quad \forall x \in Q \tag{2.48}$$

where

$$G(x,y) = \frac{e^{jk|x-y|}}{4\pi|x-y|} \tag{2.49}$$

This operator is well defined [2] as operator

$$\tilde{\mathcal{L}} : H^{-1/2}(Div, \partial D) \longrightarrow H(curl, Q) \tag{2.50}$$

Define

$$\mathcal{L} := \gamma_t \tilde{\mathcal{L}} \tag{2.51}$$

The operator $\mathcal{L}$ is well defined and bounded [2] as:

$$\mathcal{L} : H^{-1/2}(Div, \partial D) \longrightarrow H^{-1/2}(Div, \partial D) \tag{2.52}$$

We consider positive constants $\epsilon > 0$, $\mu > 0$,$\omega > 0$ and $k = \omega\sqrt{\mu\epsilon}$ and $f \in H^{-1/2}(Div, \partial D)$.

Theorem 2.2.2. *[2] Let $f \in H^{-\frac{1}{2}}(Div, \partial D)$. Assume that k^2 is not an eigenvalue of $curl^2$ in D. Then the exterior boundary value problem has a unique solution (E, H) and there exists $a \in H^{-1/2}(Div, \partial D)$ such that $\mathcal{L}a = f$ and $E = \tilde{\mathcal{L}}a$.*

We report also a theorem for smoother domains.

Theorem 2.2.3. *[10] Let $D \subseteq \mathbb{R}^3$ a bounded, connected open set with C^∞-smooth boundary ∂D and $\epsilon, \mu, \omega > 0$ positive real constants and $k = \omega\sqrt{\mu\epsilon}$ such that k^2 is not an eigenvalue of $curl^2$. Then $\mathcal{L}$ is an isomorphism in $H^s(Div, \partial D)$ $\forall s \in \mathbb{R}$.*

Chapter 3

The method

The aim of the method is to determine the (far) electric field $\mathbf{E}_{tgt}$ radiated by an antenna placed over a structure using M measured near field samples $\{\mathbf{E}_{tgt}(r_m)\}_{1\leq m\leq M}$ of the electric field and numerically constructed expansion functions $\{\boldsymbol{\psi}_n\}$. We want to find an approximation of the exact $\mathbf{E}_{tgt}$ in the form

$$\mathbf{E}_{tgt} = \sum_{n=1}^{N} \alpha_n \boldsymbol{\psi}_n + \boldsymbol{\epsilon}_N \tag{3.1}$$

where $\boldsymbol{\epsilon}_N$ is the reconstruction error; the coefficients $\{\alpha_n\} \subset \mathbb{C}$ are computed in order to minimize the error $\boldsymbol{\epsilon}_N$ in a least squares sense

$$\min_{\{\alpha_n\}} ||\mathbf{E}_{tgt} - \sum_{n=1}^{N} \alpha_n \boldsymbol{\psi}_n|| \tag{3.2}$$

The method works in the following manner:

- the antenna is enclosed in a virtual surface B that we will call "Box". The box has to enclose the antenna but not the whole structure. For equivalence theory we know that exist electric and magnetic currents $\mathbf{J}$ and $\mathbf{M}$ placed on B such that the field radiated by these currents in the outer region of B is equal to the field $\mathbf{E}_{tgt}$. The problem now is to reconstruct the currents $\mathbf{J}$ and $\mathbf{M}$ defined on B.

- currents are decomposed as linear combination of piecewise linear basis functions $\{\mathbf{f}_n\}$ with limited support (rwg basis functions), for example for the electric current $\mathbf{J}$

  $$\mathbf{J} = \sum_n \alpha_n \mathbf{f}_n \tag{3.3}$$

  A function $\mathbf{f}_n$ is called an "elementary source" and the field radiated by it "elementary field".

- the field $\Psi(\mathbf{J}, \mathbf{M})$ radiated by currents $\mathbf{J}$ and $\mathbf{M}$ is the linear combination of the fields radiated by each elementary source

  $$\Psi(\mathbf{J}, \mathbf{M}) = \sum_n \alpha_n \Psi(\mathbf{f}_n, 0) + \sum_n \beta_n \Psi(0, \mathbf{f}_n) \tag{3.4}$$

where $\Psi(\mathbf{J}, \mathbf{M})$ denotes the electric field given by an electric current $\mathbf{J}$ and magnetic current $\mathbf{M}$.

- the elementary field $\boldsymbol{\psi}_n := \Psi(\mathbf{f}_n,0)$ (or $\boldsymbol{\phi}_n := \Psi(0, \mathbf{f}_n)$) radiated by an electric (or magnetic) elementary source $\mathbf{f}_n$ has to take into account the presence of the structure. For this reason each elementary field $\boldsymbol{\psi}_n$ is decomposed as sum of two fields: $\boldsymbol{\psi}_n = \boldsymbol{\psi}_n^0 + \boldsymbol{\psi}_n^s$ where $\boldsymbol{\psi}_n^0$ is the field given by the elementary source in isolation (without the structure) and $\boldsymbol{\psi}_n^s$ is the scattered field due to the reflection of the structure. More precisely $\boldsymbol{\psi}_n^s$ is the field generated by a current over S due to an elementary source over B.

- computed each elementary field $\boldsymbol{\psi}_n$ and $\boldsymbol{\phi}_n$ we determine coefficients imposing matching (in least squares sense) between the field radiated by the currents and the near-field samples acquired. Once coefficients are determined, we know the currents defined over B and are then able to evaluate the field radiated by these currents in any point of the space outside S. In particular, we are able to evaluate the far-field.

We denote with $Ext(B)$ the space outside the surface B. To compute the scattered part $\boldsymbol{\psi}^s$ (or $\boldsymbol{\phi}^s$) of the electric field due to an elementary source $\mathbf{J}$ or $\mathbf{M}$ we

- compute the incident field $\mathbf{E}^{inc}$ that the electric current $\mathbf{J}$ (or magnetic $\mathbf{M}$) generates on the scatterer through

$$\mathbf{E}^{inc}(\mathbf{r}) = -\mathcal{L}_B(\mathbf{J})(\mathbf{r}) + \mathcal{K}_B(\mathbf{M})(\mathbf{r}) \quad \mathbf{r} \in Ext(B) \tag{3.5}$$

  where (differently from the previous chapter we denote with $\mathcal{L}$ the operator without $\mathbf{n}\times$)

$$\mathcal{L}_B(\mathbf{J})(\mathbf{r}) = j\omega\mu \int_B G(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')ds(\mathbf{r}') + \frac{j}{\epsilon\omega}\nabla \int_B G(\mathbf{r}, \mathbf{r}') \text{ Div } \mathbf{J}(\mathbf{r}')ds(\mathbf{r}') \tag{3.6}$$

$$\mathcal{K}_B(\mathbf{M})(\mathbf{r}) = \int_B \mathbf{M}(\mathbf{r}') \times \nabla G(\mathbf{r}, \mathbf{r}')ds(\mathbf{r}') \tag{3.7}$$

  The field in isolation $\boldsymbol{\psi}^0$ is computed through the same formula.

- compute the induced current $\mathbf{J}^s$ generated by $\mathbf{E}^{inc}$ on the scatterer S (modeled as a PEC) through

$$\mathbf{n}(\mathbf{r}) \times \mathbf{E}^{inc}(\mathbf{r}) = \mathbf{n}(\mathbf{r}) \times \mathcal{L}_S(\mathbf{J}^s)(\mathbf{r}) \quad \mathbf{r} \in S \tag{3.8}$$

- compute the electric field $\boldsymbol{\psi}^s$ generated by $\mathbf{J}^s$ through

$$\boldsymbol{\psi}^s(\mathbf{r}) = -\mathcal{L}_S(\mathbf{J}^s)(\mathbf{r}) \quad \mathbf{r} \in Ext(S) \tag{3.9}$$

- compute the total field $\boldsymbol{\psi} = \boldsymbol{\psi}^0 + \boldsymbol{\psi}^s$ due to an elementary source.

3.0.1 A second glance on the method

We first consider the simpler case of a radiating source in vacuum, without a scattering structure and then we will consider the standard case in presence of a scatterer S. The *surface equivalence principle* states that given electric and magnetic fields $\mathbf{E}, \mathbf{H}$ in $\mathbb{R}^3$ and a closed smooth surface B with unit normal vector $\mathbf{n}$, separating the outer region Ω^+ and the inner region Ω^-, exist an electric current $\mathbf{J}$ and a magnetic current $\mathbf{M}$ defined over B that radiate the same electric and magnetic fields $\mathbf{E}, \mathbf{H}$ in Ω^+ and radiate fields $\mathbf{E}', \mathbf{H}'$ in Ω^- such that $\forall \mathbf{x} \in B$

$$\Big[\mathbf{E}_+(\mathbf{x}) - \mathbf{E}'_-(\mathbf{x})\Big] \times \mathbf{n}(\mathbf{x}) = \mathbf{M}(\mathbf{x}) \qquad \mathbf{n}(\mathbf{x}) \times \Big[\mathbf{H}_+(\mathbf{x}) - \mathbf{H}'_-(\mathbf{x})\Big] = \mathbf{J}(\mathbf{x}) \tag{3.10}$$

and such that the electric field radiatied by $\mathbf{J}$ and $\mathbf{M}$ is equal to $\mathbf{E}$ in Ω^+, i.e.

$$\mathbf{E}(\mathbf{r}) = -\mathcal{L}_B(\mathbf{J})(\mathbf{r}) + \mathcal{K}_B(\mathbf{M})(\mathbf{r}) \quad \forall \mathbf{r} \in \Omega^+ \tag{3.11}$$

where

$$\mathbf{E}_\pm(\mathbf{x}) := \lim_{\mathbf{x}^\pm \in \Omega^\pm, \mathbf{x}^\pm \to \mathbf{x}} \mathbf{E}(\mathbf{x}^\pm) \tag{3.12}$$

In the theorem $\mathbf{E}', \mathbf{H}'$ in Ω^- are degrees of freedom and the simplest choice is to enforce $\mathbf{E}' = \mathbf{H}' = 0$; in this case the theorem is called also *Love's equivalence theorem* and the currents satisfy

$$\mathbf{E}_+ \times \mathbf{n} = \mathbf{M} \quad \mathbf{n} \times \mathbf{H}_+ = \mathbf{J} \tag{3.13}$$

A source in free space - no scatterer

We now analyze the case where a source generates an electric field $\mathbf{E}^{tgt}$. We enclose the source with a surface B and then using the equivalence principle there exist an electric current $\mathbf{J}$ and a magnetic current $\mathbf{M}$ defined on B that radiate $\mathbf{E}^{tgt}$ in the outer region of B. We consider a mesh over B with N interior edges and we search the unknown currents in the form (with abuse of notation we still denote with $\mathbf{J}$ and $\mathbf{M}$ their approximations):

$$\mathbf{J} = \sum_{n=1}^{N} \alpha_n \mathbf{f}_n \quad \mathbf{M} = \sum_{n=1}^{N} \beta_n \mathbf{f}_n \tag{3.14}$$

For linearity of operators $\mathcal{L}_B$ and $\mathcal{K}_B$ it holds that

$$\Psi^{NF}(\mathbf{J}, \mathbf{M}) = \sum_{n=1}^{N} \alpha_n \Psi^{NF}(\mathbf{f}_n, 0) + \sum_{n=1}^{N} \beta_n \Psi^{NF}(0, \mathbf{f}_n) \tag{3.15}$$

We compute $\boldsymbol{\psi}_n^{NF} := \Psi(\mathbf{f}_n, 0)$ and $\boldsymbol{\phi}_n^{NF} := \Psi(0, \mathbf{f}_n)$ for $1 \leq n \leq N$ and then find coefficients $\{\alpha_n\}, \{\beta_n\} \subset \mathbb{C}$ through the least squares problem

$$\min_{\{\alpha_n\}, \{\beta_n\}} || \sum_{n=1}^{N} \alpha_n \boldsymbol{\psi}_n^{NF} + \sum_{n=1}^{N} \beta_n \boldsymbol{\phi}_n^{NF} - \mathbf{E}^{NF} || \tag{3.16}$$

Found a possible choice of coefficients we are able to evaluate the electric field in any point outside the box B, in particular in the far field region.

Implementation: We use the K samples of $\mathbf{E}^{tgt}$ measured at some points of the near field sphere related to angles $\{\alpha_k = (\theta_k, \phi_k)\}_{1\leq k\leq K}$. We build vectors such that the first K components are related to $\hat{\theta}$ and the others to $\hat{\phi}$, for example $\mathbf{E}^{NF} = [\mathbf{E}^{tgt}_{\theta}(\alpha_k); \mathbf{E}^{tgt}_{\phi}(\alpha_k)]_{1\leq k\leq K}$. The least squares problem is solved with the iterative solver LSQR. The matrix generated of the previous least squares problem generally is not full rank, thus there exist more then one choice of the coefficients $\{\alpha_n\}, \{\beta_n\}$ that minimize the previous quantity; in this case we chose the minimum norm solution to the least squares problem. It can be proved that the solution to the least squares problem plus the minimum norm condition is unique. One problem is that such a choice of coefficients (minimum norm solution) does not ensure that currents radiate null electric and magnetic fields in the interior part of the box B [11].

With scatterer

The previous procedure is theoretically applyable also to the case of an antenna with a reflecting structure. It can be considered a surface B that encloses both antenna and the structure and currents $\mathbf{J}$ and $\mathbf{M}$ over the virtual surface B. The problem is that usually the scattering structure is much bigger then the antenna so we easily obtain a system with too many unknowns. The method consider a surface B that encloses only the antenna and not the whole structure, the field radiated by the antenna can be reconstructed through equivalent sources on B but now the field radiated by an elementary source has to take into account the presence of the scatterer. The advantage of this procedure is that the complexity of the whole structure is now reduced to the complexity of the single antenna.

Adding $\mathbf{E}_0$

In the case we have also the electric field $\mathbf{E}_0$ of the antenna in isolation we can use it as a basis function in the reconstruction

$$\min_{\alpha,\{\beta_n\},\{\gamma_n\}} ||\alpha\mathbf{E}_0^{NF} + \sum_{n=1}^{N}\beta_n\boldsymbol{\psi}_n^{NF} + \sum_{n=1}^{N}\gamma_n\boldsymbol{\phi}_n^{NF} - \mathbf{E}^{NF}|| \tag{3.17}$$

where α, β_n, $\gamma_n \in \mathbb{C}$.

Method 2

In the previous section we decomposed each elementary field $\boldsymbol{\psi}_n = \boldsymbol{\psi}_n^0 + \boldsymbol{\psi}_n^s$ and $\boldsymbol{\phi}_n = \boldsymbol{\phi}_n^0 + \boldsymbol{\phi}_n^s$ where $\boldsymbol{\psi}_n^0$ is the field given by an electric elementary source in isolation (without the structure) and $\boldsymbol{\psi}_n^s$ is the scattered field due to the reflection of the structure. Now we decompose each elementary field as a linear combination of the field in isolation and the scattered field, for example $\boldsymbol{\psi}_n = a_n\boldsymbol{\psi}_n^0 + b_n\boldsymbol{\psi}_n^s$, for some $a_n, b_n \in \mathbb{C}$. We arrive to a different least squares problem

$$\min_{\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\xi_n\}} || \sum_{n=1}^{N}\alpha_n\boldsymbol{\psi}_n^{0,NF} + \sum_{n=1}^{N}\beta_n\boldsymbol{\phi}_n^{0,NF} + \sum_{n=1}^{N}\gamma_n\boldsymbol{\psi}_n^{s,NF} + \sum_{n=1}^{N}\xi_n\boldsymbol{\phi}_n^{s,NF} - \mathbf{E}^{NF}|| \tag{3.18}$$

Chapter 4

Discretization of the problem

The method of moments (MOM) uses the electric field integral equations (EFIE) to compute the electric current $\mathbf{J}$ induced over a surface by an incident field $\mathbf{E}^{inc}$. We will treat the case of a PEC. Using the boundary condition $\mathbf{n} \times \mathbf{E} = 0$ we have that the magnetic current $\mathbf{M}$ vanish identically over the surface. We consider a mesh over the surface of the scatterer S and we consider a set of basis functions $\{\mathbf{f}_m, \quad m = 1, ..., M\}$ where M is the number of interior edges of the mesh over S and $\mathbf{f}_m : S \longrightarrow \mathbb{R}^3$ is a function associated to an interior edge of the mesh, for $m = 1, ..., M$. These functions were introduced by Rao Wilton Glisson [4] and are called RWG basis functions or briefly RWGs and are strictly linked with the finite elements of Raviart and Thomas.

A connection with lowest order Raviart and Thomas elements RT_0

We have seen in previous chapters that integral equations that involve the electric current $\mathbf{J}$ suggest us to search a solution in the space $H^1(div, \Omega)$ i.e. the space of functions in $L^2(\Omega, \mathbb{C}^3)$ with a variational divergence. In this section we analyze a simpler case in a planar domain $\Omega \subset \mathbb{R}^2$. The space of Raviart and Thomas elements can be used to approximate functions in $H^1(div, \Omega)$. If we consider a triangle $T \subset \mathbb{R}^2$ the space $RT_0(T)$ is defined as

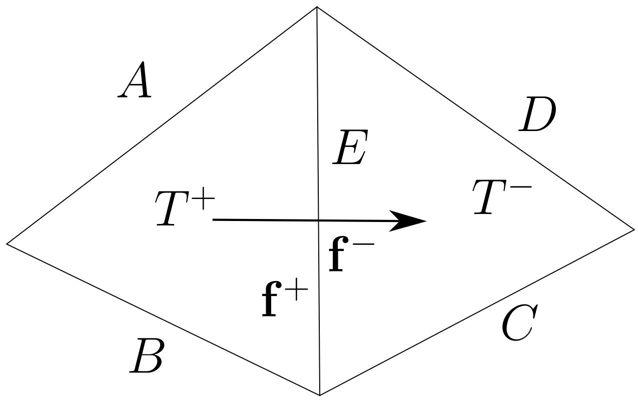
$$RT_0(T) = \{\mathbf{f} : T \longrightarrow \mathbb{R}^2 \text{ s.t. } \exists \alpha \in \mathbb{R}^2,\ \beta \in \mathbb{R} \ \ \mathbf{f}(\mathbf{x}) = \alpha + \beta \mathbf{x} \quad \forall \mathbf{x} \in T\} \tag{4.1}$$

We define the degrees of freedom Σ_E for each edge E of the triangle T as

$$\Sigma_E(\mathbf{f}) = \frac{1}{|E|} \int_E \mathbf{f} \cdot \mathbf{n} d\gamma = \frac{1}{|E|} \int_a^b \mathbf{f}(\gamma(t)) \cdot \mathbf{n}(\gamma(t)) ||\gamma'(t)|| dt \tag{4.2}$$

where $\gamma : [a, b] \longrightarrow E$ is a parametrization of the edge E and $\mathbf{n}$ the unit outward-pointing normal vector of the edge E of T. It can be proved that the element $(T, RT_0(T), \{\Sigma_j,\ j = 1,2,3\})$ is unisolvent.

In order to build a RWG basis function $\mathbf{f}$ we consider two triangles T^+ and T^- with a common edge E and free edges A, B for T^+ and C, D for T^-. Then we consider $\mathbf{f}^+ \in RT_0(T^+)$ and $\mathbf{f}^- \in RT_0(T^-)$ and enforce



- $\Sigma_A(\mathbf{f}^+) = \Sigma_B(\mathbf{f}^+) = 0$
- $\Sigma_C(\mathbf{f}^-) = \Sigma_D(\mathbf{f}^-) = 0$
- $\Sigma_E(\mathbf{f}^+) = 1$ and $\Sigma_E(\mathbf{f}^-) = -1$

The third condition ensures continuity of the current normal to the edge E. We define the basis function as

$$\mathbf{f} = \begin{cases} \mathbf{f}^+ \text{ in } T^+ \\ \mathbf{f}^- \text{ in } T^- \end{cases} \tag{4.3}$$

In the next section we write an explicit formula for the basis function $\mathbf{f}$ for a generic triangle of $\mathbb{R}^3$.

4.1 Mom

The method of moments (MOM) employs RWG basis functions to build an approximation of the electric current $\mathbf{J}$. Each function $\mathbf{f}_m$ has compact support and vanishes on S except in the two triangles attached to the edge m. We denote the interior part (the triangle without adges) of these two triangles with T_m^+ and T_m^-, their area with A_m^+ and A_m^- , their free vertexes (vertexes not on the common edge) with x_m^+ and x_m^- and the length of the common edge with l_m; then

$$\mathbf{f}_m(\mathbf{r}) = \begin{cases} \frac{l_m}{2A_m^+}(\mathbf{r} - x_m^+) \ \mathbf{r} \in T_m^+ \\ \frac{l_m}{2A_m^-}(x_m^- - \mathbf{r}) \ \mathbf{r} \in T_m^- \\ \quad 0 \text{ otherwise} \end{cases}$$

See figure 4.1 where we defined $\rho_m^+ := \mathbf{r} - x_m^+$ for $\mathbf{r} \in T_m^+$ and $\rho_m^- := x_m^- - \mathbf{r}$ for $\mathbf{r} \in T_m^-$. RWGs have several nice properties:

- $\mathbf{f}_m$ has no normal component across free edges (edges that are not the common edge m)
- the normal component of $\mathbf{f}_m$ across the common edge is continuous across that edge

- the surface divergence of $\mathbf{f}_m$ is

$$\text{Div } \mathbf{f}_m(\mathbf{r}) = \begin{cases} l_m/A_m^+ & \mathbf{r} \in T_m^+ \\ -l_m/A_m^- & \mathbf{r} \in T_m^- \\ 0 \text{ otherwise} \end{cases}$$

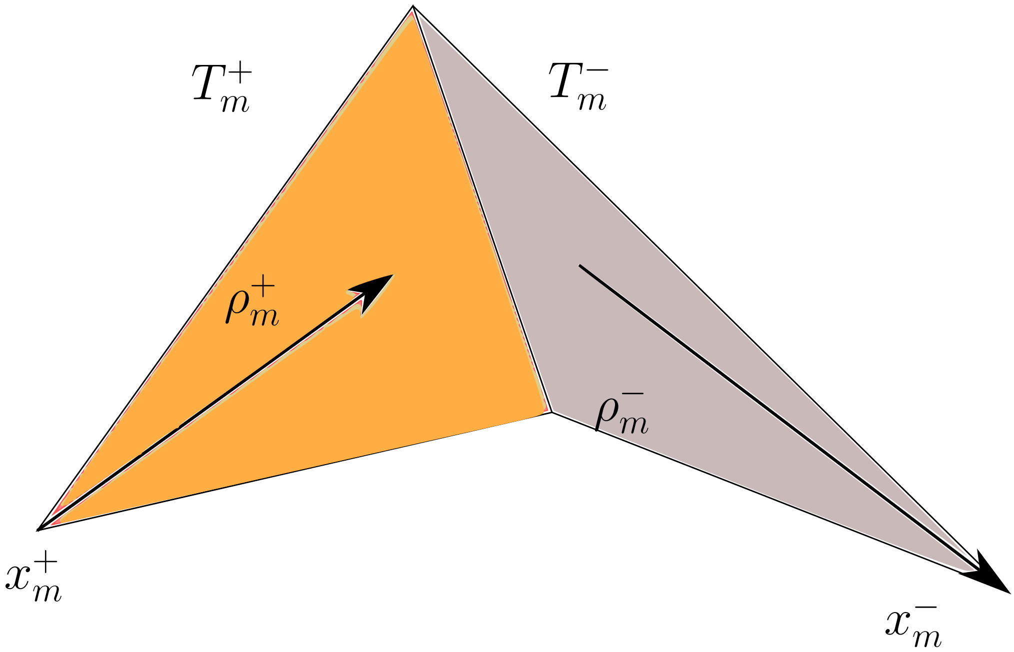


Figure 4.1. The basis function $\mathbf{f}_m$

The current $\mathbf{J}$ is expressed as linear combination of the basis functions $\{\mathbf{f}_m\}$; with abuse of notation we continue to denote with $\mathbf{J}$ the current that solves the discrete problem and that hopefully is a good approximation of the exact current that solves the continuous problem.

$$\mathbf{J} = \sum_{m=1}^{M} I_m \mathbf{f}_m \tag{4.4}$$

The electric scattered field $\mathbf{E}^s$ is linked with the current $\mathbf{J}$ through

$$\mathbf{E}^s = -j\omega\mathbf{A} - \nabla\phi \tag{4.5}$$

now applying the boundary condition $\mathbf{n} \times (\mathbf{E}^{inc} + \mathbf{E}^s) = 0$ we obtain that

$$-\mathbf{n}(\mathbf{r}) \times \mathbf{E}^{inc}(\mathbf{r}) = -j\omega\mathbf{n}(\mathbf{r}) \times \mathbf{A}(\mathbf{r}) - \mathbf{n}(\mathbf{r}) \times \nabla\phi(\mathbf{r}) \quad \forall \mathbf{r} \in S \tag{4.6}$$

We define the operator $\mathcal{L}$ as

$$\mathcal{L}(\mathbf{J}) = j\omega\mathbf{n} \times \mathbf{A} + \mathbf{n} \times \nabla\phi \tag{4.7}$$

or explicitly $\forall \mathbf{r} \in S$

$$\mathcal{L}(\mathbf{J})(\mathbf{r}) = j\omega\mu\mathbf{n}(\mathbf{r}) \times \int_S \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') + \frac{j}{\epsilon\omega}\mathbf{n}(\mathbf{r}) \times \nabla \int_S \text{ Div } \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \quad (4.8)$$

we obtain the following integral equation $\forall \mathbf{r} \in S$

$$(\mathbf{n} \times \mathbf{E}^{inc})(\mathbf{r}) = j\omega\mu\mathbf{n}(\mathbf{r}) \times \int_S \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') + \frac{j}{\epsilon\omega}\mathbf{n}(\mathbf{r}) \times \nabla \int_S \text{ Div } \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \quad (4.9)$$

Given two functions $\mathbf{h}, \mathbf{g} : S \longrightarrow \mathbb{C}^3$ we define the following symmetric product as:

$$(\mathbf{h}, \mathbf{g}) = \int_S \mathbf{h}(\mathbf{r}') \cdot \mathbf{g}(\mathbf{r}')ds(\mathbf{r}') = \int_S \sum_{i=1}^{3} h_i(\mathbf{r}')g_i(\mathbf{r}')ds(\mathbf{r}') \quad (4.10)$$

Multiplying $\mathcal{L}(\mathbf{J}) = \mathbf{n} \times \mathbf{E}^{inc}$ for a function $\mathbf{f}_m$ we obtain

$$(\mathcal{L}(\mathbf{J}), \mathbf{f}_m) = (\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m) \quad \forall m = 1, ..., M \quad (4.11)$$

and using the expansion

$$\mathbf{J} = \sum_{m=1}^{M} I_m \mathbf{f}_m \quad (4.12)$$

we obtain that

$$\sum_{n=1}^{M} I_n(\mathcal{L}(\mathbf{f}_n), \mathbf{f}_m) = (\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m) \quad \forall m = 1, ..., M \quad (4.13)$$

and we arrive to the linear system

$$\mathbf{ZI} = \mathbf{V} \quad (4.14)$$

where $\mathbf{I} = [I_n]$ and $V_m = (\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m)$ and $Z_{m,n} = (\mathcal{L}(\mathbf{f}_n), \mathbf{f}_m)$ or explicitly

$$(\mathcal{L}(\mathbf{f}_n), \mathbf{f}_m) = j\omega\mu \int_S \mathbf{f}_m(\mathbf{r}) \cdot \left[\int_S \mathbf{f}_n(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \right] ds(\mathbf{r}) +$$
$$+ \frac{j}{\epsilon\omega} \int_S \mathbf{f}_m(\mathbf{r}) \cdot \nabla \left[\int_S \text{ Div } \mathbf{f}_n(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \right] ds(\mathbf{r})$$

We note that if $supp(\mathbf{f}_n) \cap supp(\mathbf{f}_m) \neq \emptyset$ then the Green's function has a singularity in $\mathbf{r} = \mathbf{r}'$ so one must take care in the computation of integrals. It holds that

$$\int_S \mathbf{f}_n(\mathbf{r}) \cdot \nabla \left(\int_S \text{ Div } \mathbf{f}_m(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \right) ds(\mathbf{r}) = \int_S \mathbf{f}_m(\mathbf{r}) \cdot \nabla \left(\int_S \text{ Div } \mathbf{f}_n(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \right) ds(\mathbf{r}) \quad (4.15)$$

and

$$(\mathbf{f}_n, \mathcal{L}(\mathbf{f}_m)) = (\mathcal{L}(\mathbf{f}_n), \mathbf{f}_m) \quad (4.16)$$

so the matrix $\mathbf{Z}$ is symmetric (but not hermitian).

Resolution of MOM

Each linear system arising by MOM

$$\mathbf{Zx} = \mathbf{b} \tag{4.17}$$

is solved with the iterative solver GMRES coupled with a fast algorithm to evaluate matrix-vector product [5]. The $\mathbf{Z}$ is splitted as sum of two matrices $\mathbf{Z} = \mathbf{Z}^{near} + \mathbf{Z}^{far}$. The domain is subdivided with an octree, interactions between each subgroup of basis functions and its neighbors are computed exactly with the standard formula of the MOM and are considered in the matrix $\mathbf{Z}^{near}$. Interactions between far groups (groups that are not near) are considered in the matrix $\mathbf{Z}^{far}$. More precisely we used a right-preconditioned version of GMRES i.e. Flexible-GMRES (FGMRES) [9]. It is known that in the classic GMRES, given a linear system $\mathbf{Zx} = \mathbf{b}$ and a vector $\mathbf{x}_0$ and defined $\mathbf{r}_0 = \mathbf{Zx}_0 - \mathbf{b}$ we search the solution in the space $\mathbf{x}_0 + K_m(\mathbf{Z}, \mathbf{v}_1)$ where $\mathbf{v}_1 = \mathbf{r}_0/||\mathbf{r}_0||$ and $K_m(\mathbf{Z}, \mathbf{v}_1)$ is the m-th Krylov subspace related to the matrix $\mathbf{Z}$ and vector $\mathbf{v}_1$. At each iteration we have to compute a matrix-vector product $\mathbf{Zv}$ in order to compute (when it is possible) a basis $\{\mathbf{v}_1, ..., \mathbf{v}_m\}$ of the Krylov subspace K_m. Instead in FGMRES at the $j-th$ iteration, given a preconditioner $\mathbf{M}_j$, we have to compute

$$\mathbf{p}_j = \mathbf{M}_j^{-1}\mathbf{v}_j \tag{4.18}$$

and then the matrix-vector product $\mathbf{Zp}_j$. The main idea is to consider a preconditioner $\mathbf{M}_j$ such that $\mathbf{M}_j \simeq (\mathbf{Z}^{near})^{-1}$. We solve this inner linear system 4.18 with an iterative method with an high residual in order to approximate the matrix-vector multiplication for the inverse of the matrix due to near field interations only. To compute $\mathbf{p}_j = \mathbf{M}_j^{-1}\mathbf{v}_j$ we solved

$$\mathbf{Z}^{near}\mathbf{p}_j = \mathbf{v}_j \tag{4.19}$$

with an inner GMRES with $m' << m$ as the maximum dimension of the Krylov subspace. We have choosen $m' = 50$ while the maximum dimension of the Krylov subspace for the resolution of the main linear system is $m = 2000$.

We fixed a threshold of $\epsilon_0 = 10^{-3}$ and we allowed a solution that generates a relative residual smaller than ϵ_0 i.e. FGMRES stopped when it found a solution $\mathbf{x}$ such that

$$\frac{||\mathbf{Zx} - \mathbf{b}||}{||\mathbf{b}||} < \epsilon_0 \tag{4.20}$$

4.2 Computation of the incident field $\mathbf{E}^{inc}$

In this section we explain how we can determine the incident field $\mathbf{E}^{inc}$ on the scatterer S generated by an electric current $\mathbf{J}$ or a magnetic current $\mathbf{M}$ on the Box B using the MOM matrix related to the system "Box+Scatterer". More precisely we will compute the term

$$(\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_s) \tag{4.21}$$

for every $\mathbf{f}_s$ with support contained on the scatterer S.

From J to $\mathbf{E}^{inc}$

We consider the case of an elementary electric source $\mathbf{J}_l$ defined on the box B that radiates an electric field $\mathbf{E}^{inc}$ on the scatterer S. We consider a mesh over the box B and over the scatterer S, then we put rwg basis functions over these surfaces. Our aim is to compute $(\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m)$ for every $\mathbf{f}_m$ defined over S using the Mom matrix $\mathbf{Z}$ related to both B and S. The electric field $\mathbf{E}^{inc}$ satisfies

$$-\mathcal{L}(\mathbf{J}) = \mathbf{n} \times \mathbf{E}^{inc} \tag{4.22}$$

where $\mathbf{n}(\mathbf{r})$ is the unit normal vector to S in $\mathbf{r} \in S$ and

$$\mathcal{L}(\mathbf{J})(\mathbf{r}) = j\omega\mu\mathbf{n}(\mathbf{r}) \times \int_S \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') + \frac{j}{\epsilon\omega}\mathbf{n}(\mathbf{r}) \times \nabla \int_S \text{ Div } \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \tag{4.23}$$

We use the decomposition $\mathbf{J}_l = \sum_{n=1}^{N} \delta_{nl}\mathbf{f}_n = \mathbf{f}_l$ where $\mathbf{f}_l$ is an rwg function with support in B. We obtain

$$-Z_{s,l} =: (\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_s) \tag{4.24}$$

for all $\mathbf{f}_s$ with support in S and where $\mathbf{Z}$ is the Mom matrix

$$Z_{s,l} = (\mathcal{L}(\mathbf{f}_l), \mathbf{f}_s) \tag{4.25}$$

From M to $\mathbf{E}^{inc}$

As before we consider the case of an elementary magnetic source $\mathbf{M}_l$ defined on the box B that radiates an electric field $\mathbf{E}^{inc}$ on the scatterer S. The electric field $\mathbf{E}^{inc}$ radiated is

$$\mathbf{E}^{inc} = -\frac{1}{\epsilon} \text{ curl } \mathbf{F} \tag{4.26}$$

so multiplying for $\mathbf{n}\times$

$$\mathbf{n} \times \mathbf{E}^{inc} = -\frac{1}{\epsilon}\mathbf{n} \times \text{ curl } \mathbf{F} \tag{4.27}$$

or more explicitly $\forall \mathbf{r} \in S$

$$\mathbf{n}(\mathbf{r}) \times \mathbf{E}^{inc}(\mathbf{r}) = -\mathbf{n}(\mathbf{r}) \times \text{ curl } \int_S \mathbf{M}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds(\mathbf{r}') \tag{4.28}$$

For the following equality

$$\text{curl } (G\mathbf{M}) = G \text{ curl } \mathbf{M} + \nabla G \times \mathbf{M} \tag{4.29}$$

we have that

$$\mathbf{n}(\mathbf{r}) \times \mathbf{E}^{inc}(\mathbf{r}) = -\mathbf{n}(\mathbf{r}) \times \int_S \nabla_{\mathbf{r}} G(\mathbf{r},\mathbf{r}') \times \mathbf{M}(\mathbf{r}')ds(\mathbf{r}') \tag{4.30}$$

multiplying for a rwg function $\mathbf{f}_m$ with support over S we obtain

$$(\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m) = -(\mathbf{n}(\cdot) \times \int_S \nabla G(\cdot,\mathbf{r}') \times \mathbf{M}(\mathbf{r}')ds(\mathbf{r}'), \mathbf{f}_m) \tag{4.31}$$

Now using the decomposition of $\mathbf{M}$ as

$$\mathbf{M} = \sum_{n=1}^{M} \delta_{nl}\mathbf{f}_n = \mathbf{f}_l \tag{4.32}$$

we obtain

$$(\mathbf{n} \times \mathbf{E}^{inc}, \mathbf{f}_m) = \left(\mathbf{n}(\cdot) \times \int_S \mathbf{f}_l(\mathbf{r}') \times \nabla G(\cdot, \mathbf{r}')ds(\mathbf{r}')\ ,\ \mathbf{f}_m\right) = Z_{m,l} \tag{4.33}$$

where $\mathbf{Z}$ is the Mom matrix related to a PMC (perfect magnetic conductor) i.e. the matrix related only to the part of magnetic current.

4.3 Model order reduction

Given the n-th elementary electric source on the box B for $n = 1, ..., N$, where N is the number of interior edges of the mesh on B, that generates an incident field $\mathbf{E}_n^{inc}$ on the scatterer S we have to solve the linear system for $\mathbf{J}_n^s$

$$\mathbf{Z}\mathbf{J}_n^s = \mathbf{E}_n^{inc} \quad n = 1, ..., N \tag{4.34}$$

where $\mathbf{Z} \in \mathbb{C}^{M\times M}$ is the MOM matrix and M is the number of rwg basis functions over S. We can rewrite previous equations as

$$\mathbf{Z}\mathbf{J} = \mathbf{E} \tag{4.35}$$

where $\mathbf{J} \in \mathbb{C}^{M\times N}$ is $\mathbf{J}(:, n) = \mathbf{J}_n^s$ and $\mathbf{E} \in \mathbb{C}^{M\times N}$ is $\mathbf{E}(:, n) = \mathbf{E}_n^{inc}$ for $n = 1, ..., N$. Our aim is to reduce the complexity of the model and solve only $k < N$ linear systems. For this reason we apply a SVD decomposition to the matrix $\mathbf{E}$. For proofs and other properties on SVD we refer to [3]. We recall that a matrix $\mathbf{U} \in \mathbb{C}^{M\times M}$ is said to be unitary if $\mathbf{U}'\mathbf{U} = \mathbf{U}\mathbf{U}' = \mathbf{Id}$, where $\mathbf{U}'$ is the conjugate transpose of $\mathbf{U}$. We denote with $||\mathbf{E}||$ the Frobenius norm of the matrix $\mathbf{E}$, i.e.

$$||\mathbf{E}||^2 = \sum_{i=1}^{M}\sum_{j=1}^{N} |E_{ij}|^2 = trace(\mathbf{E}'\mathbf{E}) \tag{4.36}$$

Theorem 4.3.1 (SVD decomposition)**.** *[3] Given $\mathbf{E} \in \mathbb{C}^{M\times N}$ there exist unitary matrices $\mathbf{U} \in \mathbb{C}^{M\times M}$ and $\mathbf{V} \in \mathbb{C}^{N\times N}$ and a diagonal matrix $\mathbf{\Sigma} \in \mathbb{R}^{M\times N}$ with $\Sigma_{mn} = \delta_{mn}\sigma_n$ with diagonal elements (called singular values) in non-increasing order i.e.*
$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p \geq 0$ with $p = min(M, N)$ such that

$$\mathbf{E} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}' \tag{4.37}$$

The singular values $\{\sigma_j\}$ are uniquely determined. Furthermore the rank of $\mathbf{E}$ coincides with the number of non-zero singular values and it holds that

$$||\mathbf{E}|| = \sqrt{\sum_{j=1}^{p} \sigma_j^2} \tag{4.38}$$

where $||\cdot||$ is the Frobenius norm of a matrix and $\mathbf{V}'$ denotes the conjugate transpose (Hermitian transponse) of the matrix $\mathbf{V}$.

We use an SVD decomposition for $\mathbf{E} = \mathbf{U\Sigma V'}$ where $\mathbf{U} \in \mathbb{C}^{M\times M}$, $\mathbf{V} \in \mathbb{C}^{N\times N}$ and $\mathbf{\Sigma} \in \mathbb{R}^{M\times N}$ with $\Sigma_{mn} = \delta_{mn}\sigma_n$ is the diagonal matrix with singular values in non-increasing order $\{\sigma_n,\ n = 1, ..., min(N, M)\}$. Fixed an $\epsilon > 0$ we consider the k such that

$$\sigma_k/\sigma_1 > \epsilon \quad \text{and} \quad \sigma_{k+1}/\sigma_1 < \epsilon \tag{4.39}$$

and we use a truncated SVD (T-SVD) for $\mathbf{E}$ truncated at the $k-$th singular value σ_k. We define $\tilde{\mathbf{E}} := \tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}'$ where $\tilde{\mathbf{U}} \in \mathbb{C}^{M\times k}$, $\tilde{\mathbf{\Sigma}} \in \mathbb{C}^{k\times k}$, $\tilde{\mathbf{V}} \in \mathbb{C}^{N\times k}$ such that $\tilde{\mathbf{U}} := \mathbf{U}(: ,1 : k)$ and $\tilde{\mathbf{V}} := \mathbf{V}(: ,1 : k)$ and $\tilde{\mathbf{\Sigma}} := \mathbf{\Sigma}(1 : k,1 : k)$. Denoting with $||\mathbf{E}||$ Frobenius norm of the matrix $\mathbf{E}$, we have that

$$||\mathbf{E} - \tilde{\mathbf{E}}|| = \sqrt{\sum_{j=k+1}^{N} \sigma_j^2} \tag{4.40}$$

The matrix $\mathbf{J}$ is approximated through

$$\mathbf{J} = \mathbf{Z}^{-1}\mathbf{E} \simeq \mathbf{Z}^{-1}\tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}' = \mathbf{Y}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}' \tag{4.41}$$

where $\mathbf{Y} \in \mathbb{C}^{M\times k}$ solves $\mathbf{ZY} = \tilde{\mathbf{U}}$. Denoting with $\mathbf{u}_j := \mathbf{U}(:, j) \in \mathbb{C}^M$ the $j - th$ column of $\mathbf{U}$ we prove that

$$||\mathbf{J} - \tilde{\mathbf{J}}|| = \sqrt{\sum_{j=k+1}^{N} \sigma_j^2 ||\mathbf{Z^{-1}}\mathbf{u}_j||^2} \tag{4.42}$$

We formalize this result in a lemma.

Lemma 4.3.1. *Let* $\mathbf{A} \in \mathbb{C}^{M\times M}$ *and* $\mathbf{E} \in \mathbb{C}^{M\times N}$ *with* $M > N$. *Consider an SVD decomposition of* $\mathbf{E} = \mathbf{U\Sigma V'}$ *and consider its k-T-SVD* $\tilde{\mathbf{E}}$

$$\tilde{\mathbf{E}} := \sum_{j=1}^{k} \sigma_j \mathbf{u}_j \mathbf{v}_j' \tag{4.43}$$

Then

$$||\mathbf{A}(\mathbf{E} - \tilde{\mathbf{E}})|| = \sqrt{\sum_{j=k+1}^{N} \sigma_j^2 ||\mathbf{A}\mathbf{u}_j||^2} \tag{4.44}$$

where $\mathbf{u}_j$ *and* $\mathbf{v}_j$ *are the* $j-th$ *column of* $\mathbf{U}$ *and* $\mathbf{V}$ *respectively.*

Proof. Define $\mathbf{B} := \mathbf{A}(\mathbf{E} - \tilde{\mathbf{E}})$, our aim is to compute $||\mathbf{B}||^2 = tr(\mathbf{B}'\mathbf{B})$ where we denote with $tr(\mathbf{B})$ the trace of the matrix $\mathbf{B}$. It holds that

$$\mathbf{B} = \mathbf{A}(\mathbf{E} - \tilde{\mathbf{E}}) = \sum_{j=k+1}^{N} \sigma_j \mathbf{A}\mathbf{u}_j \mathbf{v}_j' \tag{4.45}$$

Because $\{\mathbf{u}_j, \quad j = 1, ..., M\}$ is an orthonormal basis of $\mathbb{C}^M$ then for each $\mathbf{A}\mathbf{u}_j \in \mathbb{C}^M$ exists $\{\lambda_z^j \in \mathbb{C}, \quad z = 1, ..., M\}$ such that

$$\mathbf{A}\mathbf{u}_j = \sum_{z=1}^{M} \lambda_z^j \mathbf{u}_z \tag{4.46}$$

Thus

$$\mathbf{B} = \sum_{j=k+1}^{N} \sum_{z=1}^{M} \sigma_j \lambda_z^j \mathbf{u}_z \mathbf{v}_j' \qquad \mathbf{B}' = \sum_{i=k+1}^{N} \sum_{t=1}^{M} \sigma_i \overline{\lambda}_t^i \mathbf{v}_i \mathbf{u}_t' \tag{4.47}$$

In the second equality we use that $\mathbf{u}_t'\mathbf{u}_z = \delta_{tz}$

$$\mathbf{B}'\mathbf{B} = \sum_{i,j=k+1}^{N} \sum_{z,t=1}^{M} \sigma_i \sigma_j \lambda_z^j \overline{\lambda}_t^i \mathbf{v}_i \mathbf{u}_t' \mathbf{u}_z \mathbf{v}_j' = \sum_{i,j=k+1}^{N} \sum_{z,t=1}^{M} \sigma_i \sigma_j \lambda_z^j \overline{\lambda}_t^i \mathbf{v}_i \delta_{tz} \mathbf{v}_j' =$$
$$= \sum_{i,j=k+1}^{N} \sum_{z=1}^{M} \sigma_i \sigma_j \lambda_z^j \overline{\lambda}_z^i \mathbf{v}_i \mathbf{v}_j'$$

Denoting with tr the trace of a matrix, it holds that

$$tr(\mathbf{B}'\mathbf{B}) = \sum_{i,j=k+1}^{N} \sum_{z=1}^{M} \sigma_i \sigma_j \lambda_z^j \overline{\lambda}_z^i tr(\mathbf{v}_i \mathbf{v}_j') = \sum_{i,j=k+1}^{N} \sum_{z=1}^{M} \sigma_i \sigma_j \lambda_z^j \overline{\lambda}_z^i \delta_{ij} =$$
$$= \sum_{i=k+1}^{N} \sum_{z=1}^{M} \sigma_i^2 |\lambda_z^i|^2 = \sum_{i=k+1}^{N} \sigma_i^2 \sum_{z=1}^{M} |\lambda_z^i|^2 = \sum_{i=k+1}^{N} \sigma_i^2 ||\mathbf{A}\mathbf{u}_j||^2$$

because $tr(\mathbf{v}_i \mathbf{v}_j') = \mathbf{v}_j' \mathbf{v}_i = \delta_{ij}$ and for $i = 1, ..., M$ we have $||\mathbf{A}\mathbf{u}_i||^2 = \sum_{z=1}^{M} |\lambda_z^i|^2$ for the orthonormality of $\{\mathbf{u}_j\}$. Now using that $||\mathbf{B}||^2 = tr(\mathbf{B}'\mathbf{B})$ we arrive to the desired result. □

The previous lemma can be applied with $\mathbf{A} = \mathbf{Z}^{-1}$ to find

$$||\mathbf{J} - \tilde{\mathbf{J}}|| = \sqrt{\sum_{j=k+1}^{N} \sigma_j^2 ||\mathbf{Z}^{-1}\mathbf{u}_j||^2} \tag{4.48}$$

This procedure is applied also to incident fields on S due to elementary magnetic current on the box B.

4.4 Computation of the far field

The value of the electric far field $\mathbf{E}$ depends obviusly on the distance between the source and the observation point. We are interested in variations in module of the electric field $\mathbf{E}$ and not in exact values of $\mathbf{E}$ because we want to find the directions where antenna generates an high electric field in module and directions where the field generated is weak in module. Thus for our purposes is not restrictive to study the far field $\mathbf{E}(r, \theta, \phi)$ on the sphere S^2, thus we put $r = 1$ and briefly write $\mathbf{E}(\theta, \phi)$ instead of $\mathbf{E}(r, \theta, \phi)$.

Electric current J

If we have an electric current $\mathbf{J}$ defined over the surface S while the magnetic current $\mathbf{M}$ vanishes on S then

$$\mathbf{E}(\theta,\phi) = -\frac{j\omega\mu}{4\pi}e^{-jk}(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi})\cdot\int_S \mathbf{J}(\mathbf{r}')e^{jk\hat{r}(\theta,\phi)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.49}$$

We consider a mesh of S, rwg basis functions $\{\mathbf{f}_m\}$ and suppose to have the current $\mathbf{J}$ in the form of

$$\mathbf{J} = \sum_{m=1}^{M} I_m \mathbf{f}_m \tag{4.50}$$

for some coefficients $\{I_m\} \subset \mathbb{C}$. We obtain that

$$\mathbf{E}(\theta,\phi) = -\frac{j\omega\mu}{4\pi}e^{-jk}(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi})\cdot\sum_{m=1}^{M} I_m\int_S \mathbf{f}_m(\mathbf{r}')e^{jk\hat{r}(\theta,\phi)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.51}$$

We are interested in values of $\mathbf{E}(\theta,\phi)$ for some values of $\{(\theta_l,\phi_s)\}_{l,s}$. Renumbering these pairs of angles and defining $\alpha_p := (\theta_l,\phi_s)$ for $p = 1,...,P$ we compute $E_\theta(\alpha_p) = \mathbf{E}(\alpha_p)\cdot\hat{\theta}(\alpha_p)$ i.e. the component of $\mathbf{E}(\alpha_p)$ over $\hat{\theta}(\alpha_p)$

$$E_\theta(\alpha_p) = -\frac{j\omega\mu}{4\pi}e^{-jk}\sum_{m=1}^{M} I_m\int_S \hat{\theta}(\alpha_p)\cdot\mathbf{f}_m(\mathbf{r}')e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.52}$$

and the component $E_\phi(\alpha_p) = \mathbf{E}(\alpha_p)\cdot\hat{\phi}(\alpha_p)$

$$E_\phi(\alpha_p) = -\frac{j\omega\mu}{4\pi}e^{-jk}\sum_{m=1}^{M} I_m\int_S \hat{\phi}(\alpha_p)\cdot\mathbf{f}_m(\mathbf{r}')e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.53}$$

We define the matrices $\mathbf{F}^\theta$ and $\mathbf{F}^\phi$ as

$$F^\theta_{pm} = -\frac{j\omega\mu}{4\pi}e^{-jk}\int_S \hat{\theta}(\alpha_p)\cdot\mathbf{f}_m(\mathbf{r}')e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.54}$$

$$F^\phi_{pm} = -\frac{j\omega\mu}{4\pi}e^{-jk}\int_S \hat{\phi}(\alpha_p)\cdot\mathbf{f}_m(\mathbf{r}')e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.55}$$

and defining the matrix $\mathbf{F} = [\mathbf{F}^\theta;\mathbf{F}^\phi]$ we obtain

$$\mathbf{FI} = \mathbf{E} \tag{4.56}$$

where $\mathbf{F} \in \mathbb{C}^{2P\times M}$ and $\mathbf{I} = [I_m] \in \mathbb{C}^M$ and $\mathbf{E} = [E_j] \in \mathbb{C}^{2P}$ where E_j is the component of the electric far field over $\hat{\theta}$ for $j = 1,...,P$ and the component over $\hat{\phi}$ for $j = P+1,...,2P$.

Magnetic current M

If we have a magnetic current $\mathbf{M}$ defined over the surface S and the electric current $\mathbf{J}$ vanishes over S then

$$\mathbf{E}(\theta,\phi) = \frac{j\omega\eta\epsilon}{4\pi}e^{-jk}(\hat{\phi}\hat{\theta}-\hat{\theta}\hat{\phi})\cdot\int_S \mathbf{M}(\mathbf{r}')e^{jk\hat{r}(\theta,\phi)\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{4.57}$$

We consider a mesh of S, rwg basis functions $\{\mathbf{f}_m\}$ and suppose to have the current $\mathbf{M}$ in the form of

$$\mathbf{M} = \sum_{m=1}^{M} I_m \mathbf{f}_m \tag{4.58}$$

for some coefficients $\{I_m\} \subset \mathbb{C}$. Similarly to the case of electric current $\mathbf{J}$ we obtain $E_\theta(\alpha_p) = \mathbf{E}(\alpha_p) \cdot \hat{\theta}(\alpha_p)$ as

$$E_\theta(\alpha_p) = -\frac{j\omega\eta\epsilon}{4\pi} e^{-jk} \sum_{m=1}^{M} I_m \int_S \hat{\phi}(\alpha_p) \cdot \mathbf{f}_m(\mathbf{r}') e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'} ds(\mathbf{r}') \tag{4.59}$$

and $E_\phi(\alpha_p) = \mathbf{E}(\alpha_p) \cdot \hat{\phi}(\alpha_p)$ as

$$E_\phi(\alpha_p) = \frac{j\omega\eta\epsilon}{4\pi} e^{-jk} \sum_{m=1}^{M} I_m \int_S \hat{\theta}(\alpha_p) \cdot \mathbf{f}_m(\mathbf{r}') e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'} ds(\mathbf{r}') \tag{4.60}$$

and defining

$$F^\theta_{pm} = -\frac{j\omega\eta\epsilon}{4\pi} e^{-jk} \int_S \hat{\phi}(\alpha_p) \cdot \mathbf{f}_m(\mathbf{r}') e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'} ds(\mathbf{r}') \tag{4.61}$$

$$F^\phi_{pm} = \frac{j\omega\eta\epsilon}{4\pi} e^{-jk} \int_S \hat{\theta}(\alpha_p) \cdot \mathbf{f}_m(\mathbf{r}') e^{jk\hat{r}(\alpha_p)\cdot\mathbf{r}'} ds(\mathbf{r}') \tag{4.62}$$

and defining $\mathbf{F} = [\mathbf{F}^\theta; \mathbf{F}^\phi]$ we obtain that

$$\mathbf{F}\mathbf{I} = \mathbf{E} \tag{4.63}$$

where $\mathbf{F} \in \mathbb{C}^{2P\times M}$ and $\mathbf{I} = [I_m] \in \mathbb{C}^M$ and $\mathbf{E} = [E_j] \in \mathbb{C}^{2P}$ where E_j is the component of the electric far field over $\hat{\theta}$ for $j = 1, ..., P$ and the component over $\hat{\phi}$ for $j = P+1, ...,2P$.

Far field of a translated body

We consider an electric current $\mathbf{J}_0$ defined over Ω_0 and its far electric field $\mathbf{E}^\infty_0$. We consider the translate of Ω_0 by $\bar{\mathbf{r}}$

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} - \bar{\mathbf{r}} \in \Omega_0\} \tag{4.64}$$

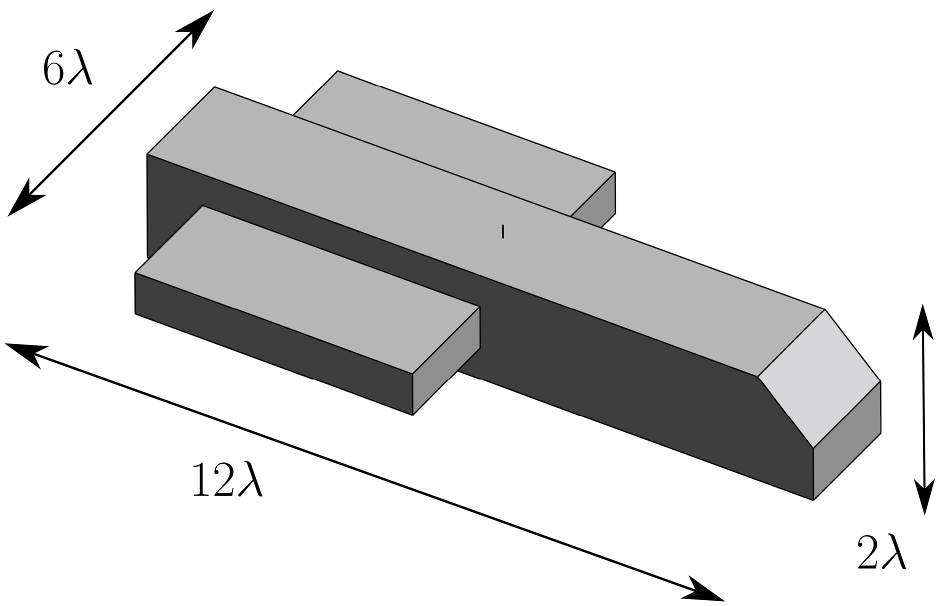
and define $\mathbf{J}(\mathbf{r}) := \mathbf{J}_0(\mathbf{r} - \bar{\mathbf{r}})$ and its far field $\mathbf{E}^\infty$. It holds that $\forall r, \theta, \phi$

$$\mathbf{E}^\infty(r, \theta, \phi) = e^{jk\hat{\mathbf{r}}(\theta,\phi)\cdot\bar{\mathbf{r}}} \mathbf{E}^\infty_0(r, \theta, \phi) \tag{4.65}$$

Chapter 5

Plane 3GHz

In this chapter we show results of the reconstruction of the far electric field of a dipole placed on a plane mock-up. The reference field $\mathbf{E}^{tgt}$ produced by the antenna is obtained through a simulation so the method is tested on synthetic data. The frequency of the dipole is 3GHz. The minimum sphere that encloses the whole structure has radius $r = 0.8m \simeq 8\lambda$ with $\lambda = 0.1m$. The lower bound on the number N of measures given by Nyquist criterion is $N = 4\pi r^2/(\lambda/2)^2 = 3215$ that corresponds to a sampling step $\Delta\phi = 4$ degrees; for this reason samples were measured at the sphere with radius $r = 0.8m$ with sampling step $\Delta\phi = \Delta\theta = 3$ degrees. We have 617 RWGs on the mesh surrounding the antenna and around 50.000 RWGs on the plane. Computing the sampling step for the dipole with the



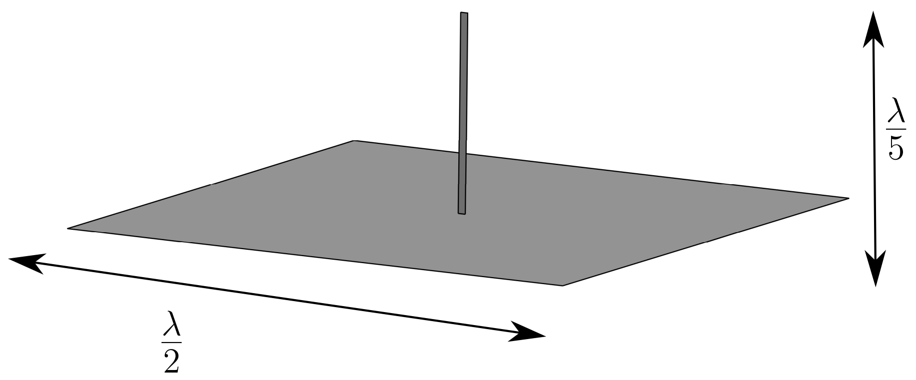


Figure 5.1. The dipole

formula [6]

$$\Delta\phi = \frac{1}{\frac{2r}{\lambda} + \frac{10}{\pi}} rad = \frac{1}{\frac{2r}{\lambda} + \frac{10}{\pi}} \frac{180}{\pi} degrees \tag{5.1}$$

we obtain a step of $\Delta\phi = 14$ degrees. A generic measuring system that acquires spherical

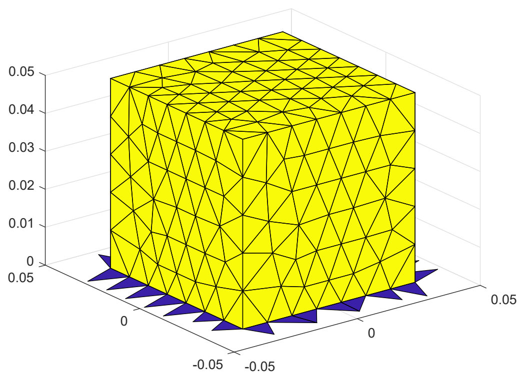


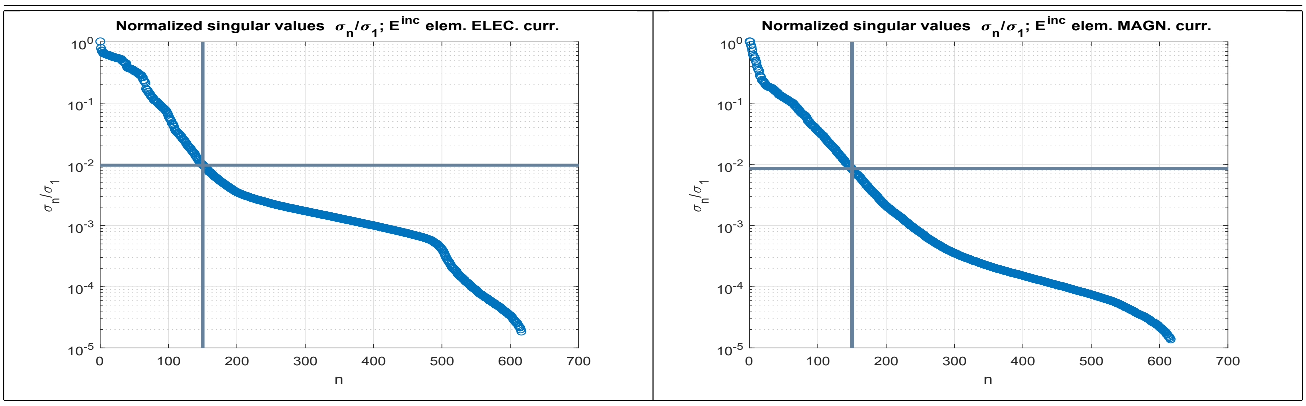
Figure 5.2. The box that encloses the antenna

near field samplings, for example with a uniform sampling step $\Delta\theta = \Delta\phi = 1$ degree, fixes one of the angles θ or ϕ and then varies the second one. For example when it acquires measures at the equator it fixes $\theta = 90$ degrees and then varies $\phi = 0, ..., 359$ degrees. Through this procedure when it measures the field $\mathbf{E}^{NF}$ at north pole it takes 360 measures for $\theta = 0$ and $\phi = 0, ..., 359$ (the same situation happens at south pole). It is clear that

$\mathbf{E}^{NF}$ is projected onto different versors $\hat{\theta}, \hat{\phi}$ but it is always the same vector $\mathbf{E}^{NF}$. The problem is that in the case of a measured field $\mathbf{E}^{NF}$ each measure can be different from an other one due to the presence of noise (the different configuration of the measuring system, etc...).

SVD decomposition

We apply an SVD decomposition to the matrix of incident fields $\mathbf{E}$ and we truncate the decomposition at $k = 150$ for both fields related to electric and magnetic elementary sources. Through this truncation we solve only 150 linear systems instead of 617.



Resolution of MOM

Each linear system arising by MOM

$$\mathbf{ZI} = \mathbf{V} \tag{5.2}$$

is solved with the iterative solver FGMRES coupled with a fast algorithm to evaluate matrix-vector product [5]. We fix a threshold of $\epsilon_0 = 10^{-3}$ and we allow a solution that generates a relative error smaller than ϵ_0 i.e. FGMRES stops when it finds a solution $\mathbf{I}$ such that

$$\frac{||\mathbf{ZI} - \mathbf{V}||}{||\mathbf{V}||} < \epsilon_0 \tag{5.3}$$

Reconstruction

We reconstruct the electric field $\mathbf{E}^{tgt}$ considering different sampling steps; obviously increasing the sampling step we impose equality between reference and reconstructed field in a smaller number of points so we expect that the error increases as the sampling step grows. The considered sampling steps are: 3, 6, 9, 12, 15, 18, 30, 36, 45 degrees. Built our numerical basis $\{\boldsymbol{\psi}_i\}$ we solve the following least squares problem through LSQR

$$\min_{\{\beta_i\}} || \sum_i \beta_i \boldsymbol{\psi}_i^{NF} - \mathbf{E}^{NF} || \tag{5.4}$$

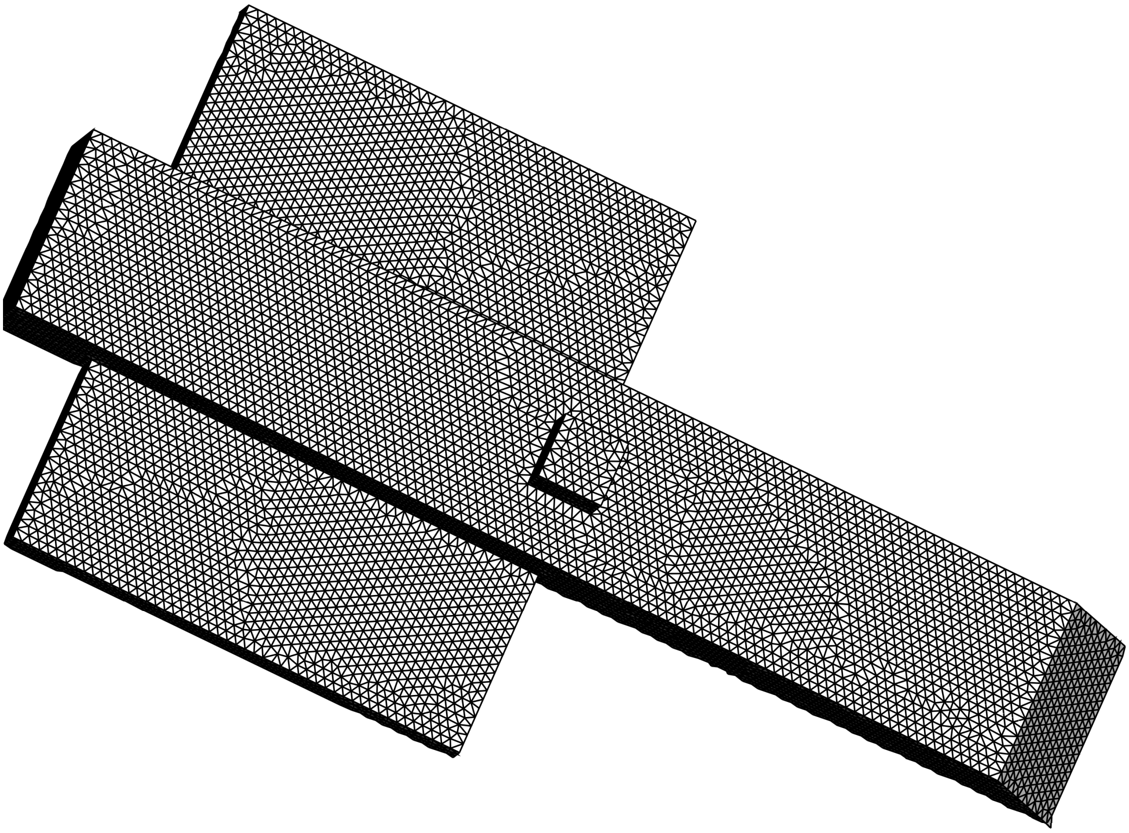


Figure 5.3. Mesh of the plane and the box

where $\mathbf{E}^{NF}$ is the vector of the measured near field samplings. We report the singular values of the matrix $\boldsymbol{\psi}^{NF} = [\boldsymbol{\psi}_i^{NF}]$ and the relative residual

$$\frac{||\boldsymbol{\psi}^{NF}\boldsymbol{\beta}_n - \mathbf{E}^{NF}||}{||\mathbf{E}^{NF}||} \tag{5.5}$$

at iteration n of LSQR.

Found a possible choice of coefficients $\{\overline{\beta_i}\}$ we build the reconstructed far field as

$$\mathbf{E}^{rec} = \sum_i \overline{\beta_i}\boldsymbol{\psi}_i^{FF} \tag{5.6}$$

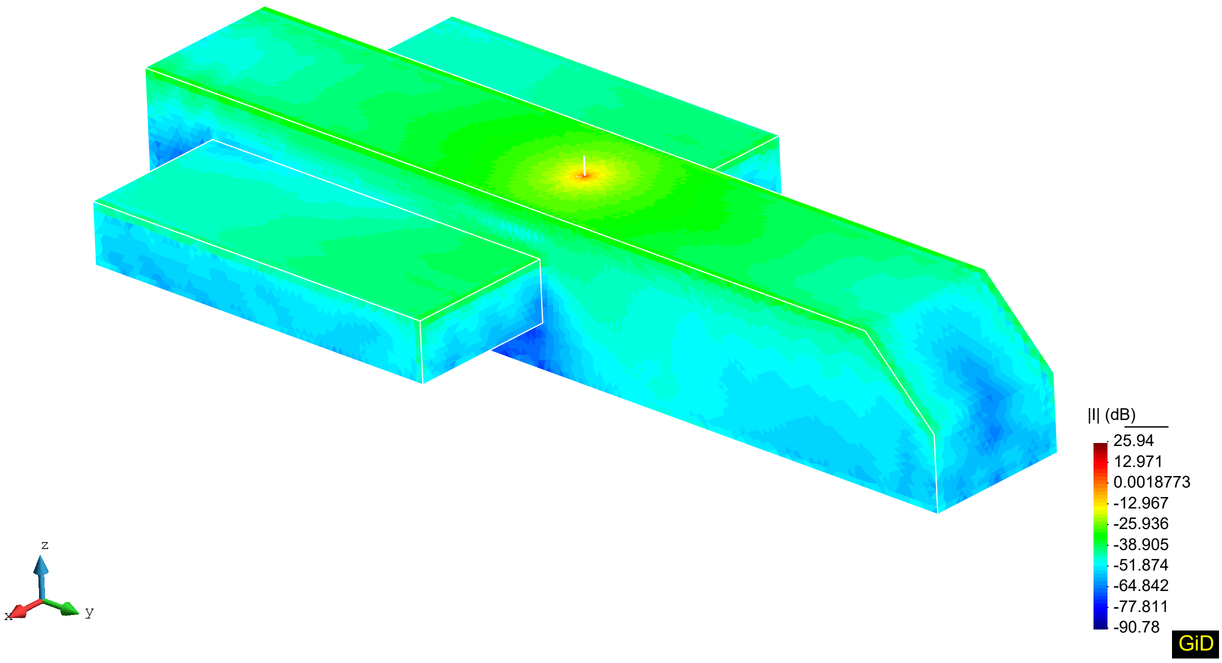
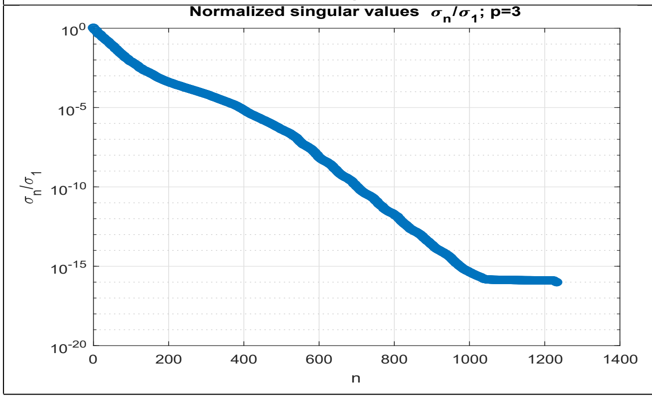
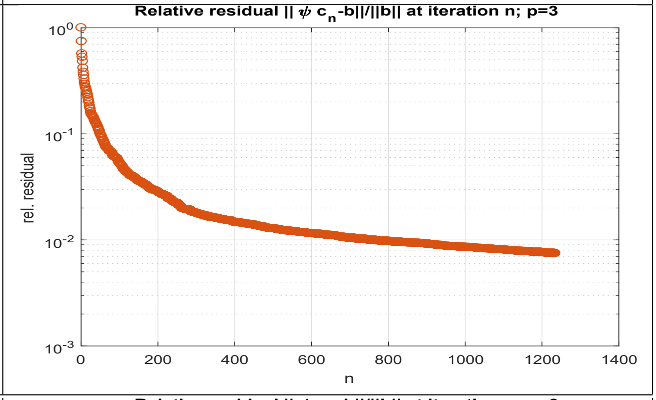
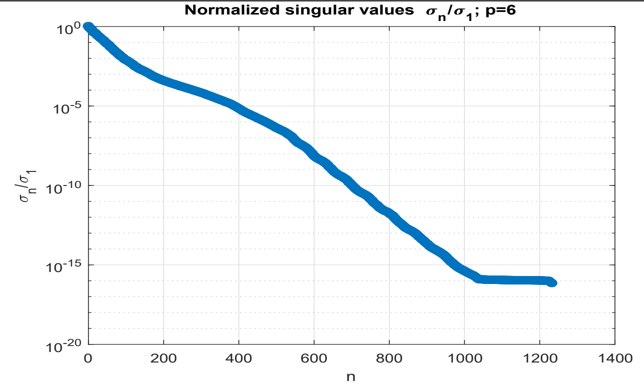


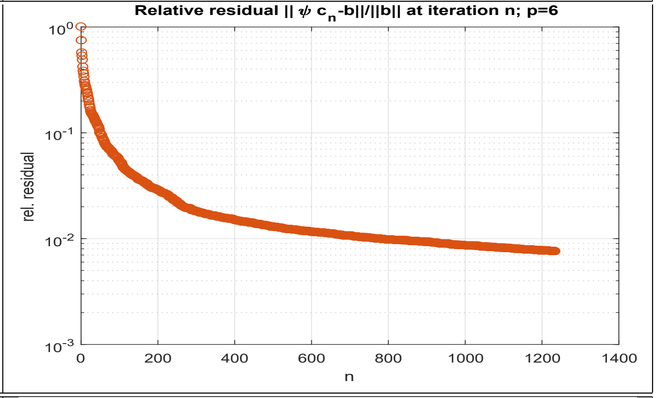
Figure 5.4. Distribution of the electric current generated by the dipole over the plane

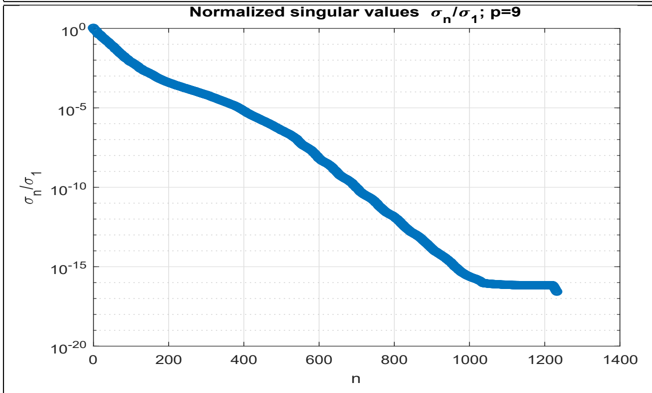
Normalized singular values	Relative residual

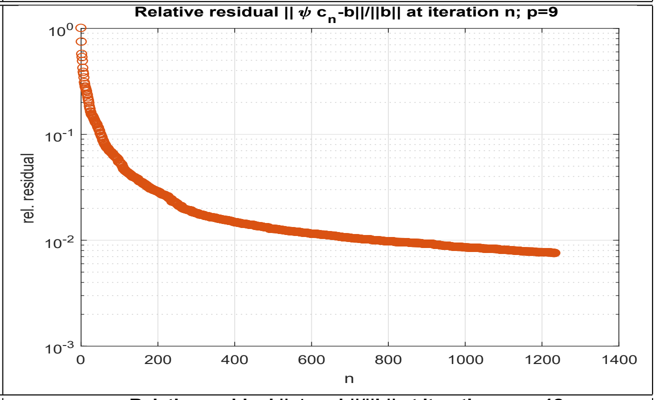


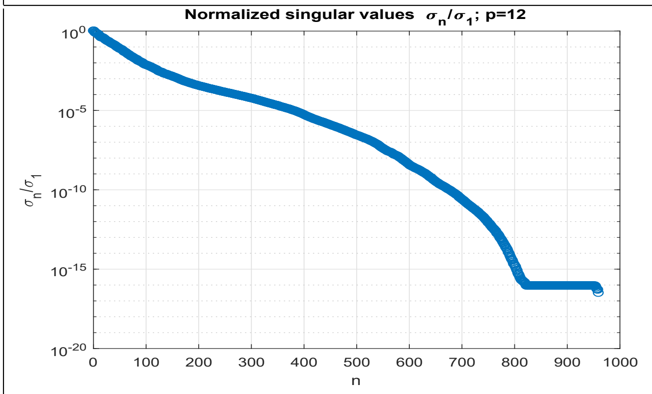


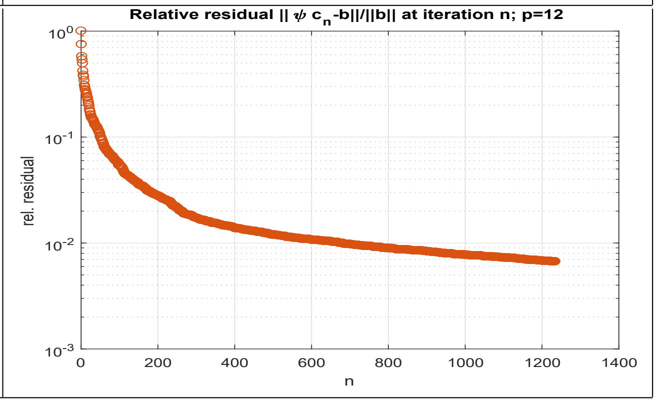




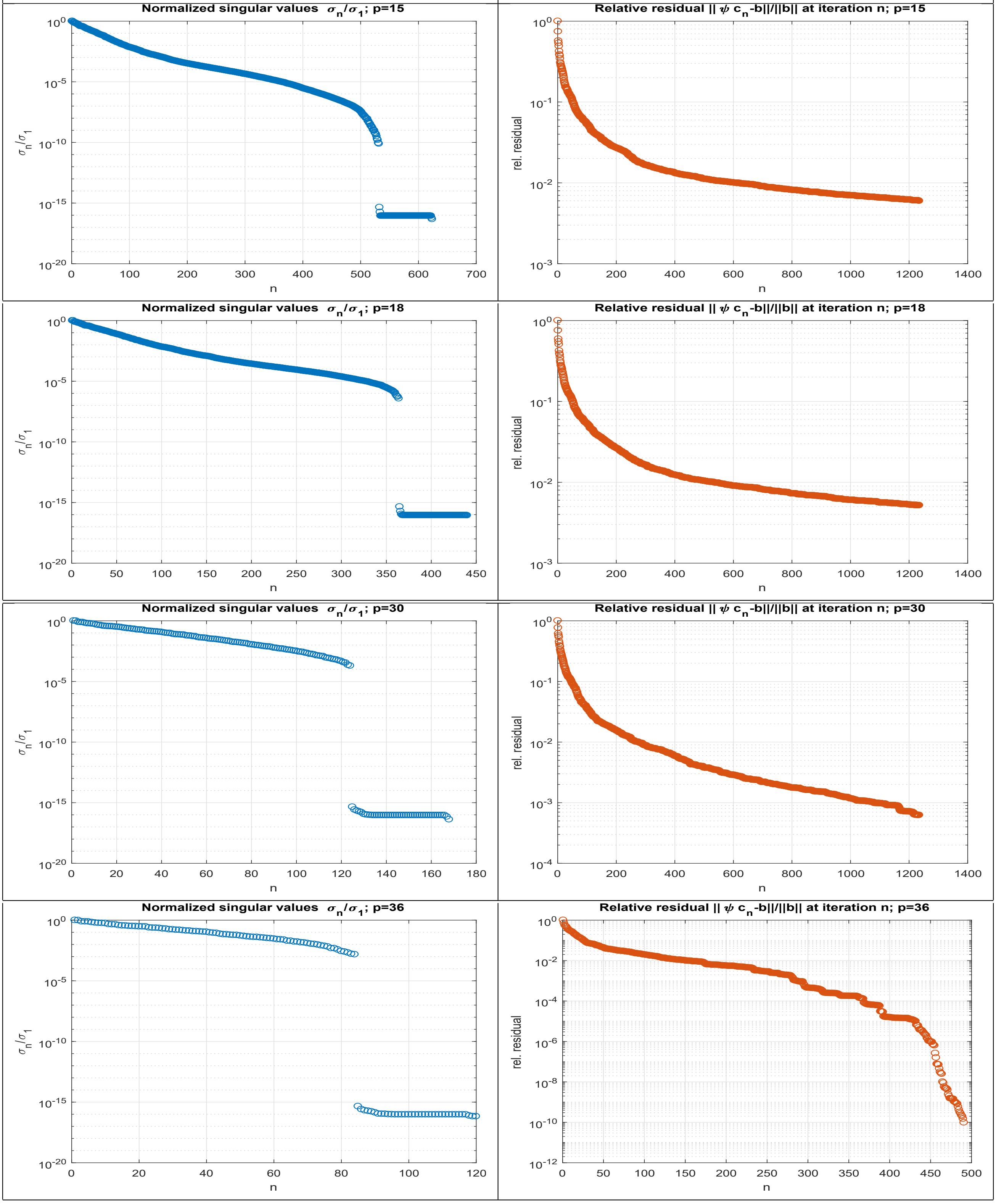


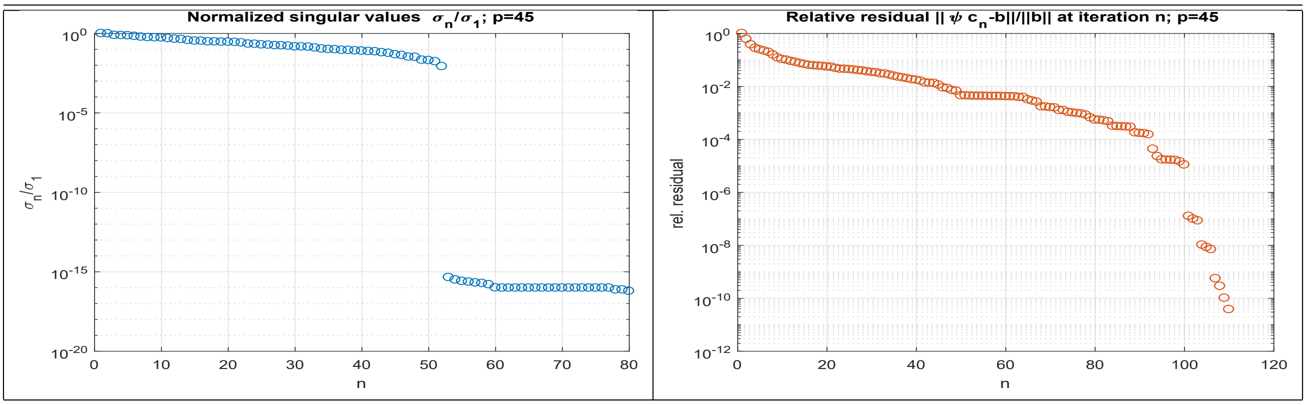






Normalized singular values	Relative residual





Far Field error

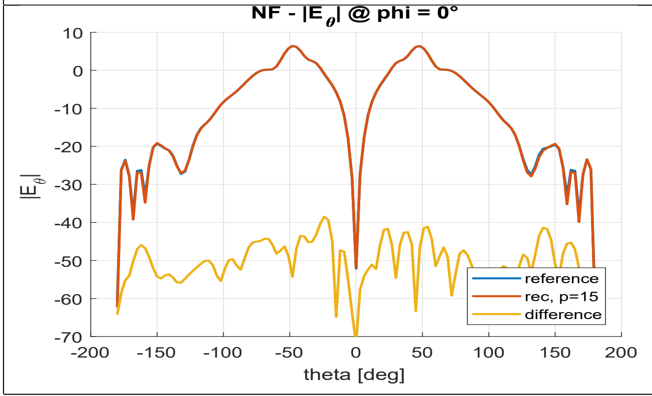
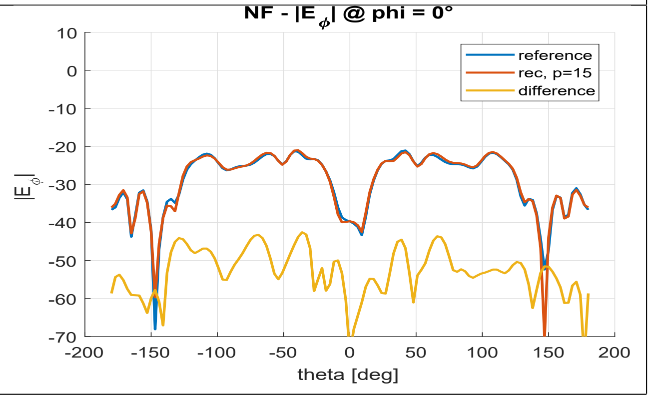
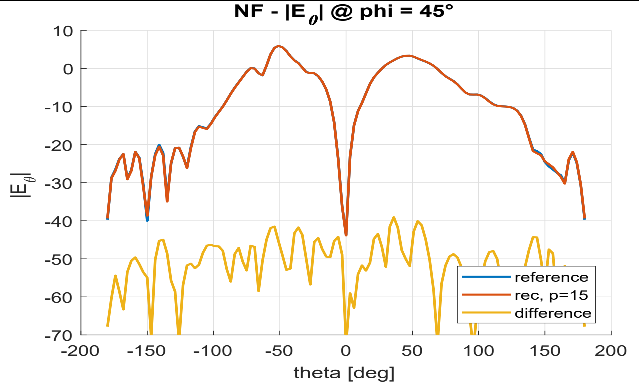
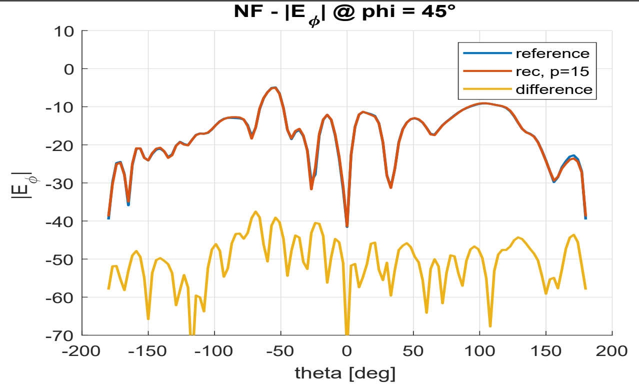
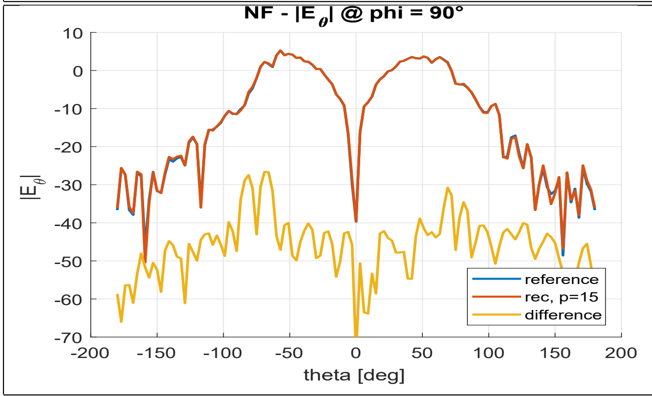
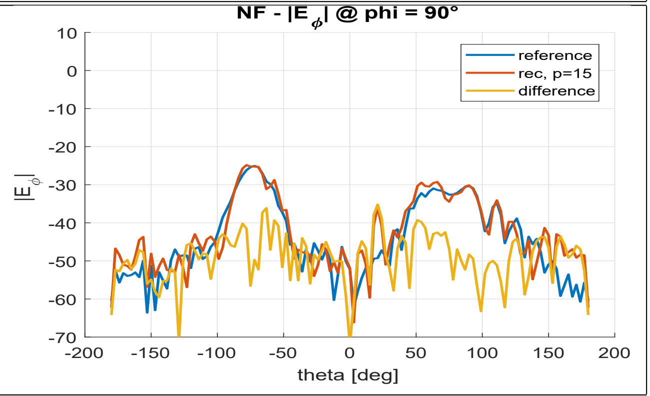
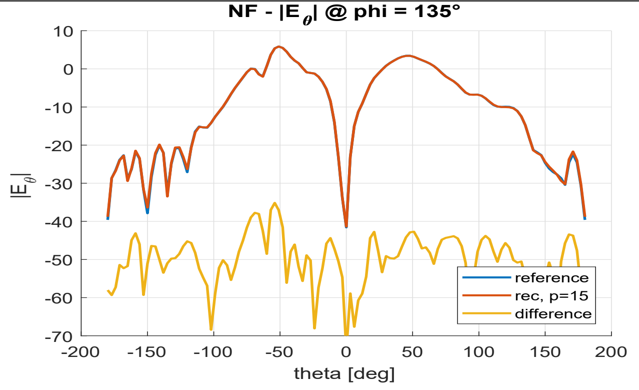
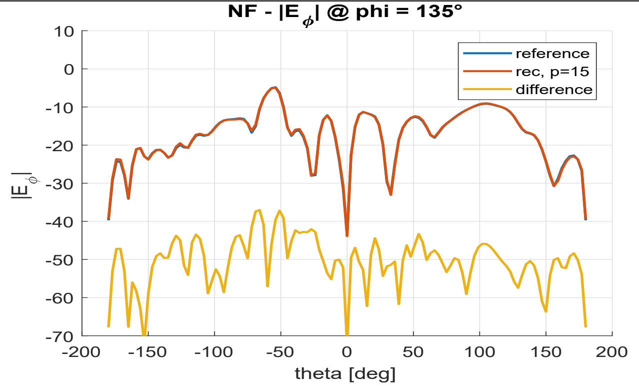
The reconstruction error considered, on the far field, is the relative error with the norm of $L^2(S^2)$ for each tangential component E_θ, E_ϕ of the far electric field i.e.

$$e_w = \frac{||E_w^{tgt} - E_w^{rec}||_{L^2(S^2)}}{||E_w^{tgt}||_{L^2(S^2)}} \tag{5.7}$$

where $w = \theta, \phi$ and

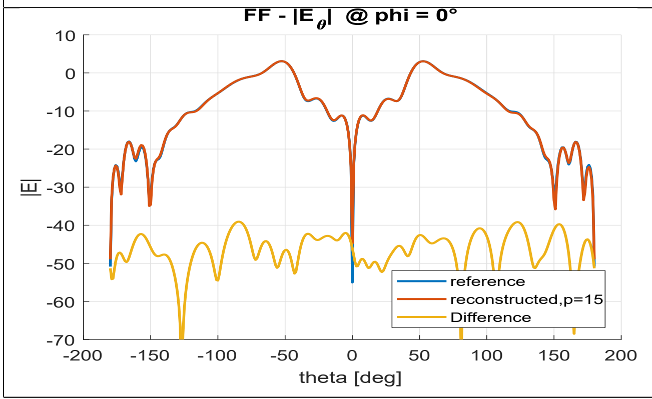
$$||E_w||_{S^2}^2 = \int_0^{2\pi} \int_0^{\pi} |E_w(\theta, \phi)|^2 \sin(\theta) d\theta d\phi \tag{5.8}$$

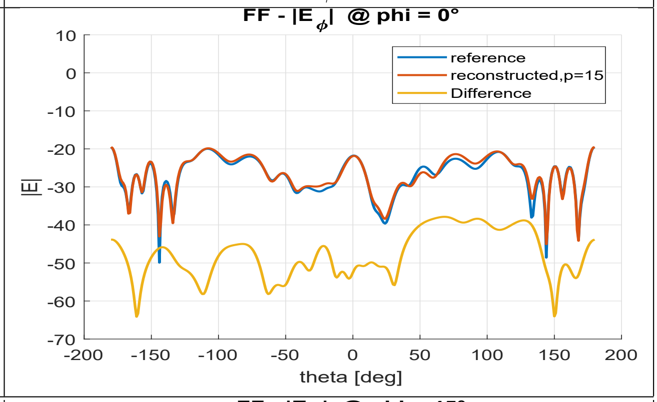
The far field is sampled with a uniform step $\Delta\theta = \Delta\phi = 1$ degree. In the following we show some plot of the reference and reconstructed near field and some plot of the reference and reconstructed far field with a scale $20\log_{10}$ for y-axis. The reconstrution is made in near field with a sampling step of $\Delta\theta = \Delta\phi = 15$ degrees.

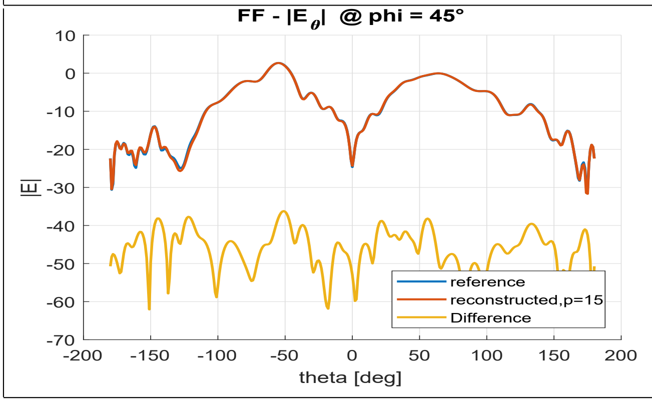
θ	ϕ
	
	
	
	

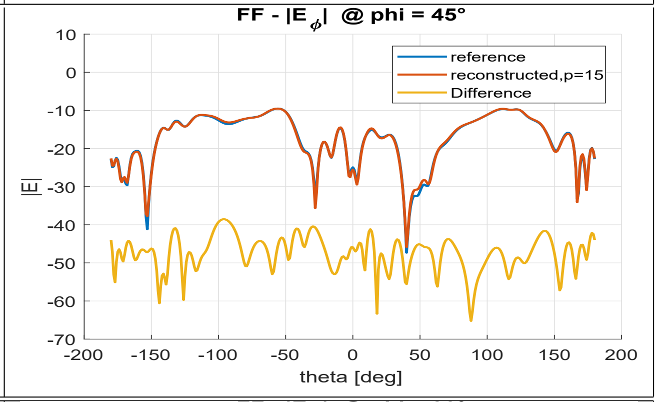
NF - |E_θ| @ phi = 0°
NF - |E_ϕ| @ phi = 0°
NF - |E_θ| @ phi = 45°
NF - |E_ϕ| @ phi = 45°
NF - |E_θ| @ phi = 90°
NF - |E_ϕ| @ phi = 90°
NF - |E_θ| @ phi = 135°
NF - |E_ϕ| @ phi = 135°
|E_θ|
|E_ϕ|
theta [deg]
reference
rec, p=15
difference

θ	ϕ


FF - |E$_\theta$| @ phi = 0°
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\phi$| @ phi = 0°
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\theta$| @ phi = 45°
|E|
theta [deg]
reference
reconstructed,p=15
Difference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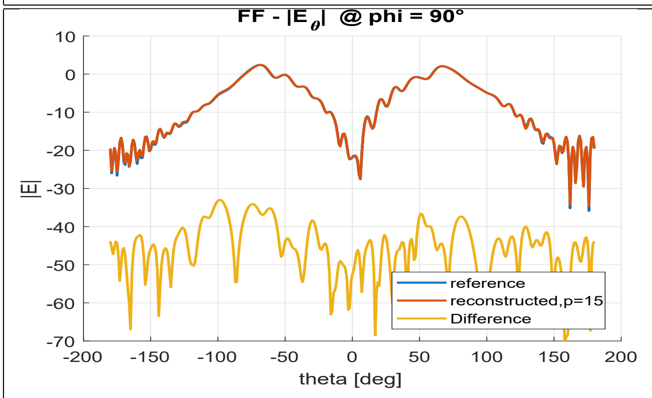
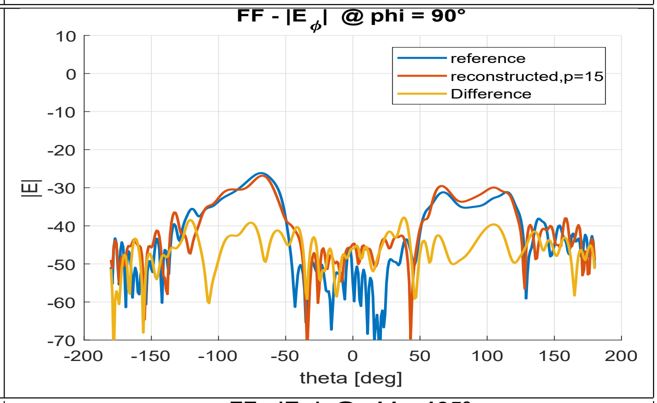
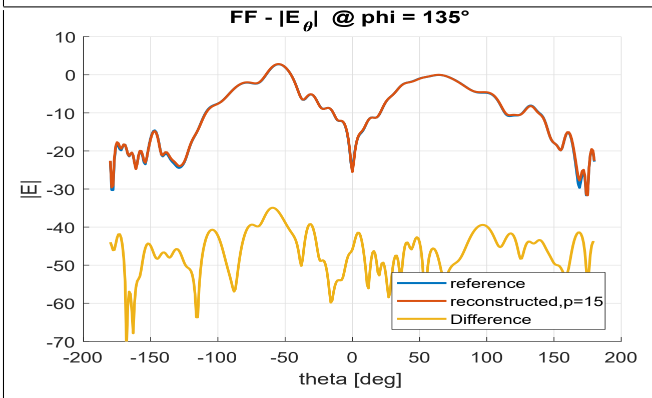
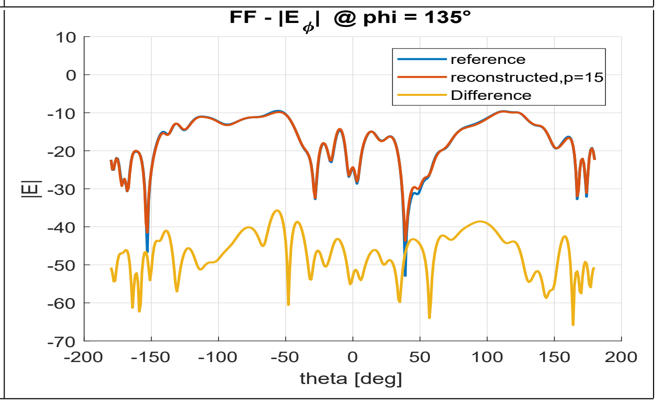
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\theta$| @ phi = 90°
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\phi$| @ phi = 90°
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\theta$| @ phi = 135°
|E|
theta [deg]
reference
reconstructed,p=15
Difference


FF - |E$_\phi$| @ phi = 135°
|E|
theta [deg]
reference
reconstructed,p=15
Difference

Noise

We studied the response of the method to introduction of noise in near field samplings. We reconstructed the field through NF corrupted samplings and then we compute the reconstruction error with respect to the original noise-free far field $\mathbf{E}^{tgt}$. We corrupt NF samplings with noise with signal to noise ratio (SNR) levels of 40 and 30 dB. More precisely if $\mathbf{E}^{NF} \in \mathbb{C}^N$ is the vector of NF-samplings we consider Gaussian vectors $\mathbf{X} = (X_1, ..., X_N)$ and $\mathbf{Y} = (Y_1, ..., Y_N)$ where $X_1, ..., X_N, Y_1, ..., Y_N \sim N(0,1)$ are i.i.d standard normal random variables. Then we define the noise $\mathbf{R}$ as

$$\mathbf{R} = 10^{-snr/20} ||\mathbf{E}^{NF}|| \frac{\mathbf{X} + j\mathbf{Y}}{||\mathbf{X} + j\mathbf{Y}||} \tag{5.9}$$

where j is the imaginary unit and $snr = 40$, 30. Through this choice of the noise $\mathbf{R}$ we have that

$$10 \log_{10} \frac{||\mathbf{E}^{NF}||^2}{||\mathbf{R}||^2} = snr \tag{5.10}$$

where $||\mathbf{E}^{NF}||^2$ represents an approximation of the power of the electric near field and $||\mathbf{R}||^2$ of the power of the noise. The corrupted NF is defined as $\mathbf{E}^{noise} := \mathbf{E}^{NF} + \mathbf{R}$.

We solve the least squares system

$$\min_{\{\beta_n\}} || \sum_n \beta_n \boldsymbol{\psi}_n^{NF} - \mathbf{E}^{noise} || \tag{5.11}$$

where $\{\boldsymbol{\psi}_n^{NF}\}$ is our numerical basis described in previous chapters. Found a possible choice of coefficients $\{\overline{\beta_n}\}$ we evaluate the relative error

$$e_w = \frac{||E_w^{tgt} - E_w^{rec}||_{L^2(S^2)}}{||E_w^{tgt}||_{L^2(S^2)}} \qquad ||E_w||_{S^2}^2 = \int_0^{2\pi} \int_0^{\pi} |E_w(\theta, \phi)|^2 \sin(\theta) d\theta d\phi \tag{5.12}$$

where $w = \theta, \phi$ and

$$\mathbf{E}^{rec} = \sum_n \overline{\beta_n} \boldsymbol{\psi}_n^{FF} \tag{5.13}$$

In the following we compare noise-free reconstruction and SNR-reconstruction through some far field cuts. The sampling step considered is $\Delta\theta = \Delta\phi = 18$ degrees for both noise free and snr reconstruction. The scale for $y-$axis is $20 \log_{10}$. We also report in the plot of relative error two vertical lines to show the Nyquist limit for the mounted antenna (3.5 degrees) and the isolated antenna (14 degrees).

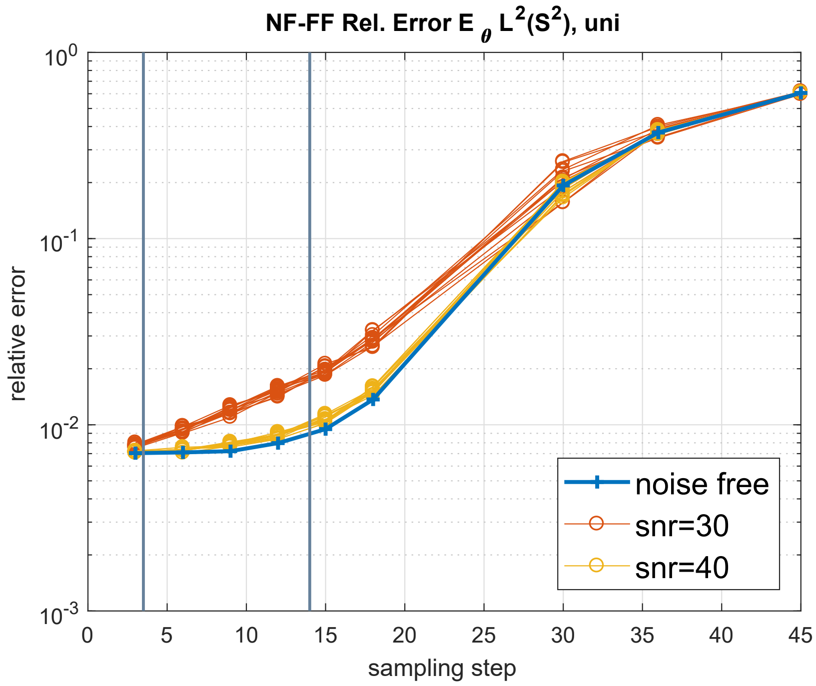


Figure 5.5. Relative error for the θ component

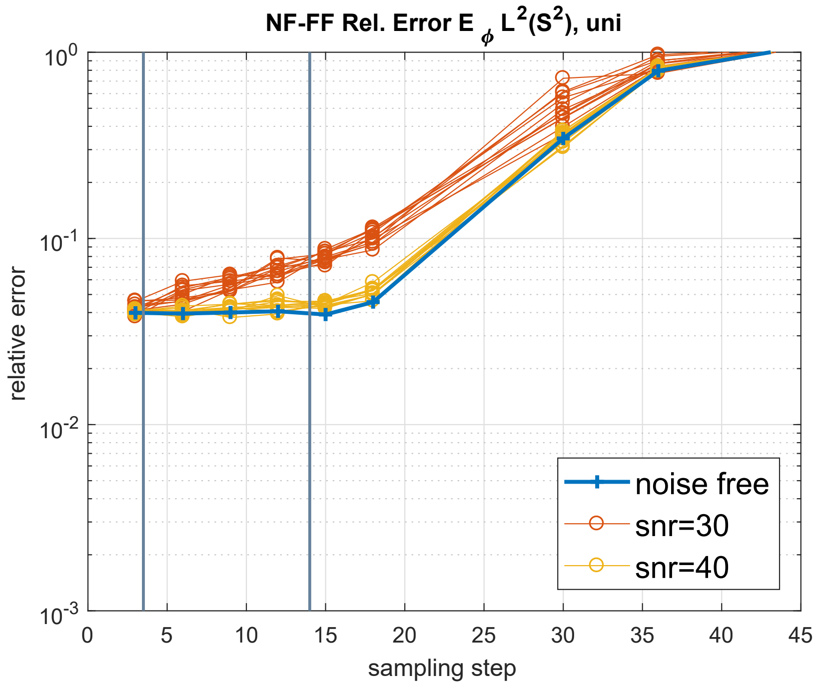
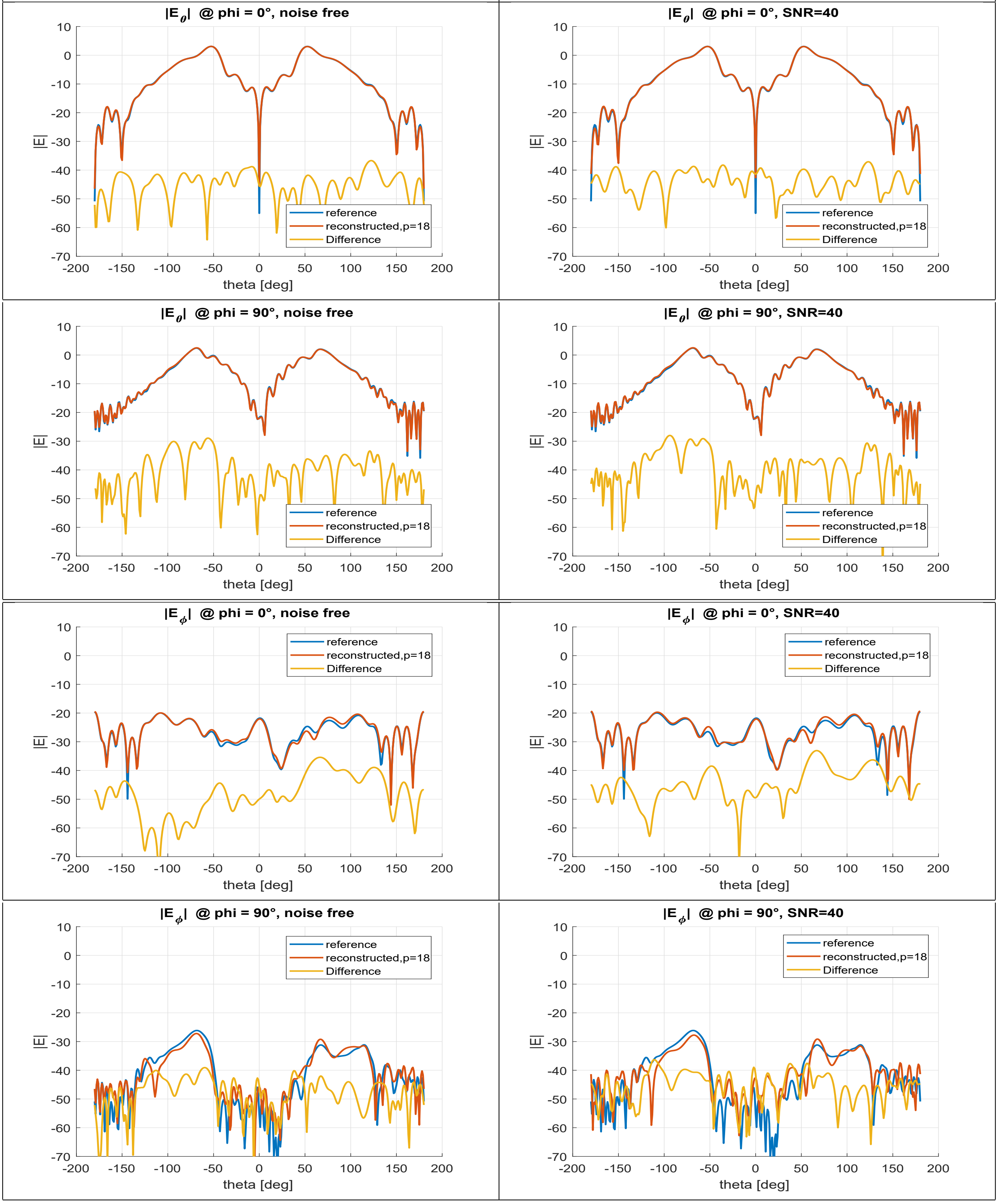
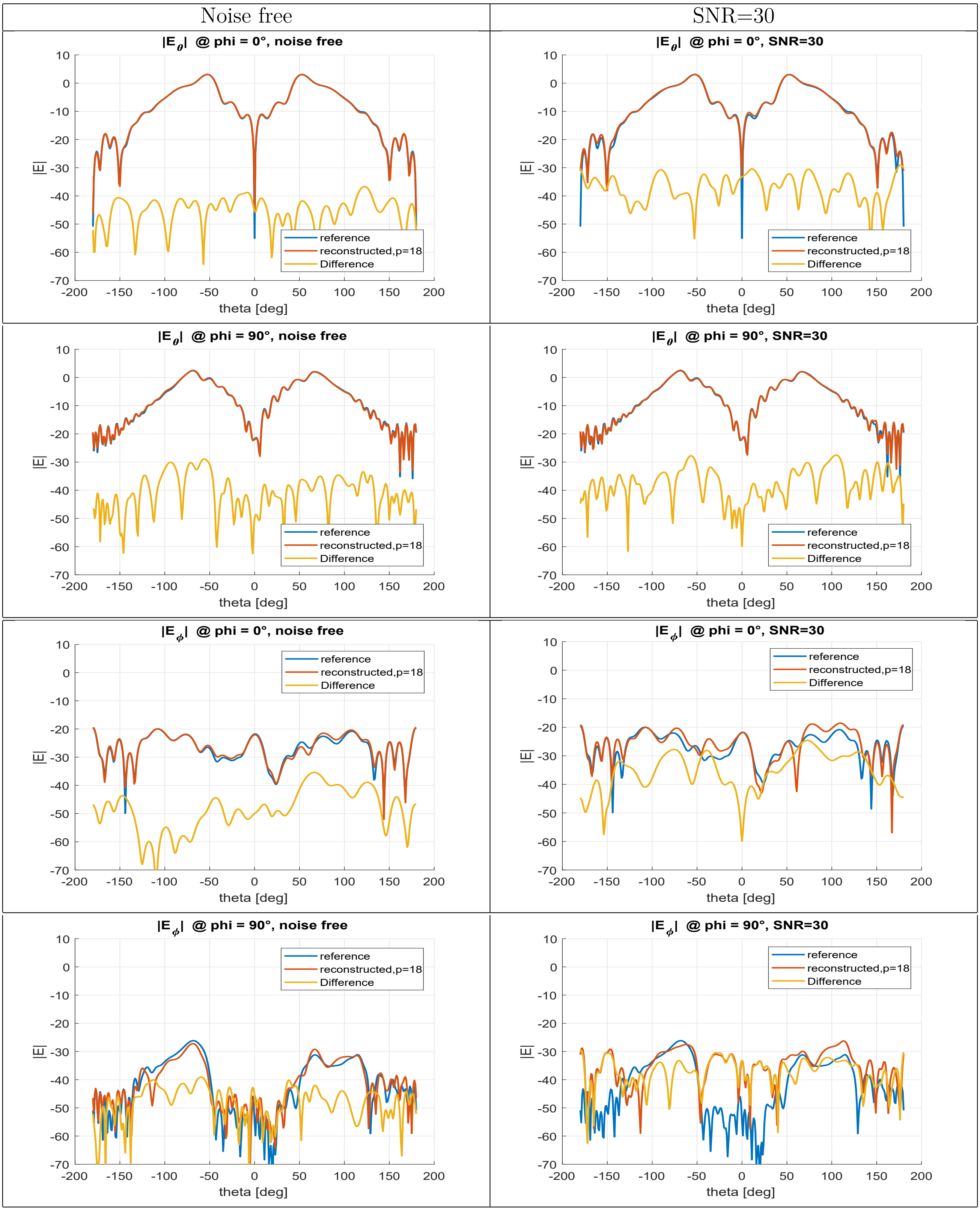


Figure 5.6. Relative error for the ϕ component

Noise free	SNR=40




Noise free
SNR=30
|E_θ| @ phi = 0°, noise free
|E_θ| @ phi = 0°, SNR=30
|E_θ| @ phi = 90°, noise free
|E_θ| @ phi = 90°, SNR=30
|E_φ| @ phi = 0°, noise free
|E_φ| @ phi = 0°, SNR=30
|E_φ| @ phi = 90°, noise free
|E_φ| @ phi = 90°, SNR=30
reference
reconstructed,p=18
Difference
|E|
theta [deg]

Method 2

We now reconstruct the target field with different coefficients for the scattered fields and fields in isolation. We also use the field of the antenna in isolation $\mathbf{E}_0$. We solve

$$\min_{\alpha,\{\beta_n\},\{\gamma_n\},\{\xi_n\},\{\eta_n\}} ||\alpha\mathbf{E}_0+\sum_{n=1}^{N}\beta_n\boldsymbol{\psi}_n^{0,NF}+\sum_{n=1}^{N}\gamma_n\boldsymbol{\phi}_n^{0,NF}+\sum_{n=1}^{N}\xi_n\boldsymbol{\psi}_n^{s,NF}+\sum_{n=1}^{N}\eta_n\boldsymbol{\phi}_n^{s,NF}-\mathbf{E}^{NF}|| \tag{5.14}$$

and then found a possible choice $\overline{\alpha}, \{\overline{\beta_n}\}, \{\overline{\gamma_n}\}, \{\overline{\xi_n}\}, \{\overline{\eta_n}\}$ of the coefficients we build the reconstructed far electric field $\mathbf{E}^{rec}$ as

$$\mathbf{E}^{rec}=\overline{\alpha}\mathbf{E}_0+\sum_{n=1}^{N}\overline{\beta_n}\boldsymbol{\psi}_n^{0,FF}+\sum_{n=1}^{N}\overline{\gamma_n}\boldsymbol{\phi}_n^{0,FF}+\sum_{n=1}^{N}\overline{\xi_n}\boldsymbol{\psi}_n^{s,FF}+\sum_{n=1}^{N}\overline{\eta_n}\boldsymbol{\phi}_n^{s,FF} \tag{5.15}$$

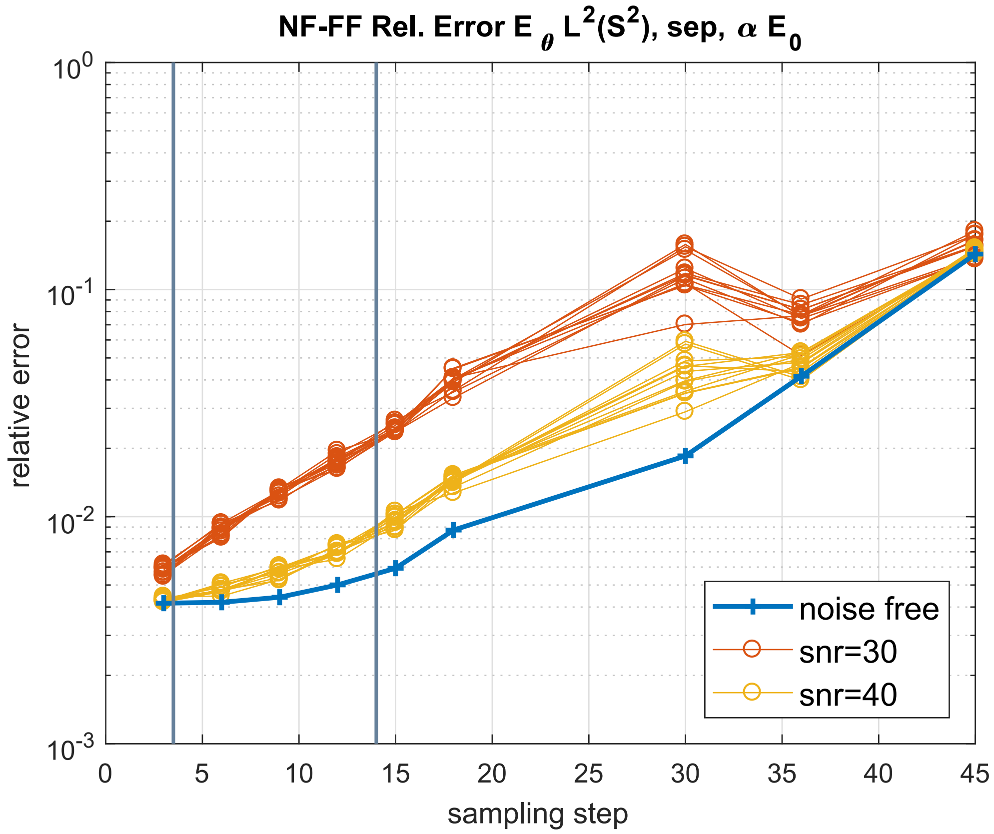


Figure 5.7. Relative error for θ component

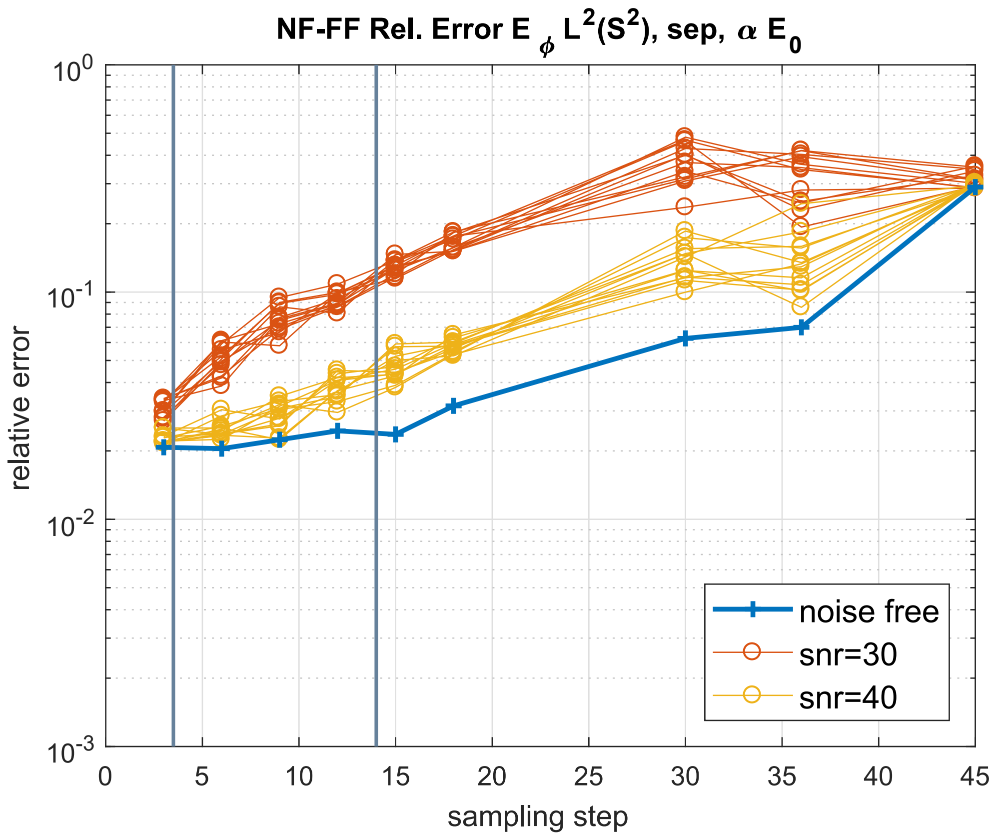
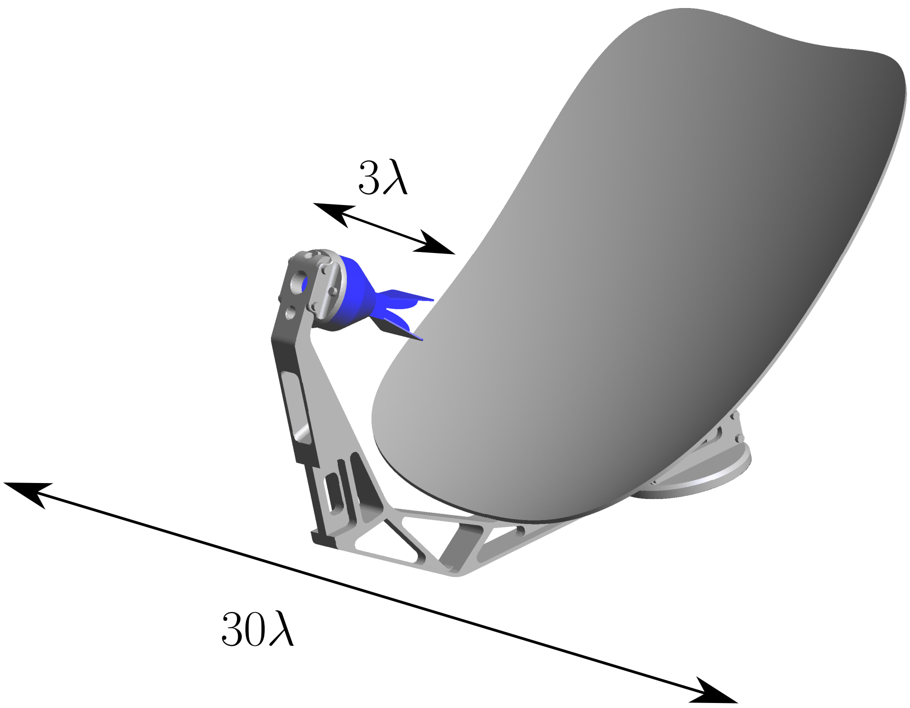


Figure 5.8. Relative error for ϕ component

Chapter 6

Reflector 8GHz

In this chapter we reconstruct the far field of a parabolic antenna. The reference field $\mathbf{E}^{tgt}$ produced by the antenna is obtained through a simulation so the method is tested on synthetic data. The frequency of the antenna is f = 8GHz. The minimum sphere that enclose the whole structure has radius $r = 0.55m \simeq 15\lambda$ with $\lambda = 0.0375m$. The lower



bound on the number N of measures given by Nyquist criterion is $N = 4\pi r^2/(\lambda/2)^2 =$

10813 that corresponds to a sampling step $\Delta\phi = 2.4$ degrees; for this reason near field samples were measured with a sampling step $\Delta\phi = \Delta\theta = 1.125$ degrees at the sphere with radius $r = 2.516m$.

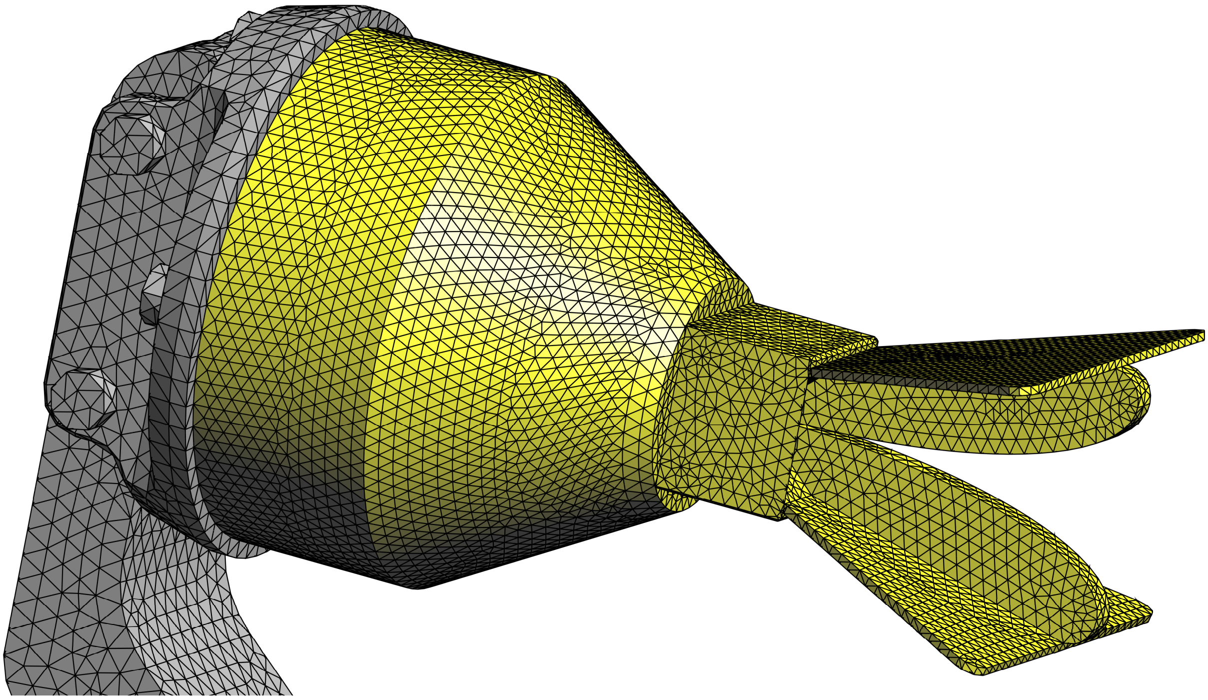


Figure 6.1. The mesh on the feed

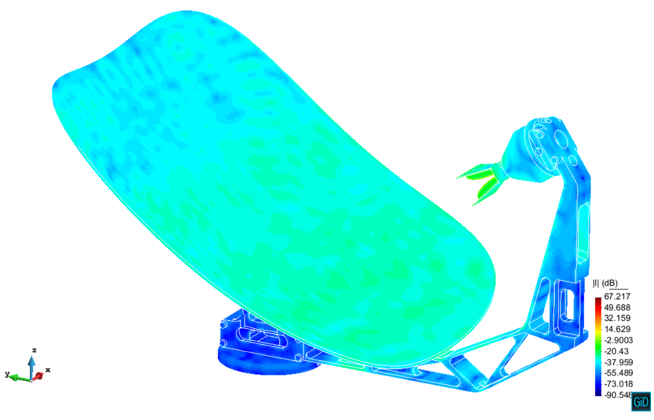


Figure 6.2. Distribution of the current on the structure

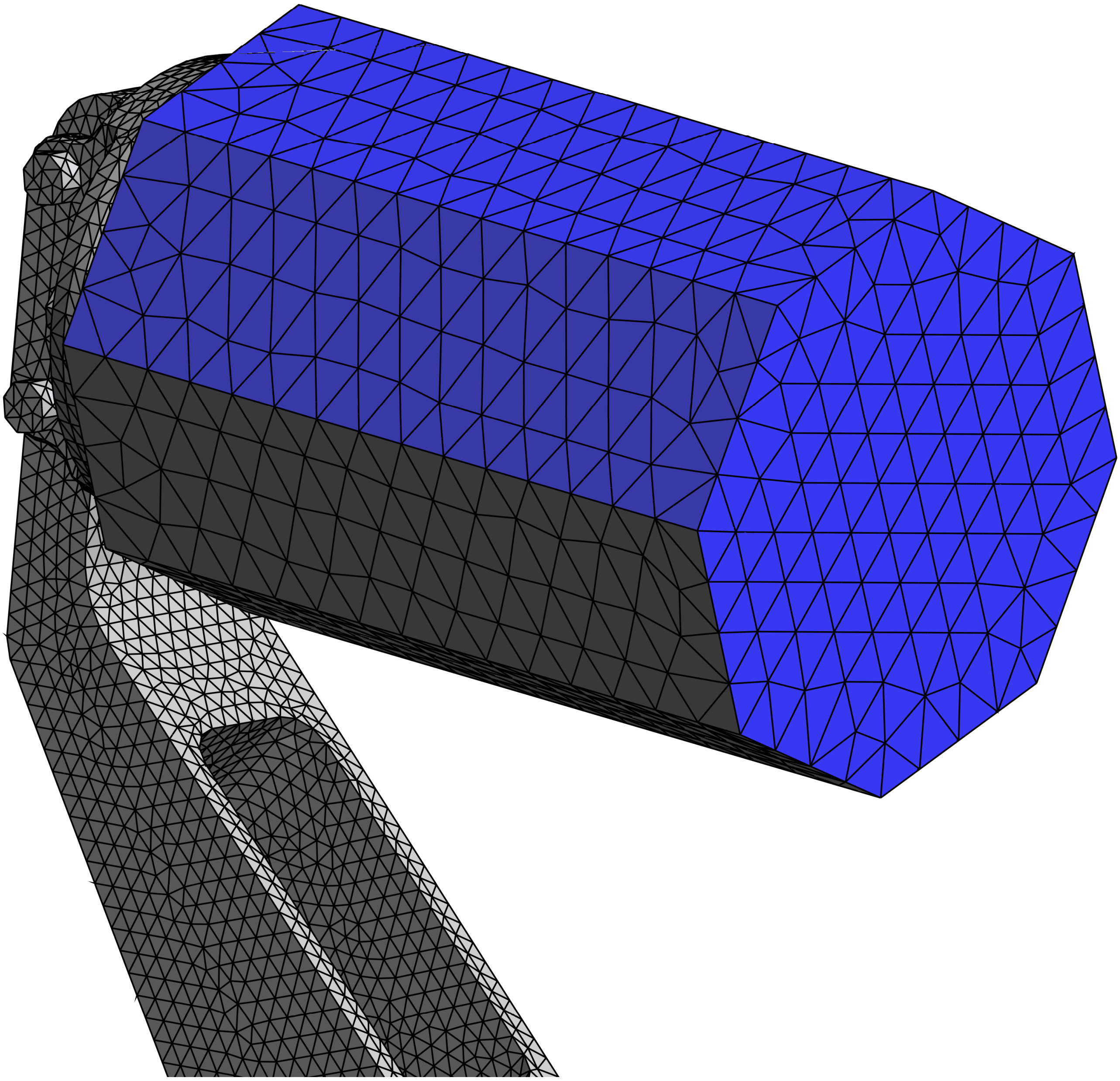
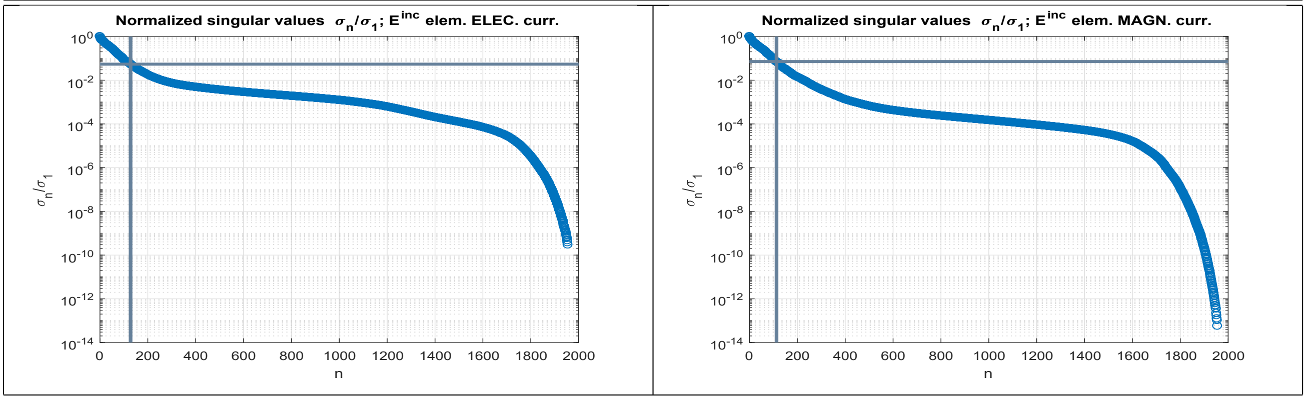


Figure 6.3. The box that encloses the antenna

SVD decomposition and MOM

We truncated the SVD of the incident fields due to elementary electric currents for $\epsilon_E = 5 \cdot 10^{-2}$ and $\epsilon_M = 7 \cdot 10^{-2}$ for incident fields due to elementary magnetic currents. The k such that $\sigma_k/\sigma_1 > \epsilon$ and $\sigma_{k+1}/\sigma_1 < \epsilon$ is $k_E = 128$ for electric sources and $k_M = 113$ for magnetic sources. Using this truncation we solved only $128 + 113 = 241$ linear systems instead of $1956 + 1956 = 3912$.



Reconstruction

We reconstructed the electric field $\mathbf{E}^{tgt}$ considering the following sampling steps: 1.125, 2.25, 4.5 ,5.625, 9, 11.25, 18, 22.5, 36 degrees. Computed the numerical basis $\{\boldsymbol{\psi}_i\}$ we solve the following least squares problem through LSQR

$$\min_{\{\beta_i\}} || \sum_i \beta_i \boldsymbol{\psi}_i^{NF} - \mathbf{E}^{NF} || \tag{6.1}$$

where $\mathbf{E}^{NF}$ is the vector of the measured near field samplings. We report the singular values of the matrix $\boldsymbol{\psi}^{NF} = [\boldsymbol{\psi}_i^{NF}]$ and the relative residual

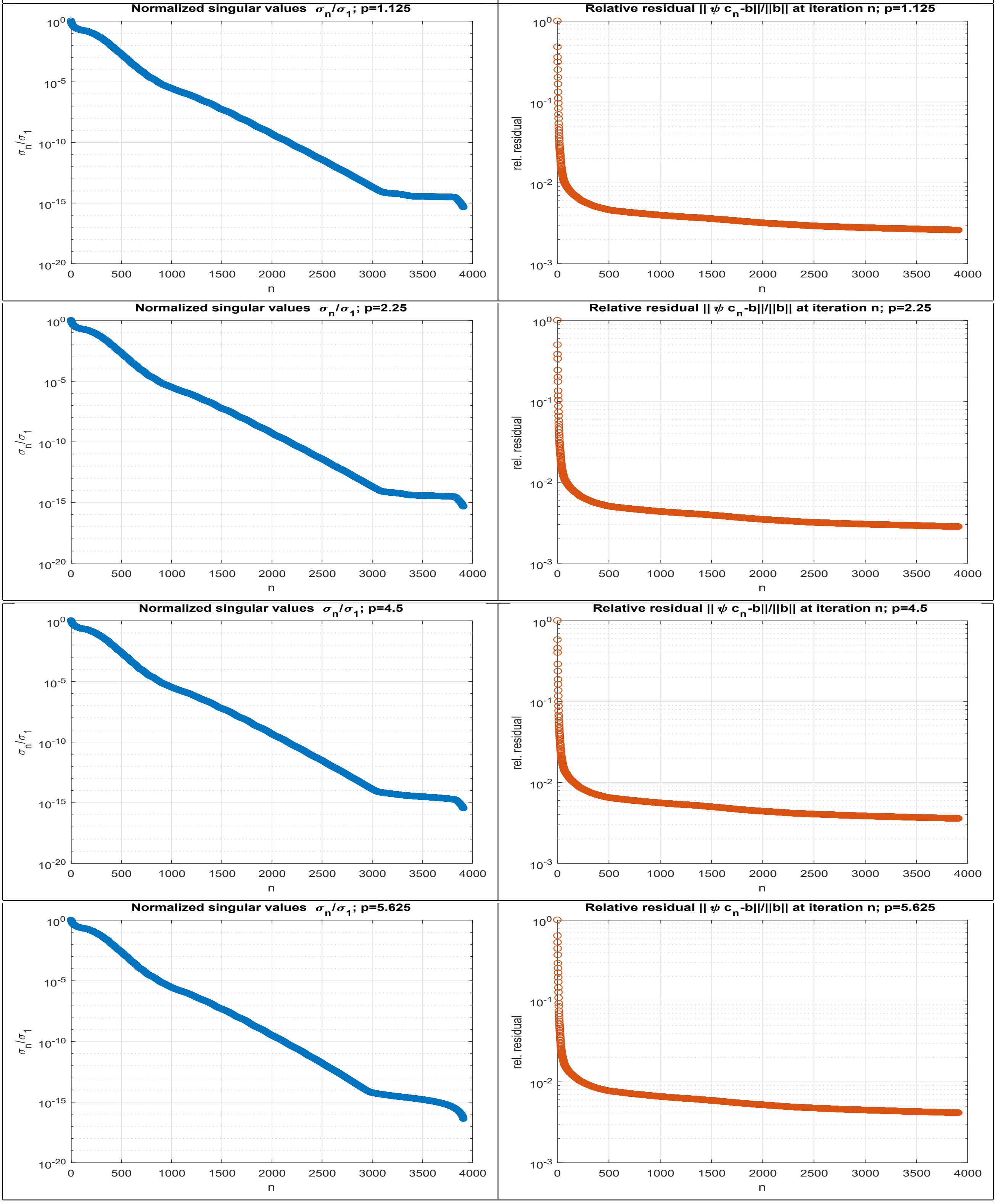
$$\frac{||\boldsymbol{\psi}^{NF}\boldsymbol{\beta}_n - \mathbf{E}^{NF}||}{||\mathbf{E}^{NF}||} \tag{6.2}$$

at iteration n of LSQR.

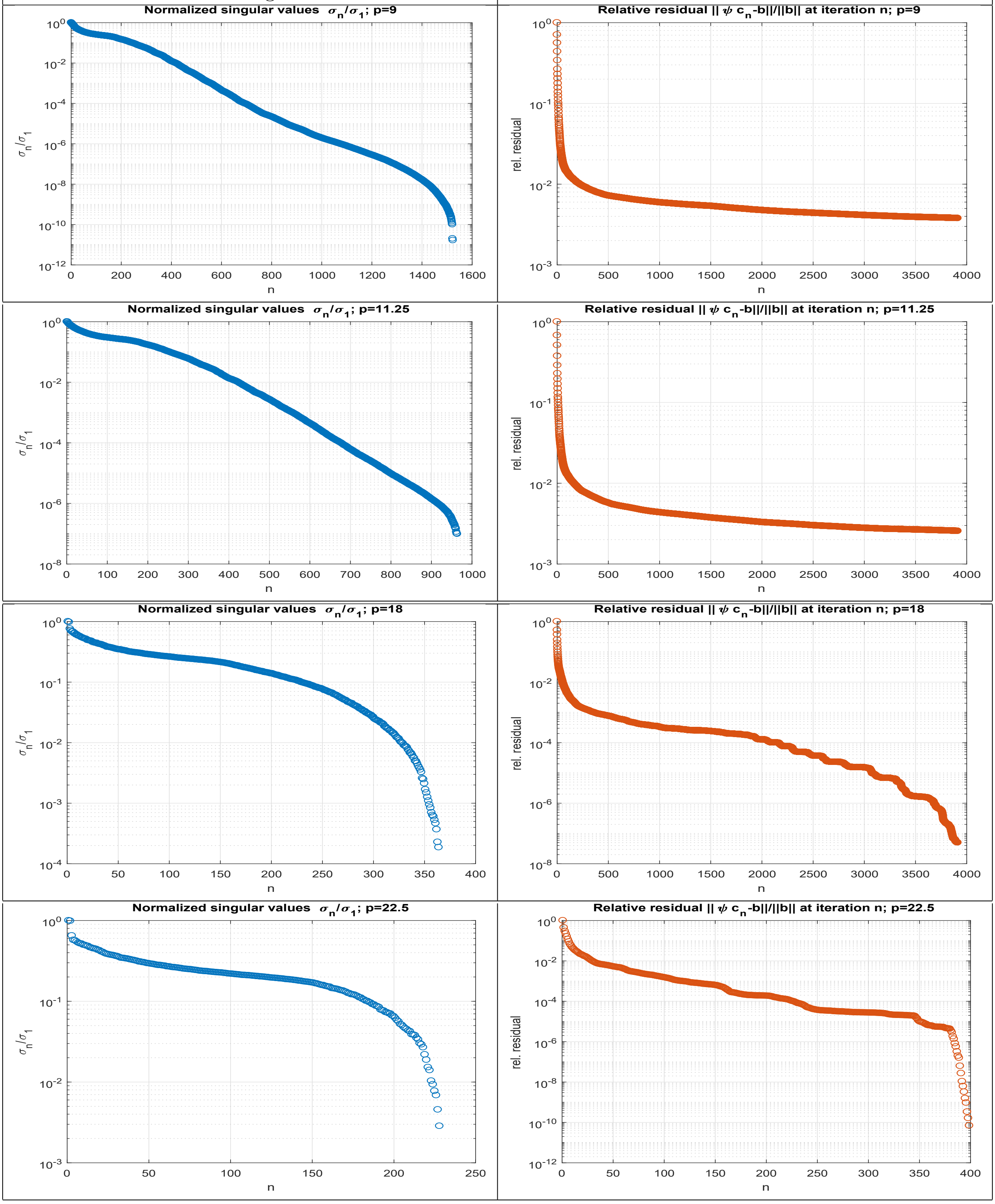
Found a possible choice of coefficients $\{\overline{\beta_i}\}$ we build the reconstructed far field as

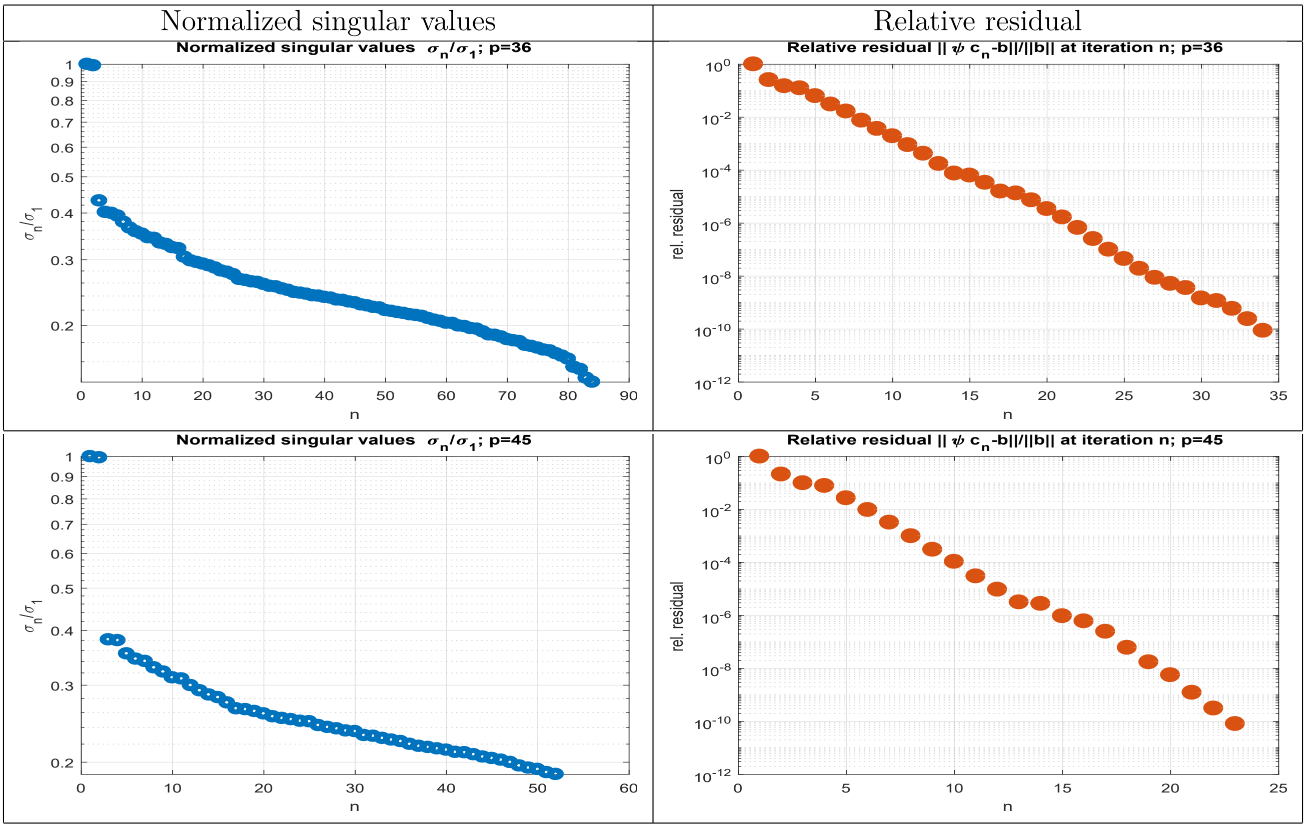
$$\mathbf{E}^{rec} = \sum_i \overline{\beta_i} \boldsymbol{\psi}_i^{FF} \tag{6.3}$$

Normalized singular values	Relative residual



Normalized singular values	Relative residual





Far Field Error

The reconstruction error considered, on the far field, is the relative error with the norm of $L^2(S^2)$ for each tangential component E_θ, E_ϕ of the far electric field i.e.

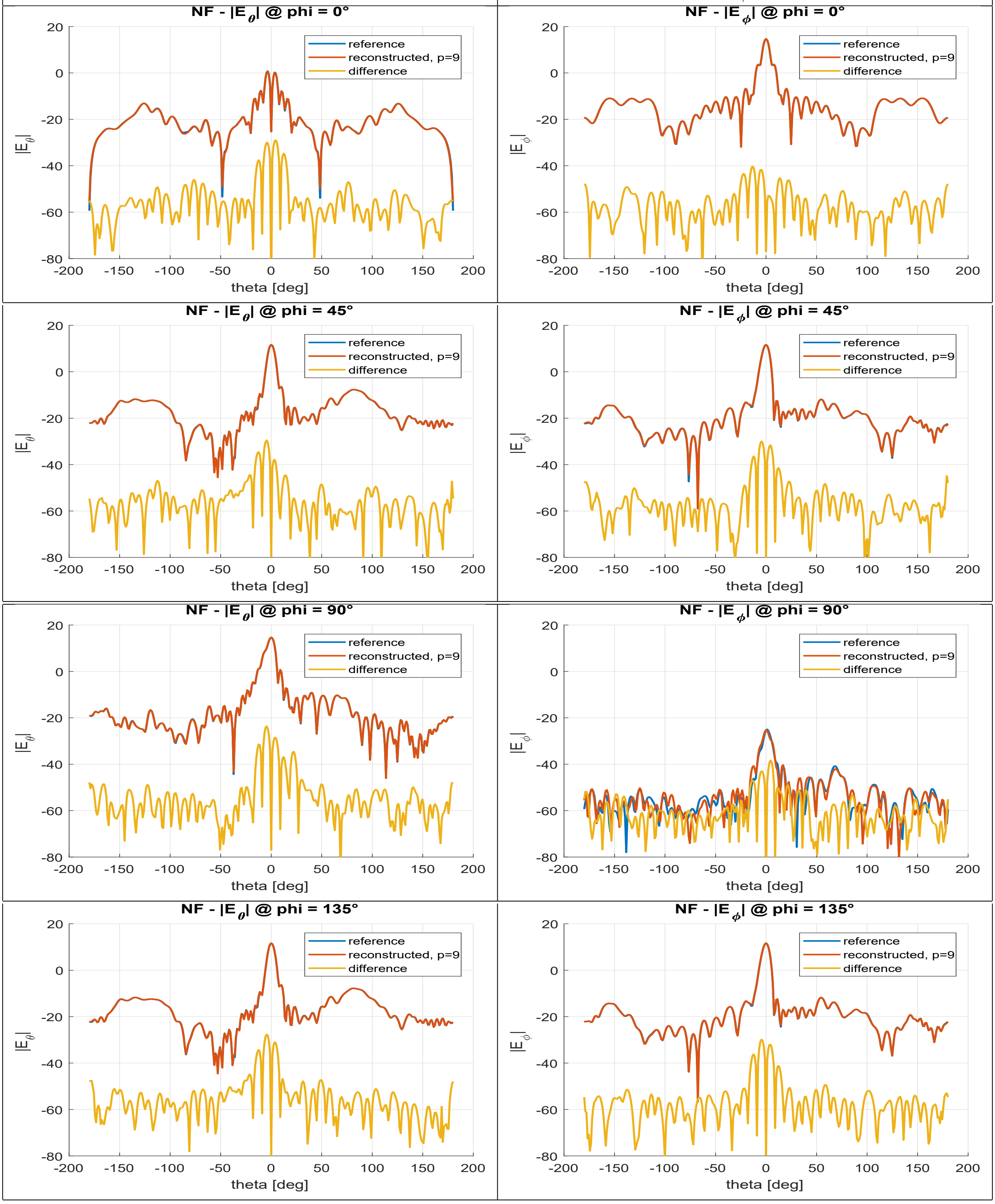
$$e_w = \frac{||E_w^{tgt} - E_w^{rec}||_{L^2(S^2)}}{||E_w^{tgt}||_{L^2(S^2)}} \tag{6.4}$$

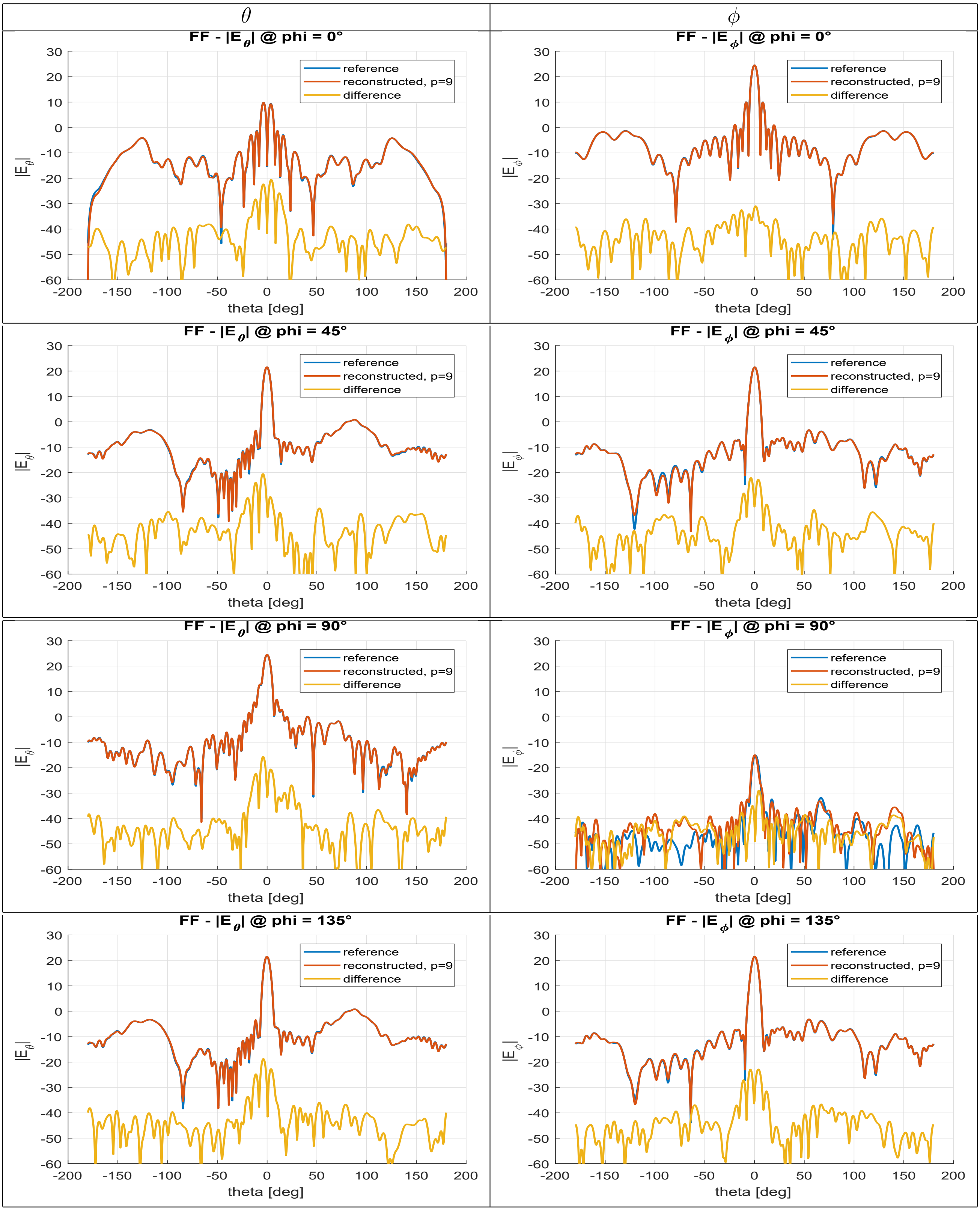
where $w = \theta, \phi$ and

$$||E_w||_{S^2}^2 = \int_0^{2\pi} \int_0^{\pi} |E_w(\theta,\phi)|^2 \sin(\theta) d\theta d\phi \tag{6.5}$$

The far field is sampled with a uniform step $\Delta\theta = \Delta\phi = 0.5$ degrees. In the following we show some plot of the reference and reconstructed near field and some plot of the reference and reconstructed far field with a scale $20\log_{10}$ for y-axis. The reconstrution is made in near field with a sampling step of $\Delta\theta = \Delta\phi = 9$ degrees.

θ	ϕ


NF - |E_θ| @ phi = 0°
NF - |E_φ| @ phi = 0°
NF - |E_θ| @ phi = 45°
NF - |E_φ| @ phi = 45°
NF - |E_θ| @ phi = 90°
NF - |E_φ| @ phi = 90°
NF - |E_θ| @ phi = 135°
NF - |E_φ| @ phi = 135°
reference
reconstructed, p=9
difference
|E_θ|
|E_φ|
theta [deg]
20
0
-20
-40
-60
-80
-200
-150
-100
-50
0
50
100
150
200


θ
ϕ
FF - |E_θ| @ phi = 0°
FF - |E_ϕ| @ phi = 0°
FF - |E_θ| @ phi = 45°
FF - |E_ϕ| @ phi = 45°
FF - |E_θ| @ phi = 90°
FF - |E_ϕ| @ phi = 90°
FF - |E_θ| @ phi = 135°
FF - |E_ϕ| @ phi = 135°
reference
reconstructed, p=9
difference
|E_θ|
|E_ϕ|
theta [deg]

6.1 Noise

We studied the response of the method to introduction of noise in near field samplings. We reconstructed the field through NF corrupted samplings and then we compute the reconstruction error with respect to the original noise-free far field $\mathbf{E}^{tgt}$. We corrupt NF samplings with noise with signal to noise ratio (SNR) levels of 40 and 30 dB. More precisely if $\mathbf{E}^{NF} \in \mathbb{C}^N$ is the vector of NF-samplings we consider Gaussian vectors $\mathbf{X} = (X_1, ..., X_N)$ and $\mathbf{Y} = (Y_1, ..., Y_N)$ where $X_1, ..., X_N, Y_1, ..., Y_N \sim N(0,1)$ are i.i.d standard normal random variables. Then we define the noise $\mathbf{R}$ as

$$\mathbf{R} = 10^{-snr/20}||\mathbf{E}^{NF}|| \frac{\mathbf{X} + j\mathbf{Y}}{||\mathbf{X} + j\mathbf{Y}||} \tag{6.6}$$

where j is the imaginary unit and $snr = 40$, 30. Through this choice of the noise $\mathbf{R}$ we have that

$$10 \log_{10} \frac{||\mathbf{E}^{NF}||^2}{||\mathbf{R}||^2} = snr \tag{6.7}$$

The corrupted NF is defined as $\mathbf{E}^{noise} := \mathbf{E}^{NF} + \mathbf{R}$.

We solve the least squares system

$$\min_{\{\beta_n\}} || \sum_n \beta_n \boldsymbol{\psi}_n^{NF} - \mathbf{E}^{noise}|| \tag{6.8}$$

where $\{\boldsymbol{\psi}_n^{NF}\}$ is our numerical basis described in previous chapters. Found a possible choice of coefficients $\{\overline{\beta_n}\}$ we evaluate the relative error

$$e_w = \frac{||E_w^{tgt} - E_w^{rec}||_{L^2(S^2)}}{||E_w^{tgt}||_{L^2(S^2)}} \qquad ||E_w||_{S^2}^2 = \int_0^{2\pi} \int_0^{\pi} |E_w(\theta, \phi)|^2 \sin(\theta) d\theta d\phi \tag{6.9}$$

where $w = \theta, \phi$ and

$$\mathbf{E}^{rec} = \sum_n \overline{\beta_n} \boldsymbol{\psi}_n^{FF} \tag{6.10}$$

In the following we compare noise-free reconstruction and SNR-reconstruction through some far field cuts where we use a scale $20 \log_{10}$ for the $y-$axis. The sampling step considered is $\Delta\theta = \Delta\phi = 9$ degrees for both noise free and snr reconstruction. We also report in the plot of relative error two vertical lines to show the Nyquist limit for the mounted antenna (2.4 degrees) and the isolated antenna (8 degrees).

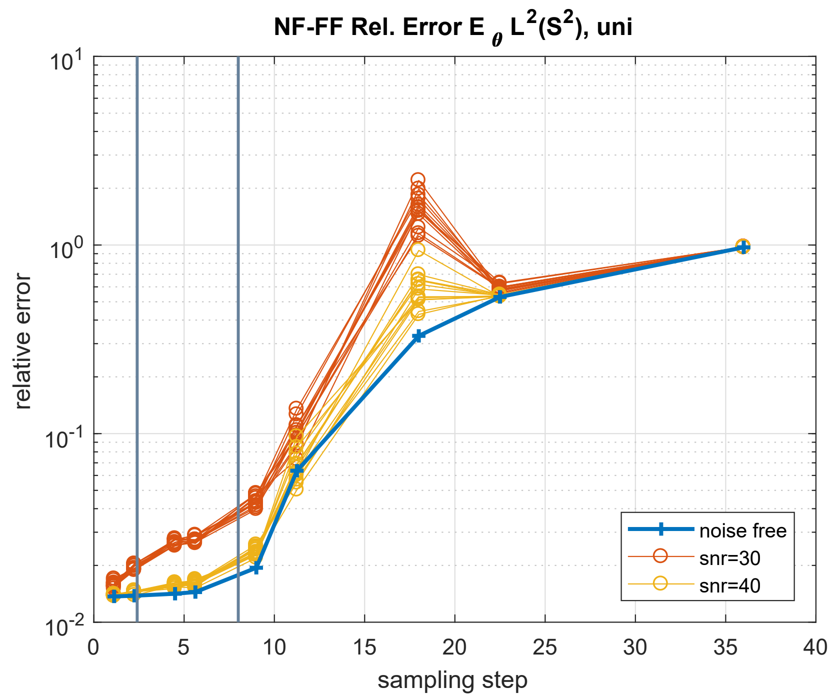


Figure 6.4. Relative error for the θ component of the electric field

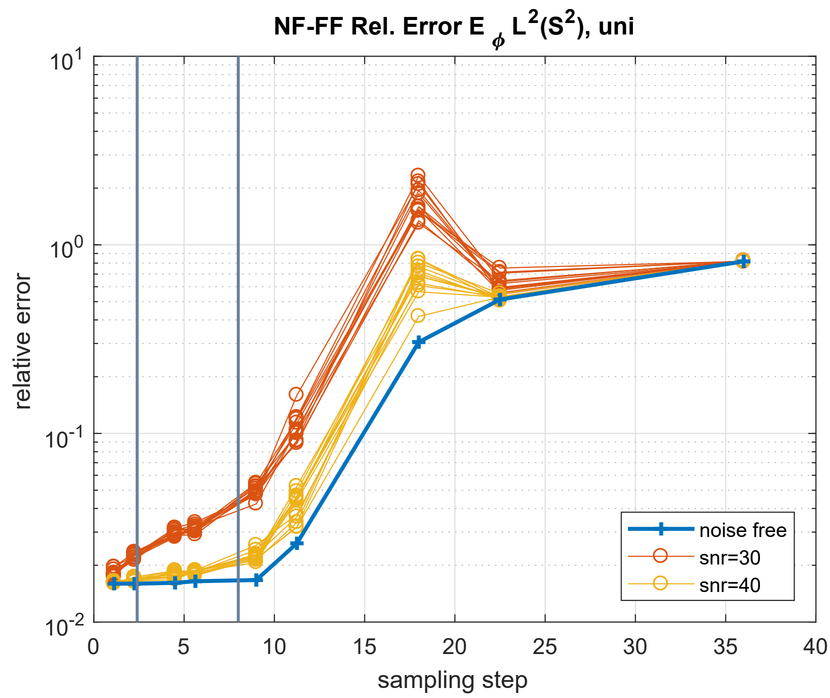
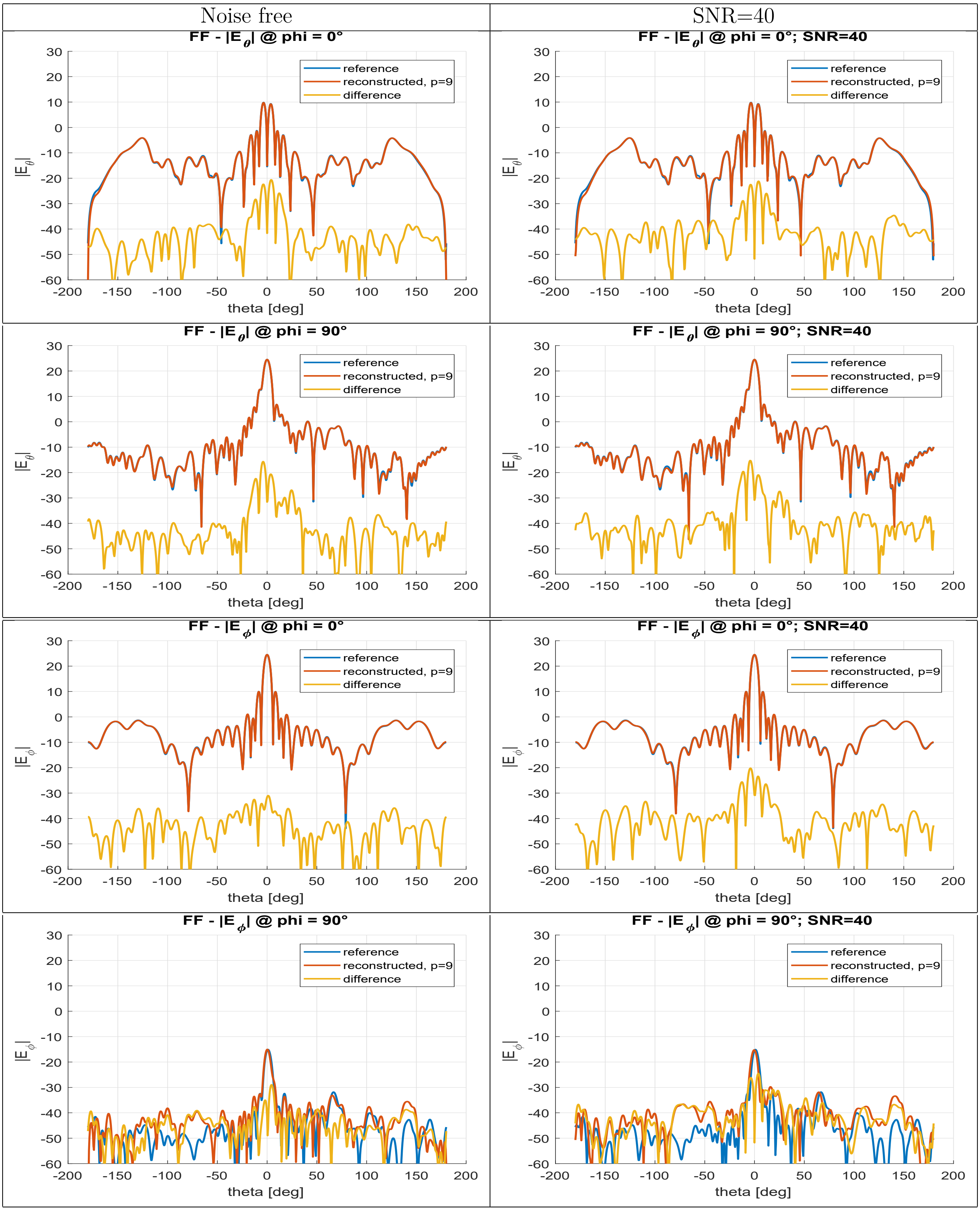
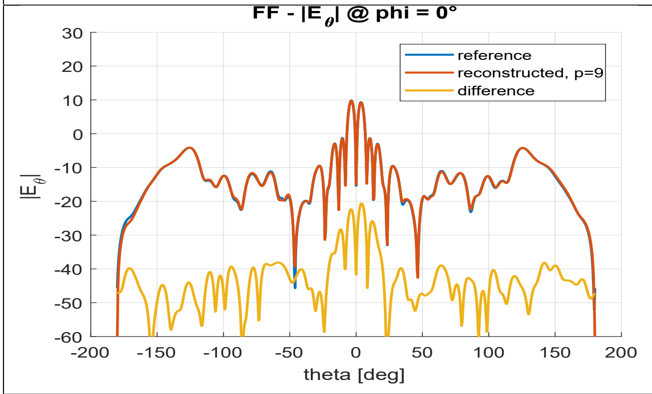
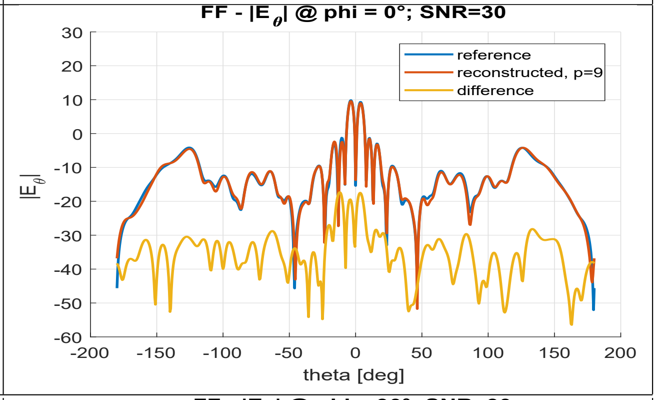
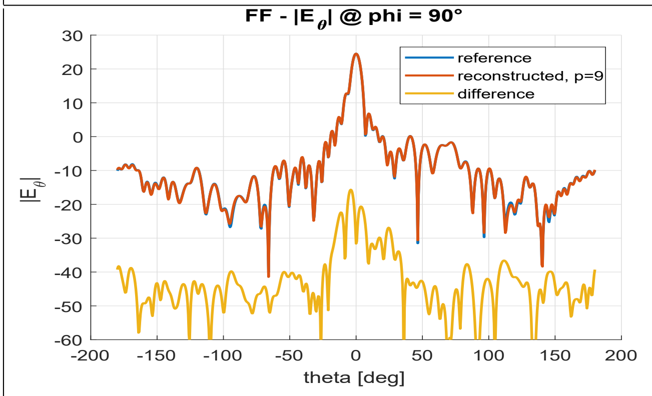
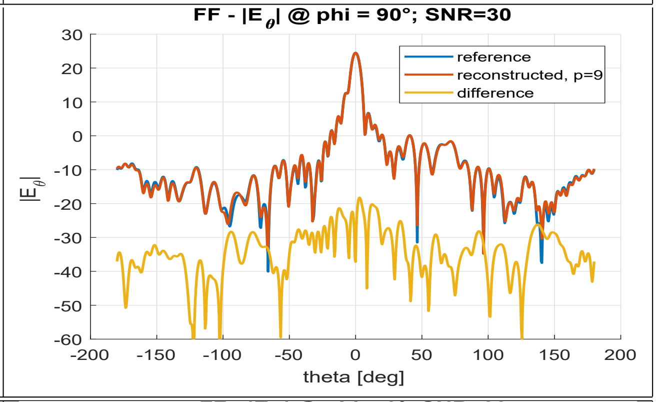
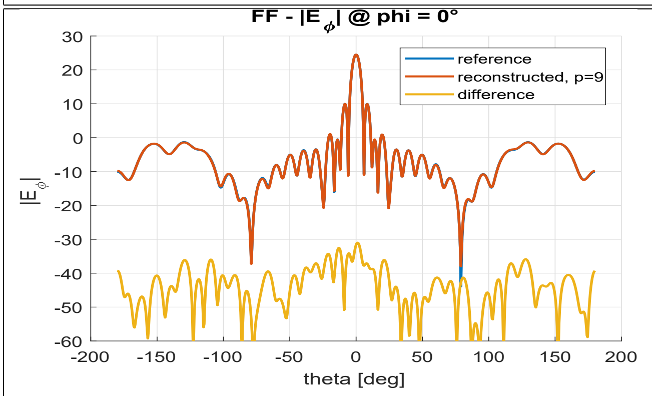
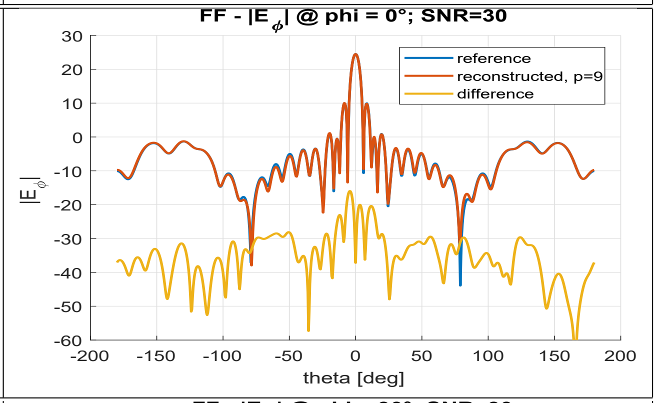
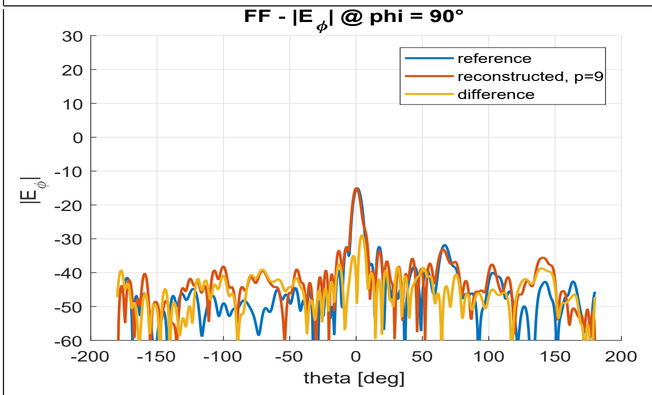
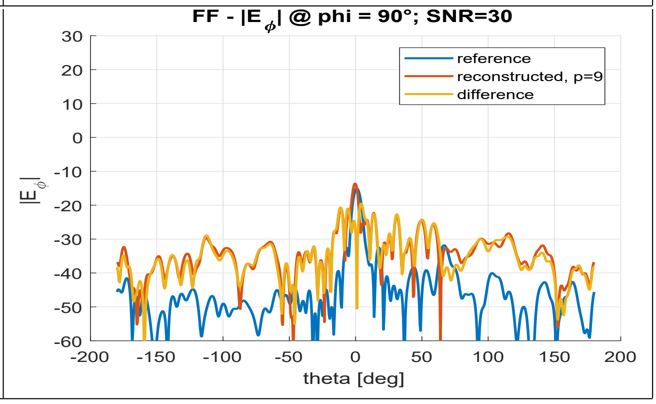


Figure 6.5. Relative error for the ϕ component of the electric field


Noise free
SNR=40
FF - |E_θ| @ phi = 0°
FF - |E_θ| @ phi = 0°; SNR=40
FF - |E_θ| @ phi = 90°
FF - |E_θ| @ phi = 90°; SNR=40
FF - |E_φ| @ phi = 0°
FF - |E_φ| @ phi = 0°; SNR=40
FF - |E_φ| @ phi = 90°
FF - |E_φ| @ phi = 90°; SNR=40
reference
reconstructed, p=9
difference
|E_θ|
|E_φ|
theta [deg]

Noise free	SNR=30
	
	
	
	

Chapter 7

Reflector 8GHz - Measured Samples

We measured near field samples of the reflector antenna (the same of the previous chapter) in the 5 x 5 x 4m anechoic chamber of LACE (Antenna and Electromagnetic Compatibility Laboratory). The frequency of the antenna is $f = 8$GHz. The walls of the chamber are covered of absorbers that operate at frequencies 700MHz-40GHz. Inside the chamber is

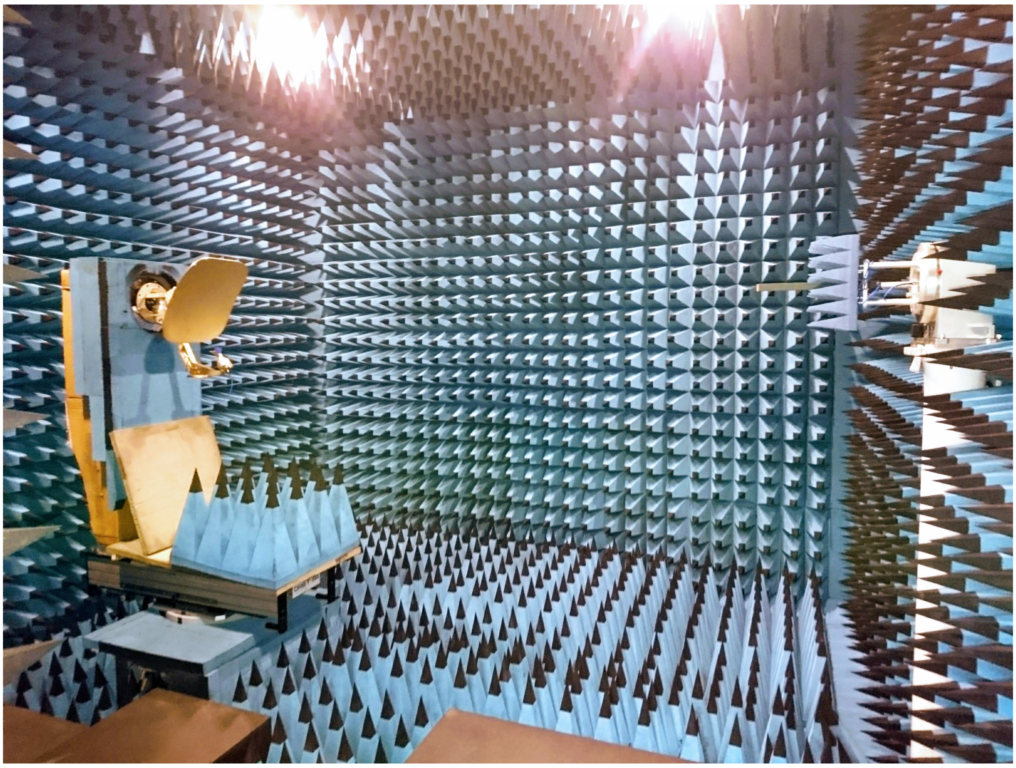


Figure 7.1. The anechoic chamber

installed a Spherical Near Field Antenna Test Range, that can be used both for Near and Far Field measurements. The minimum sphere that encloses the whole structure has radius $r = 0.55m \simeq 15\lambda$ with $\lambda = 0.0375m$. The lower bound on the number N of measures given by Nyquist criterion is $N = 4\pi r^2/(\lambda/2)^2 = 10813$ that corresponds to a sampling step $\Delta\phi = 2.4$ degrees. Near field samples are measured with a sampling step $\Delta\phi = \Delta\theta = 2.25$ degrees at the sphere with radius $r = 2.524m$ while our numerical basis is computed at the near field sphere of radius $r = 2.516m$.

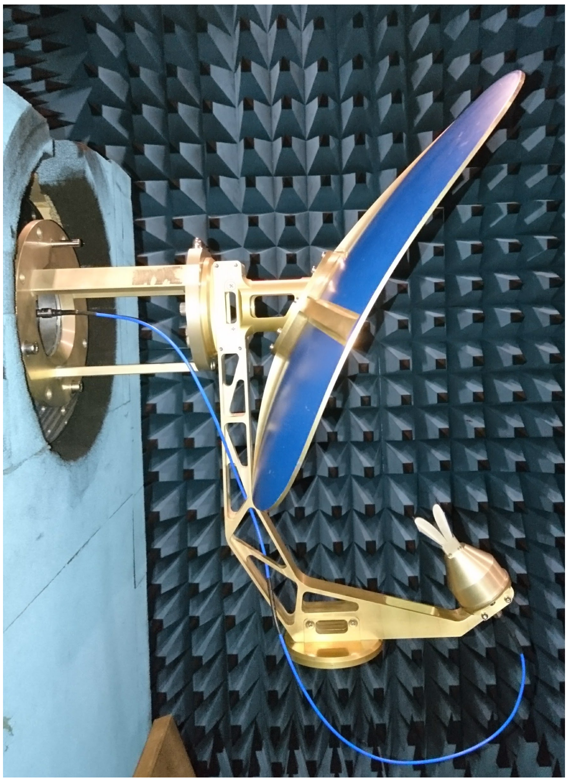


Figure 7.2. The reflector antenna

We show results for a reconstruction using a sampling step $\Delta\theta = 4.5$, $\Delta\phi = 9$ degrees that produces 1640 points on the near field sphere (it's eight times less than the number of points produced using a sampling step of $\Delta\theta = \Delta\phi = 2.25$ degrees). In next plots we don't show the module of the difference between the reconstructed and the reference far field because there was an error in the phase of near field numerical basis functions; nevertheless the basis was able to reconstruct well the module of the reference far field.

The partial directivities of the antenna for the far field are defined [12] as

$$D_w(\theta, \phi) = 4\pi \frac{|E_w(\theta, \phi)|^2}{||\mathbf{E}||^2_{L^2(S^2)}} \tag{7.1}$$

for $w = \theta, \phi$. In the sequel we show some plots of the directivity normalized electric field $\sqrt{4\pi}|E_w(\theta, \phi)|/||\mathbf{E}||_{L^2(S^2)}$ and in these plots with abuse of notation we denote this quantity

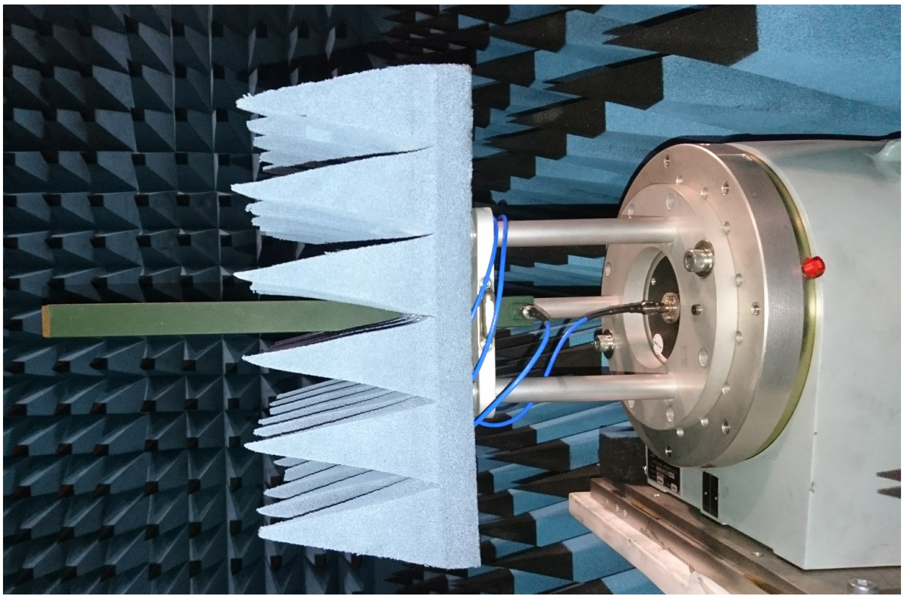


Figure 7.3. The probe

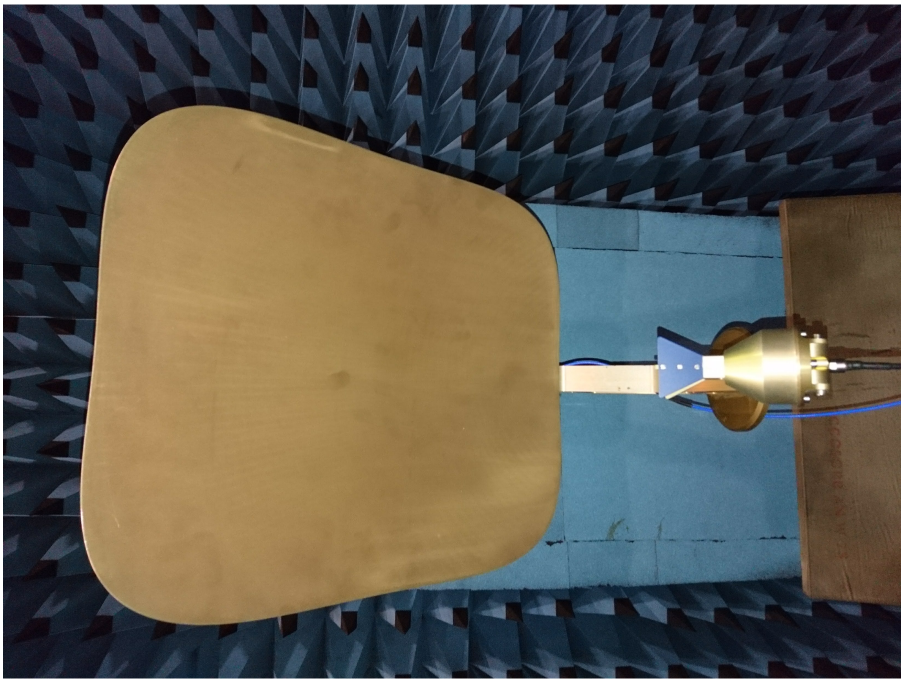
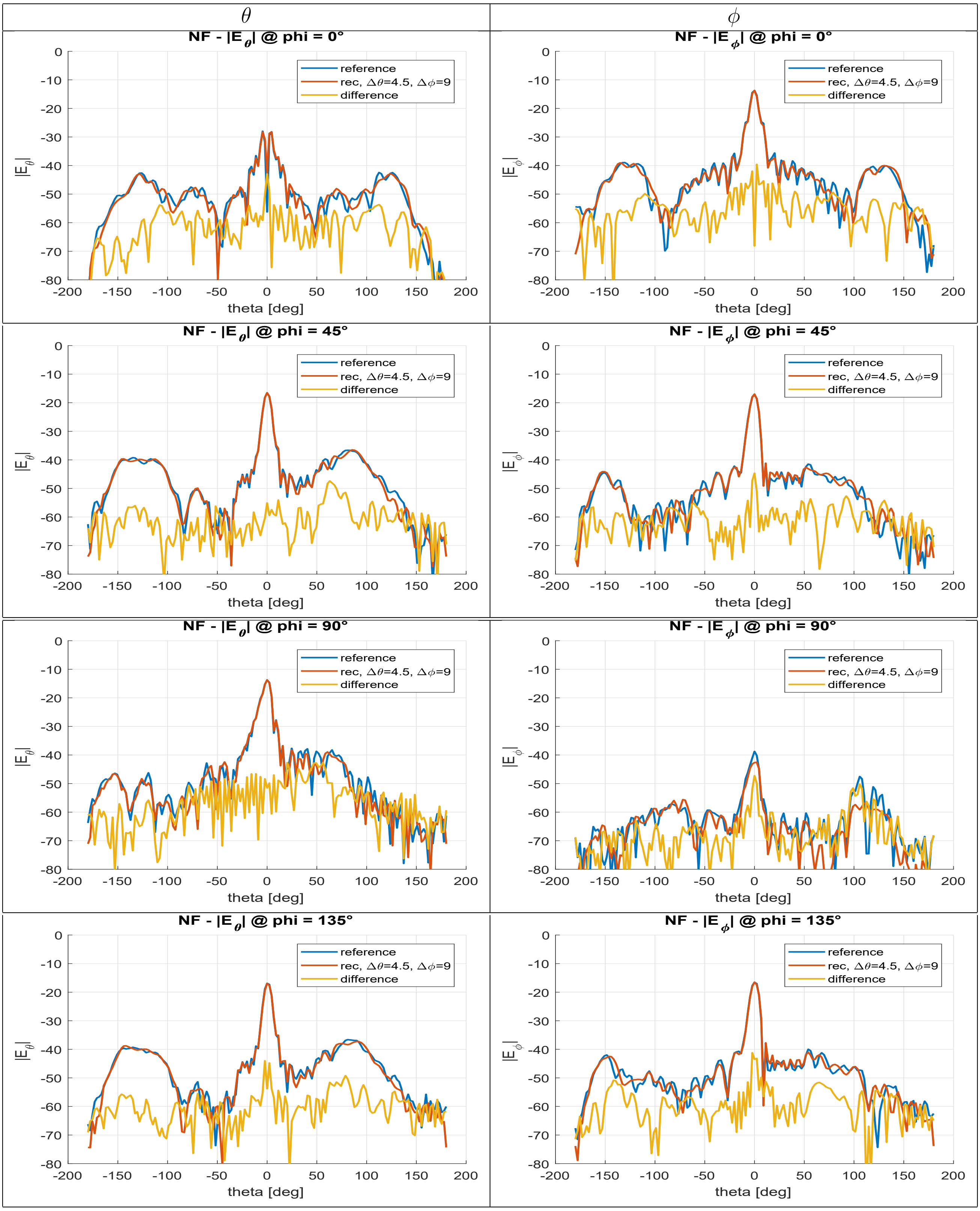
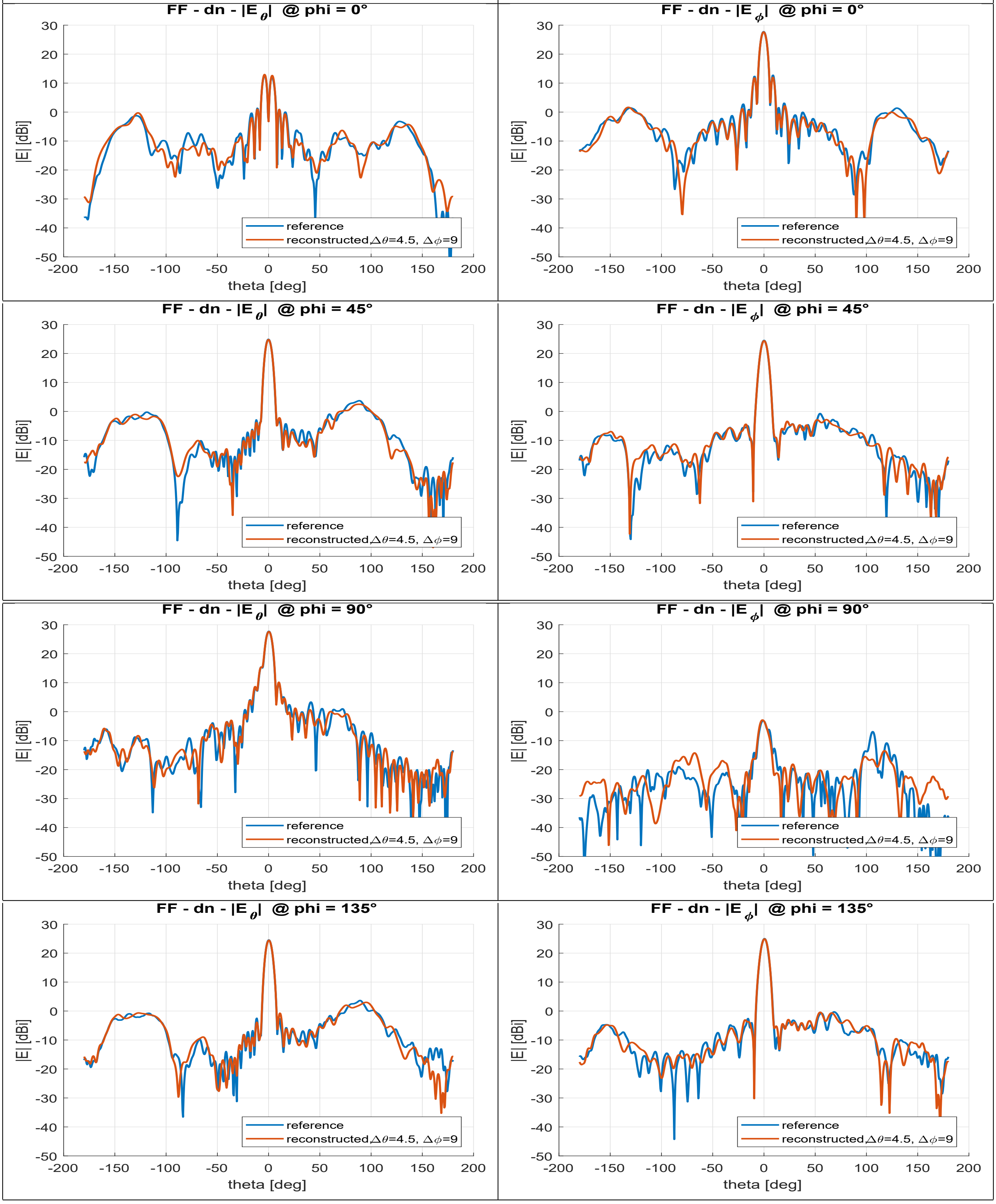


Figure 7.4. The parabolic antenna

still with **E**.


θ
ϕ
NF - |E$_\theta$| @ phi = 0°
NF - |E$_\phi$| @ phi = 0°
NF - |E$_\theta$| @ phi = 45°
NF - |E$_\phi$| @ phi = 45°
NF - |E$_\theta$| @ phi = 90°
NF - |E$_\phi$| @ phi = 90°
NF - |E$_\theta$| @ phi = 135°
NF - |E$_\phi$| @ phi = 135°
reference
rec, $\Delta\theta$=4.5, $\Delta\phi$=9
difference
|E$_\theta$|
|E$_\phi$|
theta [deg]

θ	ϕ


FF - dn - |E θ| @ phi = 0°
FF - dn - |E φ| @ phi = 0°
FF - dn - |E θ| @ phi = 45°
FF - dn - |E φ| @ phi = 45°
FF - dn - |E θ| @ phi = 90°
FF - dn - |E φ| @ phi = 90°
FF - dn - |E θ| @ phi = 135°
FF - dn - |E φ| @ phi = 135°
|E| [dBi]
theta [deg]
reference
reconstructed Δθ=4.5, Δφ=9

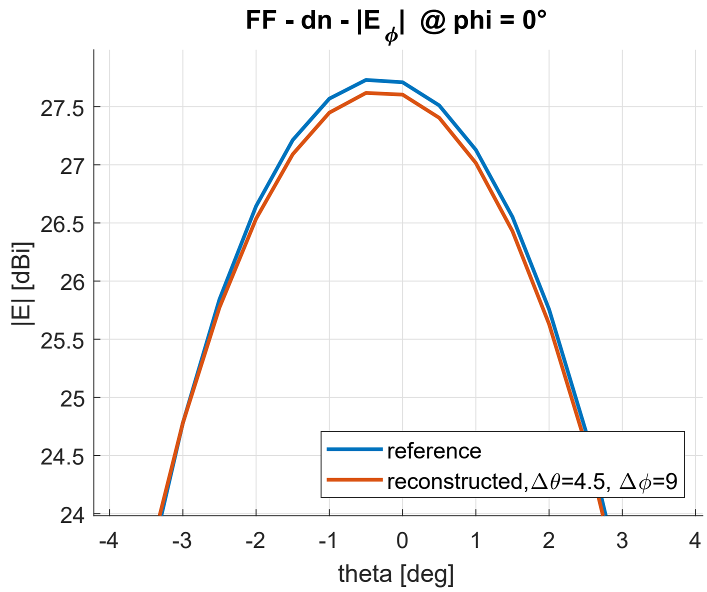


Figure 7.5. The main beam and the reconstruction with a samplig step $\Delta\theta = 4.5$, $\Delta\phi = 9$ degrees

We discarded the reconstruction made using a samplig step $\Delta\theta = \Delta\phi = 9$ degrees because there is an error greater than 0.5 dB in the reconstruction of the main beam.

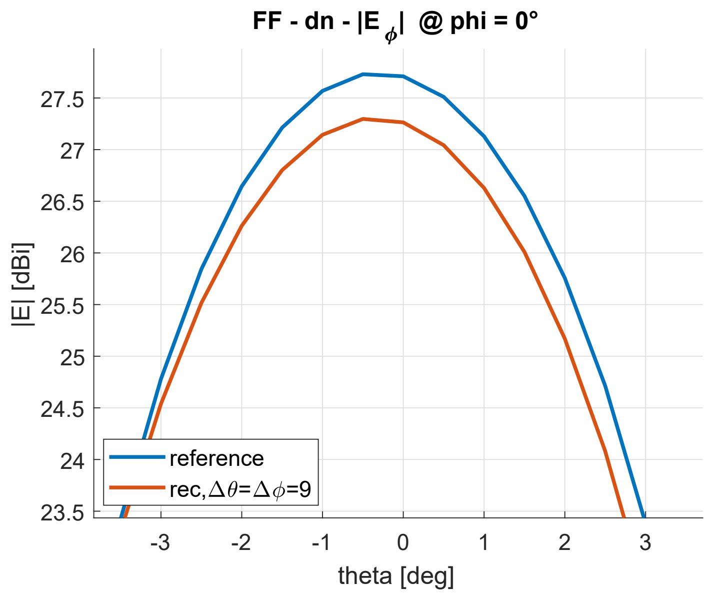


Figure 7.6. The main beam and the reconstruction with a samplig step $\Delta\theta = \Delta\phi = 9$ degrees

Chapter 8

Conclusions

The thesis analyzes a method [1] to determine the electric far-field radiated by electrically large antennas (or antennas placed over structures) using few measured near field samples of the electric field and numerically constructed expansion functions. In chapters 5 and 6 we showed results of reconstructions using synthetic data i.e. near field samples of the reference electric field acquired through simulations while in chapter 7 we used measured samplings of the electric field. We analyzed:

- a dipole placed over a plane mock-up at frequency 3 GHz (synthetic data);
- a reflector antenna at frequency 8 GHz (synthetic data);
- the same reflector antenna at frequency 8 GHz (measured samples).

In order to test the robustness of the method, in the case of synthetic data, we corrupted near field samplings with noise with signal to noise ratio (SNR) levels of 40 and 30 dB and then we used these corrupted samplings to reconstruct the noise-free reference far field. We show a table with the number of points that the analyzed method (M) and the classic method (C) that uses spherical wave functions [6] need to reconstruct well the far electric field.

	M	C
plane (synthetic data)	312	3215
reflector (synthetic data)	840	12960
reflector (measured samples)	1640	12960

In the case of measured samples we encountered some problems on the reconstruction of the phase of the reference far field. This work can be used to continue the study of antennas previously analyzed and solve such a problem. One can also improve the model using an impedance boundary condition instead of the one used for a PEC. An other problem is that the solution of the scattering equation is not unique for some frequencies called resonances. We used EFIE equations to determine the current produced by an incident field but there is a different set of equations (CFIE) that do not produce an ill conditioned matrix also when one is working at a frequency near to a resonant one. We enforced equality on points on the sphere using a uniform step in θ and ϕ but it could be useful to find a better choice of a grid on the sphere.

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