Adaptive Model Predictive Control

June 15, 2021

1 Reference Frames

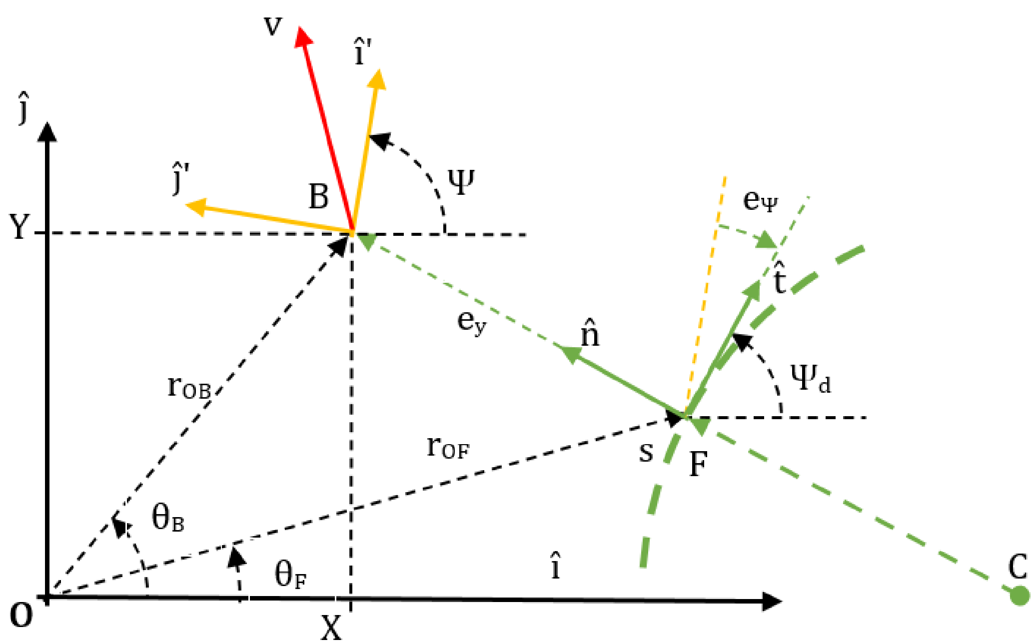


Figure 1: Global, body and track frames

It is considered a moving body frame B in a plane with respect to an inertial global frame O, and a Frenet frame F.

- Global frame $O(\hat{i}, \hat{j})$ defined as an inertial frame.

- Body frame $B(\hat{i}', \hat{j}')$ defined with the exes fixed to the body.
- Frenet frame $F(\hat{t}, \hat{n})$ defined by projecting the body to the track, with the axes tangent and normal to the curve.

The position of the bosy can be defined in cartesian coordinates (x, y) or curvilinear coordiantes (s, e_y).

1.1 Inertial-Frenet-Body transfrom

1.1.1 Rotation matrix

Rotation matrix between the inertial frame O, body frame B and Frenet frame F:

$$\begin{aligned} R_B^O(\psi) &= \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \\ R_F^O(\psi_d) &= \begin{bmatrix} \cos\psi_d & -\sin\psi_d \\ \sin\psi_d & \cos\psi_d \end{bmatrix} \\ R_B^F(e_\psi) &= \begin{bmatrix} \cos e_\psi & -\sin e_\psi \\ \sin e_\psi & \cos e_\psi \end{bmatrix} \end{aligned} \tag{1}$$

1.1.2 Curvilinear coordinates

The body can be expressed in curvilinear coordinates (s, e_y), which can be defined with respect to the Frenet frame F:

$$\begin{aligned} \vec{s} &= \vec{r}_{OF}^{F} \\ \vec{e}_y &= \vec{r}_{FB}^{F} \end{aligned} \tag{2}$$

According to the figure **??**, the curvilinear abscissa s can be expressed in polar coordinates

$$s = \rho\theta \tag{3}$$

where: $\rho :$ curve radius
$c_c = 1/\rho :$ curvature
$\theta :$ angle travelled

The angle travelled θ, taking into account the rotation direction, can be expressed as:

$$\theta = \theta_0 + \theta' = \theta_0 - (90° - \psi_d) \tag{4}$$

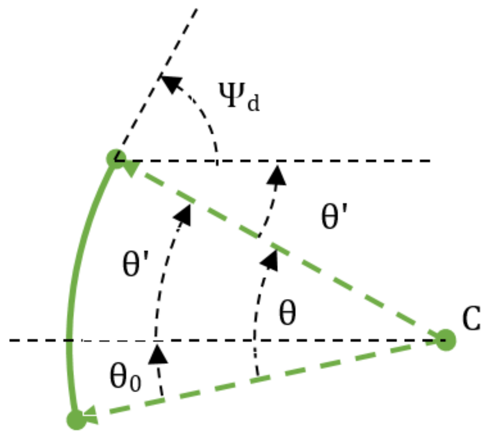


Figure 2: Angles relation

where θ_0 is an offset of the starting point to a specific position on the horizonal inertial axis.

Deriving the equation above:

$$\frac{d}{dt}(\theta) = \frac{d}{dt}(\theta_0 - (90° - \psi_d)) = \frac{d}{dt}(\psi_d) \tag{5}$$

which proves the relation of the angle travelled along the curve θ and the angle with respect to the tangent to the curve itself.

Deriving the curvilinear abscissa in polare coordinates in 3 with respect to the time, considering a constant curvature:

$$\vec{\dot{s}} = \frac{d}{dt}(\rho\theta)\hat{t} = \rho\frac{d}{dt}(\theta)\hat{t} = \rho\omega_d\hat{t} \tag{6}$$

then the angular velocity:

$$\omega_d = \frac{\dot{s}}{\rho} = \dot{s}c_c \tag{7}$$

It is defined the deviation e_ψ from the body heading to the curve (desired) heading, is:

$$e_\psi = \psi - \psi_d \tag{8}$$

1.1.3 Body with respect to the global frame

The position of the body frame B with respect the global frame O is:

$$\vec{r}_{OB}^{O} = \begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} r_{OB} \cos \theta_B \\ r_{OB} \sin \theta_B \end{bmatrix} \tag{9}$$

The velocity of the body in its reference frame $\vec{v}_B^B$, can be expressed in the global frame O as:

$$\vec{v}_B^O = R_B^O(\psi)\vec{v}_B^B \Rightarrow \vec{v}_B^B = R_O^B(\psi)\vec{v}_B^O \tag{10}$$

The acceleration of the body in its reference frame $\vec{a}_B^B$, can be expressed in the global frame O, deriving the equation above:

$$\begin{aligned} \vec{a}_B^O &= \frac{d}{dt}(\vec{v}_B^O) = \frac{d}{dt}(R_B^O(\psi)\vec{v}_B^B) = \dot{R}_B^O(\psi)\vec{v}_B^B + R_B^O(\psi)\dot{\vec{v}}_B^B \\ &= \vec{\omega}_B^O \times R_B^O(\psi)\vec{v}_B^B + R_B^O(\psi)\dot{\vec{v}}_B^B = R_B^O(\psi)(\vec{\omega}_B^O \times \vec{v}_B^B + \dot{\vec{v}}_B^B) \\ \Rightarrow & R_O^B\vec{a}_B^O = \vec{\omega}_B^O \times \vec{v}_B^B + \dot{\vec{v}}_B^B \end{aligned} \tag{11}$$

then:

$$\vec{a}_B^B = \vec{\omega}_B^O \times \vec{v}_B^B + \dot{\vec{v}}_B^B \tag{12}$$

where: $\vec{a}_B^B = R_O^B\vec{a}_B^O$

1.1.4 Frenet frame

The position of the Frenet frame F with respect the global frame O is:

$$\vec{r}_{OF}^{O} = \begin{bmatrix} x_F \\ y_F \end{bmatrix} = \begin{bmatrix} r_{OF} \cos \theta_F \\ r_{OF} \sin \theta_F \end{bmatrix} \tag{13}$$

The velocity of the body in its reference frame $\vec{v}_B^B$, can be expressed in the Frenet frame F as:

$$\vec{v}_B^F = R_B^F(e_\psi)\vec{v}_B^B \Rightarrow \vec{v}_B^B = R_F^B(e_\psi)\vec{v}_B^F \tag{14}$$

1.1.5 Body with respect to the Frenet frame

The position of the body can be expressed as:

$$\vec{r}_{OB}^O = \vec{r}_{OF}^O + \vec{r}_{FB}^O = \vec{r}_{OF}^O + R_F^O(\psi_d)\vec{r}_{FB}^F \tag{15}$$

Deriving the equation above with respect the time:

$$
\begin{aligned}
&\frac{d}{dt}(\vec{r}^O_{OB}) = \frac{d}{dt}(\vec{r}^O_{OF}) + \frac{d}{dt}(R^O_F(\psi_d)\vec{r}^F_{FB}) \\
\Rightarrow &\vec{v}^O_B = \frac{d}{dt}(\vec{r}^O_{OF}) + \dot{R}^O_F(\psi_d)\vec{r}^F_{FB} + R^O_F(\psi_d)\dot{\vec{r}}^F_{FB} \\
\Rightarrow &\vec{v}^O_B = \dot{\vec{r}}^O_{OF} + \vec{\omega}^O_F \times R^O_F(\psi_d)\vec{r}^F_{FB} + R^O_F(\psi_d)\dot{\vec{r}}^F_{FB} \\
\Rightarrow &R^F_O(\psi_d)\vec{v}^O_B = R^F_O(\psi_d)\dot{\vec{r}}^O_{OF} + \vec{\omega}^O_F \times \vec{r}^F_{FB} + \dot{\vec{r}}^F_{FB} \\
\Rightarrow &R^F_O(\psi_d)R^O_B(\psi)\vec{v}^B_B = \dot{\vec{r}}^F_{OF} + \vec{\omega}^O_F \times \vec{r}^F_{FB} + \dot{\vec{r}}^F_{FB} \\
\Rightarrow &R^F_B(e_\psi)\vec{v}^B_B = \dot{\vec{r}}^F_{OF} + \vec{\omega}^O_F \times \vec{r}^F_{FB} + \dot{\vec{r}}^F_{FB}
\end{aligned}
\tag{16}
$$

where: $R^F_B(e_\psi) = R^F_O(\psi_d)R^O_B(\psi)$
$\dot{\vec{r}}^F_{OF} = \dot{s}\hat{t}$
$\dot{\vec{r}}^F_{FB} = \dot{e}_y\hat{n}$

Passing from the cartesian coordinates to the curvilinear coordinates, using the transformation in 2:

$$
\begin{aligned}
&R^F_B(e_\psi)\vec{v}^B_B = \dot{\vec{s}} + \vec{\omega}_d \times \vec{e}_y + \dot{\vec{e}}_y \\
\Rightarrow &R^F_B(e_\psi)\vec{v}^B_B = \dot{s}\hat{t} + \dot{s}c_c\hat{k} \times e_y\hat{n} + \dot{e}_y\hat{n} \\
\Rightarrow &R^F_B(e_\psi)\vec{v}^B_B = \dot{s}\hat{t} + \dot{s}e_yc_c(-\hat{t}) + \dot{e}_y\hat{n} \\
\Rightarrow &R^F_B(e_\psi)\vec{v}^B_B = \dot{s}(1 - e_yc_c)\hat{t} + \dot{e}_y\hat{n}
\end{aligned}
\tag{17}
$$

which can be decompoed in the Frenet frame axes:

$$
\begin{bmatrix} \dot{s} \\ \dot{e}_y \end{bmatrix} = R^F_B(e_\psi)\begin{bmatrix} v_x \\ v_y \end{bmatrix}\begin{bmatrix} \frac{1}{1-c_ce_y} \\ 0 \end{bmatrix} \tag{18}
$$

while the deviation angle velocity:

$$
\dot{\vec{e}}_\psi = \frac{d}{dt}(\psi - \psi_d)\hat{k} = \vec{\omega} - \vec{\omega}_d = \vec{\omega} - \dot{s}c_c\hat{k} \tag{19}
$$

2 Bicycle dynamic model

Ref: [1] (pg. 27), [3]

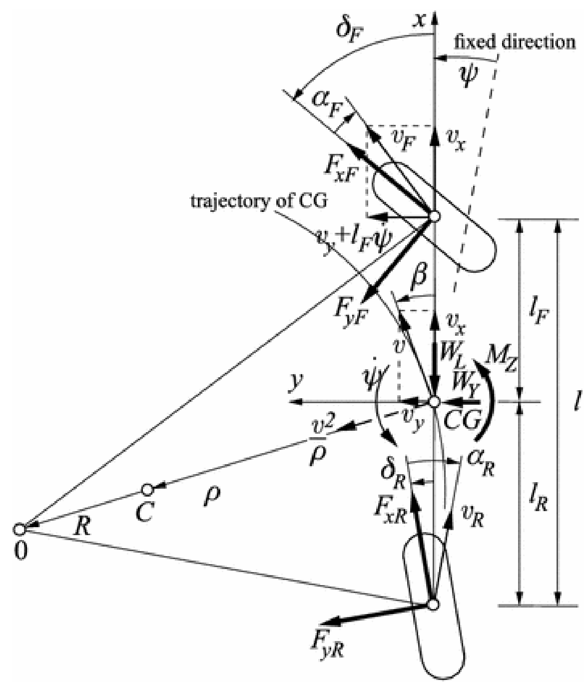


Figure 3: Bicycle dynamics

From equations 12, the acceleration of the body in its own frame B, is:

$$\begin{aligned}
\vec{a}_B &= \vec{\omega}_z \times \vec{v}_B + \dot{\vec{v}}_B = \omega_z \hat{k} \times (v_x \hat{i} + v_y \hat{j}) + (\dot{v}_x \hat{i} + \dot{v}_y \hat{j}) \\
&= \omega_z v_x \hat{j} + \omega_z v_y (-\hat{i}) + (\dot{v}_x \hat{i} + \dot{v}_y \hat{j}) \\
&(\dot{v}_x - \omega_z v_y)\hat{i} + (\dot{v}_y + \omega_z v_x)\hat{j}
\end{aligned} \tag{20}$$

then:

$$\vec{a}_B = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \dot{v}_x - \omega_z v_y \\ \dot{v}_y + \omega_z v_x \end{bmatrix} \tag{21}$$

The dynamics of a rigid body is then:

$$\begin{aligned}
m(\dot{v}_x - \omega_z v_y) &= F_{xf} + F_{xr} \\
m(\dot{v}_y + \omega_z v_x) &= F_{yf} + F_{yr} \\
I_z \dot{\omega}_z &= l_f F_{yf} - l_r F_{yr}
\end{aligned} \tag{22}$$

where: m : mass
l_f, l_r : front and rear wheel distance from the center of mass
F_{xf}, F_{xr} : Forces on x body axis of front (f) and rear (r) wheel
F_{yf}, F_{yr} : Forces on y body axis of front (f) and rear (r) wheel
in which the forces can be decomposed in the tire-fixed frame:

$$\begin{cases} F_{x,i} = F_{l,i} cos(\delta_i) - F_{s,i} sin(\delta_i) \\ F_{y,i} = F_{l,i} sin(\delta_i) + F_{s,i} cos(\delta_i) \end{cases} \tag{23}$$

where: $F_{l,i} = \{F_{lf}, F_{lr}\}$: Force on longitudinal axis of front (f) and rear (r) wheel
$F_{s,i} = \{F_{sf}, F_{sr}\}$: Force on perpendicular axis of front (f) and rear (r) wheel
$\delta_i = \{delta_f, delta_r\}$: Front (f) and rear (r) wheel steering angles

Replacing the equations 23 into 22, it is obtained the complete formulation:

$$\begin{aligned} \dot{v}_x &= \frac{1}{m}(F_{lf}\cos(\delta_f) - F_{sf}\sin(\delta_f) + F_{lr}\cos(\delta_r) - F_{sr}\sin(\delta_r)) + \omega_z v_y \\ \dot{v}_y &= \frac{1}{m}(F_{lf}\sin(\delta_f) + F_{sf}\cos(\delta_f) + F_{lr}\sin(\delta_r) + F_{sr}\cos(\delta_r)) - \omega_z v_x \\ \dot{\omega}_z &= \frac{1}{I_z}(l_f(F_{lf}\sin(\delta_f) + F_{sf}\cos(\delta_f)) - l_r(F_{lr}\sin(\delta_r) + F_{sr}\cos(\delta_r))) \end{aligned} \tag{24}$$

Sideslip angle Ref: [1] (pg. 27)

The sideslip angle is defined as:

$$\alpha_* = \delta_* - \theta_* \tag{25}$$

where: $\theta_* = \arctan\left(\frac{v_{x*}}{v_{y*}}\right)$
v_{x*}, v_{y*} : longitudinal and lateral wheel velocities in fixed body-frame
While v_{x*} coincides with the CoM longitudinal velocity, the v_{y*} is the sum of the CoM lateral velocity and the tangential velocity of the wheel respect to the CoM due to the angular velocity, then:

$$\begin{aligned} v_{xf} &= v_{xr} = v_x \\ v_{yf} &= v_y + l_f\omega_z \\ v_{yr} &= v_y - l_r\omega_z \end{aligned} \tag{26}$$

where: v_{xf}, v_{yf} : front wheel velocities in fixed body-frame
v_{xr}, v_{yr} : rear wheel velocities in fixed body-frame
Replacing the equation above in the first one, it is obtained the sideslip angle of the front

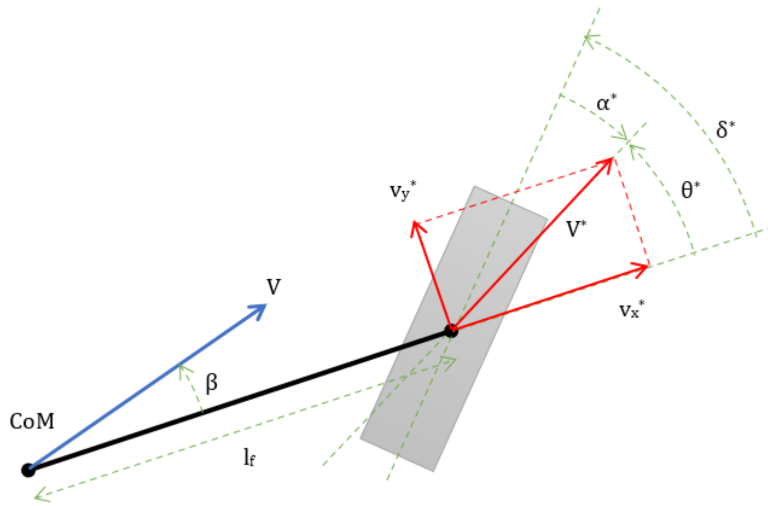


Figure 4: Sideslip angle

and the rear wheel:

$$\begin{aligned}\alpha_f &= \delta_f - \arctan\left(\frac{v_y + l_f\omega_z}{v_x}\right)\\ \alpha_r &= \delta_r - \arctan\left(\frac{v_y - l_r\omega_z}{v_x}\right)\end{aligned} \tag{27}$$

2.1 Tire's linear dynamic [3]

For small sideslip angles, the wheel lateral force can be well approximated proportional to the sideslip angle itself:

$$\begin{aligned}F_{sf} &= C_f(F_z, \mu)\alpha_f\\ F_{sr} &= C_r(F_z, \mu)\alpha_r\end{aligned} \tag{28}$$

where C_f, C_r is the "Cornering stiffness" of each wheel, depending on the normal force acting on the wheel and the friction coefficient with the surface.
Here, an estimate of this coefficient has been done evaluating the dynamic evolution of the nonlinear plant model, with different values of the cornering stiffness, until a realistic behaviour has been achieved.
To notice is that small values of the cornering stiffness, make difficult the steering, while too high values generate excessively high lateral forces also for small sideslip angles, which are incompatible with the reality.

3 Bicycle dynamics in a curvilinear reference for MPC prediction

From the previous sections, a dynamic model of the bicycle in a curvilinear reference can be implemented under the following assumptions:

- Small sideslip angles ($\alpha \leq 10°$)
- Front steering wheel command ($\delta_r = 0$)
- Rear traction ($F_{lr} = T, F_{lf} = 0$)
- The forces are multiplied by a factor of 2 in order to take into account the physics of 4 wheels

the dynamics in 24 can be semplified as:

$$\begin{aligned}
\dot{v}_x &= \frac{1}{m}(-F_{sf}\sin(\delta_f) + T) + \omega_z v_y \\
\dot{v}_y &= \frac{1}{m}(F_{sf}\cos(\delta_f) + F_{sr}) - \omega_z v_x \\
\dot{\omega}_z &= \frac{1}{I_z}(l_f F_{sf}\cos(\delta_f) - l_r F_{sr})
\end{aligned} \tag{29}$$

while the sideslip angle, from 27, under the assumption of small angles, is:

$$\begin{aligned}
\alpha_f &= \delta_f - \arctan\left(\frac{v_y + l_f\omega_z}{v_x}\right) \simeq \delta_f - \frac{v_y + l_f\omega_z}{v_x} \\
\alpha_r &= -\arctan\left(\frac{v_y - l_r\omega_z}{v_x}\right) \simeq -\frac{v_y - l_r\omega_z}{v_x}
\end{aligned} \tag{30}$$

The tire's forces are then:

$$\begin{aligned}
F_{sf} &= 2C_f\alpha_f \simeq 2C_f\left(\delta_f - \frac{v_y + l_f\omega_z}{v_x}\right) \\
F_{sr} &= 2C_r\alpha_r \simeq 2C_r\left(-\frac{v_y - l_r\omega_z}{v_x}\right)
\end{aligned} \tag{31}$$

While the link between the dynamics from the cartesian coordinate into the curvilinear cooridnate from 18 is:

$$\begin{aligned}
\dot{s}(t) &= \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} \\
\dot{e}_y(t) &= v_x \sin e_\psi + v_y \cos e_\psi \\
\dot{e}_\psi(t) &= \omega_z - \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} c_c(s)
\end{aligned} \tag{32}$$

Putting toghether 29 31 32 it is obtained the complete dynamics:

$$\begin{aligned}
\dot{v}_x(t) &= a + \frac{2}{m} \left(-C_f \left(\delta_f - \frac{v_y + l_f \omega_z}{v_x} \right) \sin(\delta_f) \right) + v_y \omega_z \\
\dot{v}_y(t) &= \frac{2}{m} \left(C_f \left(\delta_f - \frac{v_y + l_f \omega_z}{v_x} \right) \cos(\delta_f) + C_r \left(-\frac{v_y - l_r \omega_z}{v_x} \right) \right) - v_x \omega_z \\
\dot{\omega}_z(t) &= \frac{2}{I_z} \left(l_f C_f \left(\delta_f - \frac{v_y + l_f \omega_z}{v_x} \right) \cos(\delta_f) - l_r C_r \left(-\frac{v_y - l_r \omega_z}{v_x} \right) \right) \\
\dot{s}(t) &= \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} \\
\dot{e}_y(t) &= v_x \sin e_\psi + v_y \cos e_\psi \\
\dot{e}_\psi(t) &= \omega_z - \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} c_c(s)
\end{aligned} \tag{33}$$

where: $\xi = \begin{bmatrix} v_x & v_y & \omega_z & s & e_y & e_\psi \end{bmatrix}^T$
$u = \begin{bmatrix} \delta_f & a \end{bmatrix}^T$

4 MPC

4.1 Model linearization

The nonlinear system has to be linearized around the operative point $p = [\xi_0, u_0]$. The system can be approximated through the Taylor expansion, stopped to the first derivative:

$$\frac{d}{dt}(\delta\xi(t)) \approx \frac{\partial f}{\partial \xi}\delta\xi(t) + \frac{\partial f}{\partial u}\delta u(t) \Rightarrow \delta\dot{\xi}(t) \approx A(p)\delta\xi(t) + B(p)\delta u(t) \tag{34}$$

where: $\delta\xi(t) = \xi(t) - \xi_0$
$\delta u(t) = u(t) - u_0$

which can be rewritten, collecting all the constant terms in a offset matrix K:

$$\begin{aligned} \frac{d}{dt}(\xi(t) - \xi_0) = A(p)(\xi(t) - \xi_0) + B(p)(u(t) - u_0) \\ \Rightarrow \dot{\xi(t)} = A(p)\xi(t) + B(p)u(t) + K(p) \end{aligned} \tag{35}$$

where: $K = \dot{\xi_0} - A(p)\xi_0 - B(p)u_0$

4.2 Discretization

The system model is discretized according the Euler approximation:

$$\begin{aligned} A_d &= I + AT_s \\ B_d &= BT_s \\ K_d &= KT_s \end{aligned} \tag{36}$$

where: $T_s = \alpha T_{sys}$: MPC sampling time

4.3 Prediction

At k-th sampled, the predicted state is:

$$\xi_{i+1|k} = A_d(p)\xi_{i|k} + B_d(p)u_{i|k} + K_d(p), \quad \xi_{0|k} = \xi_k \tag{37}$$

State prediction:

$$\bar{\xi}_k = \bar{A}_d(p)\xi_k + \bar{B}_d(p)U_k + \bar{K}_d(p) \tag{38}$$

where: $\bar{\xi}_k = \begin{bmatrix} \xi_{1|k} \\ \vdots \\ \xi_{N_p|k} \end{bmatrix} \in \mathbb{R}^{n_x N_p}; \quad U_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N_c-1|k} \end{bmatrix} \in \mathbb{R}^{n_u N_c};$

$$\bar{A}_d(p) = \begin{bmatrix} A_d(p) \\ A_d^2(p) \\ \vdots \\ A_d^{N_p}(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x};$$

$$\bar{B}_d(p) = \begin{bmatrix} B_d(p) & 0^{n_x,n_u} & \cdots & 0^{n_x,n_u} \\ A_d(p)B_d(p) & B_d(p) & \cdots & 0^{n_x,n_u} \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_c}(p)B_d(p) & A_d^{N_c-1}(p)B_d(p) & \cdots & B_d(p) \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_p-1}(p)B_d(p) & A_d^{N_p-2}(p)B_d(p) & \cdots & A_d^{N_p-N_c}(p)B_d(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_u N_c}$$

$$\bar{K}_d(p) = \begin{bmatrix} K_d(p) \\ A_d(p)K_d(p) + K_d(p) \\ \vdots \\ \sum_{i=0}^{N_p-1} A_d(p)^i K_d(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p};$$

4.4 Optimization

4.4.1 Objective function

$$\begin{aligned} J(\xi_k) &= \sum_{i=1}^{N_p} \left\|\xi_{i|k} - ref_{i|k}\right\|_Q^2 + \sum_{i=0}^{N_c-1} \left\|u_{i|k}\right\|_R^2 \\ &= \left\|\bar{\xi}_k - ref_k\right\|_{\bar{Q}}^2 + \|U_k\|_{\bar{R}}^2 \\ &= U_k^T \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d\right) U_k + 2\left(\bar{A}_d\xi_k + \bar{K}_d - ref_k\right)^T \bar{Q}\bar{B}_d U_k \\ &+ \left(\bar{A}_d\xi_k + \bar{K}_d - ref_k\right)^T \bar{Q} \left(\bar{A}_d\xi_k + \bar{K}_d - ref_k\right) \end{aligned} \tag{39}$$

where: $ref_k = \begin{bmatrix} ref_{1|k} & \cdots & ref_{N_p|k} \end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

$$\bar{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x N_p}$$

$$\bar{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix} \in \mathbb{R}^{n_u N_c, n_u N_c}$$

4.4.2 Constraints

State constraints

$$\xi_c = \begin{bmatrix} \xi_{min} & \xi_{max} \end{bmatrix} = \begin{bmatrix} v_{x,min} & v_{x,max} \\ v_{y,min} & v_{y,max} \\ \omega_{z,min} & \omega_{z,max} \\ s_{min} & s_{max} \\ e_{y,min} & e_{y,max} \\ e_{\psi,min} & e_{\psi,max} \end{bmatrix} = \begin{bmatrix} 0m/s & +3m/s \\ -0.5m/s & +0.5m/s \\ -10°/s & +10°/s \\ 0m & s_f \\ -2m & +2m \\ -10° & +10° \end{bmatrix} \tag{40}$$

$$\xi_{min} \leq \xi_{i|k} \leq \xi_{max} \Rightarrow \begin{cases} -\bar{B}_d(p)U_k \leq -\bar{\xi}_{min} + \bar{A}_d(p)\xi_k + \bar{K}_d \\ \bar{B}_d(p)U_k \leq \bar{\xi}_{max} - \bar{A}_d(p)\xi_k - \bar{K}_d \end{cases} \tag{41}$$

where: $\bar{\xi}_{min} = \begin{bmatrix} \xi_{min} & \cdots & \xi_{min} \end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$
$\bar{\xi}_{max} = \begin{bmatrix} \xi_{max} & \cdots & \xi_{max} \end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

Input constraints

$$u_c = \begin{bmatrix} u_{min} & u_{max} \end{bmatrix} = \begin{bmatrix} \delta_{f,min} & \delta_{f,max} \\ a_{min} & a_{max} \end{bmatrix} = \begin{bmatrix} -30° & +30° \\ -1m/s^2 & +0.5m/s^2 \end{bmatrix} \tag{42}$$

$$u_{min} \leq u_{i|k} \leq u_{max} \Rightarrow \begin{cases} -U_k \leq U_{min} \\ U_k \leq U_{max} \end{cases} \tag{43}$$

where: $U_{min} = \begin{bmatrix} u_{min} & \cdots & u_{min} \end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$
$U_{max} = \begin{bmatrix} u_{max} & \cdots & u_{max} \end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$

Input rate constraints

$$\Delta u_c = [\Delta u_{min} \quad \Delta u_{max}] = \begin{bmatrix} \Delta\delta_{f,min} & \Delta\delta_{f,max} \\ \Delta a_{min} & \Delta a_{max} \end{bmatrix} = \begin{bmatrix} -15°/sT_s & +15°/sT_s \\ -0.1m/s^2T_s & +0.1m/s^2T_s \end{bmatrix} \tag{44}$$

$$\Delta u_{min} \leq \Delta u_{i|k} \leq \Delta u_{max} \Rightarrow \begin{cases} -U_k \leq -\Delta U_{min} - U_{k-1} \\ U_k \leq \Delta U_{max} + U_{k-1} \end{cases} \tag{45}$$

where: $\Delta U_{min} = \begin{bmatrix}\Delta u_{min} & \cdots & \Delta u_{min}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$
$\Delta U_{max} = \begin{bmatrix}\Delta u_{max} & \cdots & \Delta u_{max}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$

$\Delta u_{i|k} = u_{i|k} - u_{i|k-1} \rightarrow \Delta U_k = U_k - U_{k-1}$
$\Delta U_k = \begin{bmatrix}\Delta u_{0|k} & \cdots & \Delta u_{N_c-1|k}\end{bmatrix}^T$
$U_{k-1} = \begin{bmatrix}\Delta u_{0|k-1} & \cdots & \Delta u_{N_c-1|k-1}\end{bmatrix}^T$

4.5 QP formulation

$$\begin{aligned} \min_{U_k} \quad & U_k^T H(p) U_k + 2f^T(p) U_k + g(p) \\ \text{s.t.} \quad & A_{ineq}(p) U_k \leq b_{ineq}(p) \end{aligned} \tag{46}$$

where: $H(p) = \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d\right)$
$f(p) = \left(\bar{A}_d \xi_k + \bar{K}_d(p) - ref_k\right)^T \bar{Q} \bar{B}_d$
$g(p) = \left(\bar{A}_d \xi_k + \bar{K}_d(p) - ref_k\right)^T \bar{Q} \left(\bar{A}_d \xi_k + \bar{K}_d(p) - ref_k\right)$

$$A_{ineq}(p) = \begin{bmatrix} -\bar{B}_d \\ \bar{B}_d \\ -I^{n_u N_p} \\ I^{n_u N_p} \\ -I^{n_u N_p} \\ I^{n_u N_p} \end{bmatrix}, \quad b_{ineq}(p) = \begin{bmatrix} -\bar{\xi}_{min} + \bar{A}_d \xi_k + \bar{K}_d(p) \\ \bar{\xi}_{max} - \bar{A}_d \xi_k - \bar{K}_d(p) \\ -U_{min} \\ U_{max} \\ -\Delta U_{min} - U_{k-1} \\ \Delta U_{max} + U_{k-1} \end{bmatrix}$$

5 MPC - state variation

5.1 Model linearization

The nonlinear system has to be linearized around the operative point $p = [\xi_0, u_0]$. The system can be approximated through the Taylor expansion, stopped to the first derivative:

$$\frac{d}{dt}(\delta\xi(t)) \approx \frac{\partial f}{\partial \xi}\delta\xi(t) + \frac{\partial f}{\partial u}\delta u(t) \Rightarrow \delta\dot{\xi}(t) \approx A(p)\delta\xi(t) + B(p)\delta u(t) \tag{47}$$

where: $\delta\xi(t) = \xi(t) - \xi_0 = \begin{bmatrix}\delta v_x & \delta v_y & \delta\omega_z & \delta s & \delta e_y & \delta e_\psi\end{bmatrix}^T$
$\delta u(t) = u(t) - u_0 = \begin{bmatrix}\delta\delta_f & \delta a\end{bmatrix}^T$

Changing the notation for semplicity, the new state/input definition is defined as the state error with respect the operative point:

$$\dot{\xi}(t) = A(p)\xi(t) + B(p)u(t) \tag{48}$$

5.2 Discretization

The system model is discretized according the Euler approximation:

$$\begin{aligned} A_d &= I + AT_s \\ B_d &= BT_s \end{aligned} \tag{49}$$

where: $T_s = \alpha T_{sys}:$ MPC sampling time

5.3 Prediction

At k-th sampled, the predicted state is:

$$\xi_{i+1|k} = A_d(p)\xi_{i|k} + B_d(p)u_{i|k}, \quad \xi_{0|k} = \xi_k \tag{50}$$

State prediction:

$$\bar{\xi}_k = \bar{A}_d(p)\xi_k + \bar{B}_d(p)U_k \tag{51}$$

where: $\bar{\xi}_k = \begin{bmatrix} \xi_{1|k} \\ \vdots \\ \xi_{N_p|k} \end{bmatrix} \in \mathbb{R}^{n_x N_p}; \quad U_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N_c-1|k} \end{bmatrix} \in \mathbb{R}^{n_u N_c};$

$$\bar{A}_d(p) = \begin{bmatrix} A_d(p) \\ A_d^2(p) \\ \vdots \\ A_d^{N_p}(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x};$$

$$\bar{B}_d(p) = \begin{bmatrix} B_d(p) & 0^{n_x,n_u} & \cdots & 0^{n_x,n_u} \\ A_d(p)B_d(p) & B_d(p) & \cdots & 0^{n_x,n_u} \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_c}(p)B_d(p) & A_d^{N_c-1}(p)B_d(p) & \cdots & B_d(p) \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_p-1}(p)B_d(p) & A_d^{N_p-2}(p)B_d(p) & \cdots & A_d^{N_p-N_c}(p)B_d(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_u N_c}$$

5.4 Optimization

5.4.1 Objective function

$$\begin{aligned} J(\xi_k) &= \sum_{i=1}^{N_p} \left\| \xi_{i|k} - ref_{i|k} \right\|_Q^2 + \sum_{i=0}^{N_c-1} \left\| u_{i|k} \right\|_R^2 \\ &= \left\| \bar{\xi}_k - ref_k \right\|_{\bar{Q}}^2 + \left\| U_k \right\|_{\bar{R}}^2 \\ &= U_k^T \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d \right) U_k + 2 \left(\bar{A}_d \xi_k - ref_k \right)^T \bar{Q} \bar{B}_d U_k \\ &+ \left(\bar{A}_d \xi_k - ref_k \right)^T \bar{Q} \left(\bar{A}_d \xi_k - ref_k \right) \end{aligned} \tag{52}$$

where: $ref_k = \begin{bmatrix} ref_{1|k} & \cdots & ref_{N_p|k} \end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

$$\bar{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x N_p}$$

$$\bar{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix} \in \mathbb{R}^{n_u N_c, n_u N_c}$$

5.4.2 Constraints

State constraints

$$\delta\xi_c = \begin{bmatrix}\delta\xi_{min} & \delta\xi_{max}\end{bmatrix} = \begin{bmatrix}\delta v_{x,min} & \delta v_{x,max} \\ \delta v_{y,min} & \delta v_{y,max} \\ \delta\omega_{z,min} & \delta\omega_{z,max} \\ \delta s_{min} & \delta s_{max} \\ \delta e_{y,min} & \delta e_{y,max} \\ \delta e_{\psi,min} & \delta e_{\psi,max}\end{bmatrix} = \begin{bmatrix}-0.1m/s & +0.1m/s \\ -0.01m/s & +0.01m/s \\ -0.01°/s & +0.01°/s \\ -0m & +0.1m \\ -0.1m & +0.1m \\ -0.1° & +0.1°\end{bmatrix} T_s \quad (53)$$

$$\delta\xi_{min} \leq \xi_{i|k} \leq \delta\xi_{max} \Rightarrow \begin{cases}-\bar{B}_d(p)U_k \leq -\delta\bar{\xi}_{min} + \bar{A}_d(p)\xi_k \\ \bar{B}_d(p)U_k \leq \delta\bar{\xi}_{max} - \bar{A}_d(p)\xi_k\end{cases} \quad (54)$$

where: $\delta\bar{\xi}_{min} = \begin{bmatrix}\delta\xi_{min} & \cdots & \delta\xi_{min}\end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$
$\delta\bar{\xi}_{max} = \begin{bmatrix}\delta\xi_{max} & \cdots & \delta\xi_{max}\end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

Input constraints

$$\delta u_c = \begin{bmatrix}\delta u_{min} & \delta u_{max}\end{bmatrix} = \begin{bmatrix}\delta\delta_{f,min} & \delta\delta_{f,max} \\ \delta a_{min} & \delta a_{max}\end{bmatrix} = \begin{bmatrix}-0.1° & +0.1° \\ -0.1m/s^2 & +0.1m/s^2\end{bmatrix} T_s \quad (55)$$

$$\delta u_{min} \leq u_{i|k} \leq \delta u_{max} \Rightarrow \begin{cases}-U_k \leq \delta U_{min} \\ U_k \leq \delta U_{max}\end{cases} \quad (56)$$

where: $\delta U_{min} = \begin{bmatrix}\delta u_{min} & \cdots & \delta u_{min}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$
$\delta U_{max} = \begin{bmatrix}\delta u_{max} & \cdots & \delta u_{max}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$

5.5 QP formulation

$$\begin{aligned}\min_{U_k} \quad & U_k^T H(p) U_k + 2f^T(p)U_k + g(p) \\ \text{s.t.} \quad & A_{ineq}(p)U_k \leq b_{ineq}(p)\end{aligned} \quad (57)$$

where: $$\begin{aligned} H(p) &= \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d\right) \\ f(p) &= \left(\bar{A}_d \xi_k - ref_k\right)^T \bar{Q} \bar{B}_d \\ g(p) &= \left(\bar{A}_d \xi_k - ref_k\right)^T \bar{Q} \left(\bar{A}_d \xi_k - ref_k\right) \\ A_{ineq}(p) &= \begin{bmatrix} -\bar{B}_d \\ \bar{B}_d \\ -I^{n_u N_p} \\ I^{n_u N_p} \end{bmatrix}, \quad b_{ineq}(p) = \begin{bmatrix} -\delta\bar{\xi}_{min} + \bar{A}_d \xi_k \\ \delta\bar{\xi}_{max} - \bar{A}_d \xi_k \\ -\delta U_{min} \\ \delta U_{max} \end{bmatrix} \end{aligned}$$

6 Appendix

6.1 QP linearized matrices

$$A = J_f(x) = \frac{\partial f}{\partial \xi} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{v}_x}{\partial v_x} & \frac{\partial \dot{v}_x}{\partial v_y} & \frac{\partial \dot{v}_x}{\partial \omega_z} & \frac{\partial \dot{v}_x}{\partial s} & \frac{\partial \dot{v}_x}{\partial e_y} & \frac{\partial \dot{v}_x}{\partial e_\psi} \\ \frac{\partial \dot{v}_y}{\partial v_x} & \frac{\partial \dot{v}_y}{\partial v_y} & \frac{\partial \dot{v}_y}{\partial \omega_z} & \frac{\partial \dot{v}_y}{\partial s} & \frac{\partial \dot{v}_y}{\partial e_y} & \frac{\partial \dot{v}_y}{\partial e_\psi} \\ \frac{\partial \dot{\omega}_z}{\partial v_x} & \frac{\partial \dot{\omega}_z}{\partial v_y} & \frac{\partial \dot{\omega}_z}{\partial \omega_z} & \frac{\partial \dot{\omega}_z}{\partial s} & \frac{\partial \dot{\omega}_z}{\partial e_y} & \frac{\partial \dot{\omega}_z}{\partial e_\psi} \\ \frac{\partial \dot{s}}{\partial v_x} & \frac{\partial \dot{s}}{\partial v_y} & \frac{\partial \dot{s}}{\partial \omega_z} & \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial e_y} & \frac{\partial \dot{s}}{\partial e_\psi} \\ \frac{\partial \dot{e}_y}{\partial v_x} & \frac{\partial \dot{e}_y}{\partial v_y} & \frac{\partial \dot{e}_y}{\partial \omega_z} & \frac{\partial \dot{e}_y}{\partial s} & \frac{\partial \dot{e}_y}{\partial e_y} & \frac{\partial \dot{e}_y}{\partial e_\psi} \\ \frac{\partial \dot{e}_\psi}{\partial v_x} & \frac{\partial \dot{e}_\psi}{\partial v_y} & \frac{\partial \dot{e}_\psi}{\partial \omega_z} & \frac{\partial \dot{e}_\psi}{\partial s} & \frac{\partial \dot{e}_\psi}{\partial e_y} & \frac{\partial \dot{e}_\psi}{\partial e_\psi} \end{bmatrix} \tag{58}$$

$$\frac{\partial \dot{v}_x}{\partial v_x} = -\frac{2C_f \sin(\delta_{f0})(v_{y0} + l_f \omega_{z0})}{m v_{x0}^2}$$

$$\frac{\partial \dot{v}_x}{\partial v_y} = \frac{2C_f \sin(\delta_{f0})}{m v_{x0}} + \omega_{z0}$$

$$\frac{\partial \dot{v}_x}{\partial \omega_z} = \frac{2C_f l_f \sin(\delta_{f0})}{m v_{x0}} + v_{y0}$$

$$\frac{\partial \dot{v}_y}{\partial v_x} = \frac{2}{m v_{x0}^2} \left[C_f (v_{y0} + l_f \omega_{z0}) \cos(\delta_{f0}) + C_r (v_{y0} - l_r \omega_{z0}) \right] - \omega_{z0}$$

$$\frac{\partial \dot{v}_y}{\partial v_y} = -\frac{2}{m v_{x0}} \left[C_f \cos(\delta_{f0}) + C_r \right]$$

$$\frac{\partial \dot{v}_y}{\partial \omega_z} = -\frac{2}{m v_{x0}} \left[C_f l_f \cos(\delta_{f0}) - C_r l_r \right] - v_{x0}$$

$$\frac{\partial \dot{\omega}_z}{\partial v_x} = \frac{2}{I_z v_{x0}^2} \left[C_f l_f (v_{y0} + l_f \omega_{z0}) \cos(\delta_{f0}) - C_r l_r (v_{y0} - l_r \omega_{z0}) \right]$$

$$\frac{\partial \dot{\omega}_z}{\partial v_y} = \frac{2}{I_z v_{x0}} \left[-C_f l_f \cos(\delta_{f0}) + C_r l_r \right]$$

$$\frac{\partial \dot{\omega}_z}{\partial \omega_z} = \frac{2}{I_z v_{x0}} \left[-C_f l_f^2 \cos(\delta_{f0}) - C_r l_r^2 \right]$$

$$\frac{\partial \dot{s}}{\partial s} = 0$$
$$\frac{\partial \dot{s}}{\partial e_y} = \frac{v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0})}{(1 - c_c e_{y0})^2} c_c$$
$$\frac{\partial \dot{s}}{\partial e_\psi} = \frac{-v_{x0}\sin(e_{\psi 0}) - v_{y0}\cos(e_{\psi 0})}{(1 - c_c e_{y0})} \dot{e}_\psi$$
$$\frac{\partial \dot{e}_y}{\partial s} = 0$$
$$\frac{\partial \dot{e}_y}{\partial e_y} = 0$$
$$\frac{\partial \dot{e}_y}{\partial e_\psi} = \left(v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0})\right) \dot{e}_{\psi 0}$$
$$\frac{\partial \dot{e}_\psi}{\partial s} = 0$$
$$\frac{\partial \dot{e}_\psi}{\partial e_y} = -\frac{v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0})}{(1 - c_c e_{y0})^2} c_c^2$$
$$\frac{\partial \dot{e}_\psi}{\partial e_\psi} = \frac{v_{x0}\sin(e_{\psi 0}) + v_{y0}\cos(e_{\psi 0})}{(1 - c_c e_{y0})} c_c \dot{e}_{\psi 0}$$
$$\frac{\partial \dot{v}_x}{\partial s} = \frac{\partial \dot{v}_x}{\partial e_y} = \frac{\partial \dot{v}_x}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{v}_y}{\partial s} = \frac{\partial \dot{v}_y}{\partial e_y} = \frac{\partial \dot{v}_y}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{\omega}_z}{\partial s} = \frac{\partial \dot{\omega}_z}{\partial e_y} = \frac{\partial \dot{\omega}_z}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{s}}{\partial v_x} = \frac{\cos(e_{\psi 0})}{1 - c_c e_{y0}}$$
$$\frac{\partial \dot{s}}{\partial v_y} = -\frac{\sin(e_{\psi 0})}{1 - c_c e_{y0}}$$
$$\frac{\partial \dot{s}}{\partial \omega_z} = 0$$

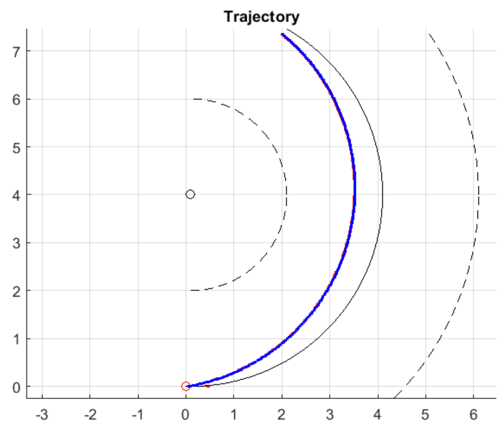
$$\frac{\partial \dot{e_y}}{\partial v_x} = \sin(e_{\psi 0})$$
$$\frac{\partial \dot{e_y}}{\partial v_y} = \cos(e_{\psi 0})$$
$$\frac{\partial \dot{e_y}}{\partial \omega_z} = 0$$
$$\frac{\partial \dot{e_\psi}}{\partial v_x} = -\frac{\cos(e_{\psi 0})}{1 - c_c e_{y0}} c_c$$
$$\frac{\partial \dot{\psi}}{\partial v_y} = \frac{\sin(e_{\psi 0})}{1 - c_c e_{y0}} c_c$$
$$\frac{\partial \dot{e_\psi}}{\partial \omega_z} = 1$$

$$B = J_f(u) = \frac{\partial f}{\partial u} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{v}_x}{\partial \delta_f} & \frac{\partial \dot{v}_x}{\partial a} \\ \frac{\partial \dot{v}_y}{\partial \delta_f} & \frac{\partial \dot{v}_y}{\partial a} \\ \frac{\partial \dot{\omega}_z}{\partial \delta_f} & \frac{\partial \dot{\omega}_z}{\partial a} \\ \frac{\partial \dot{s}}{\partial \delta_f} & \frac{\partial \dot{s}}{\partial a} \\ \frac{\partial \dot{e_y}}{\partial \delta_f} & \frac{\partial \dot{e_y}}{\partial a} \\ \frac{\partial \dot{e_\psi}}{\partial \delta_f} & \frac{\partial \dot{\psi}}{\partial a} \end{bmatrix} \tag{59}$$

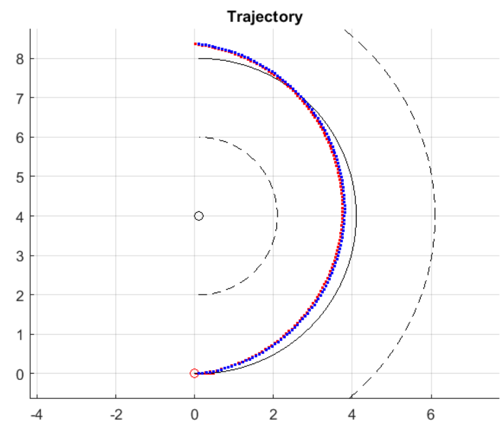
$$\frac{\partial \dot{v}_x}{\partial \delta_f} = -\frac{2C_f}{m} \left[\sin(\delta_{f0}) + \left(\delta_{f0} - \frac{v_{y0} + l_f w_{z0}}{v_{x0}} \right) \cos(\delta_{f0}) \dot{\delta}_{f0} \right]$$
$$\frac{\partial \dot{v}_x}{\partial a} = 1$$
$$\frac{\partial \dot{v}_y}{\partial \delta_f} = \frac{2C_f}{m} \left[\cos(\delta_{f0}) - \left(\delta_{f0} - \frac{v_{y0} + l_f w_{z0}}{v_{x0}} \right) \sin(\delta_{f0}) \dot{\delta}_{f0} \right]$$
$$\frac{\partial \dot{v}_y}{\partial a} = 0$$
$$\frac{\partial \dot{\omega}_z}{\partial \delta_f} = \frac{2C_f l_f}{m} \left[\cos(\delta_{f0}) - \left(\delta_{f0} - \frac{v_{y0} + l_f w_{z0}}{v_{x0}} \right) \sin(\delta_{f0}) \dot{\delta}_{f0} \right]$$
$$\frac{\partial \dot{\omega}_z}{\partial a} = 0$$

$$
\begin{aligned}
\frac{\partial \dot{s}}{\partial \delta_f} &= \frac{\partial \dot{s}}{\partial a} = 0 \\
\frac{\partial \dot{e_y}}{\partial \delta_f} &= \frac{\partial \dot{e_y}}{\partial a} = 0 \\
\frac{\partial \dot{e_\psi}}{\partial \delta_f} &= \frac{\partial \dot{e_\psi}}{\partial a} = 0
\end{aligned}
$$

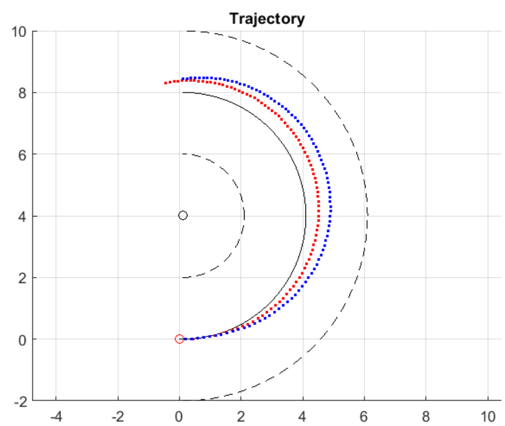
6.2 Cornering stiffness



(a) v = 0.5 m/s

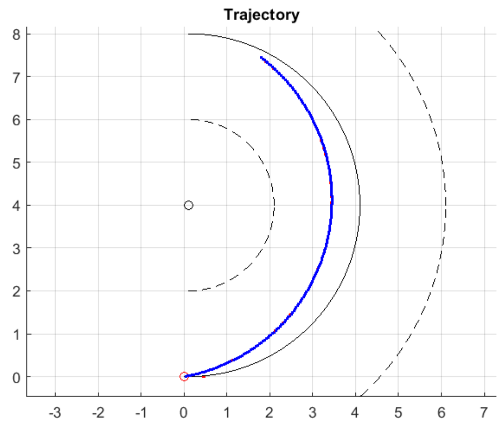


(b) v = 1 m/s

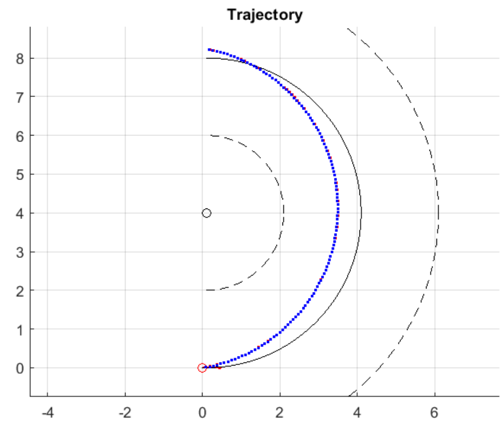


(c) v = 2 m/s

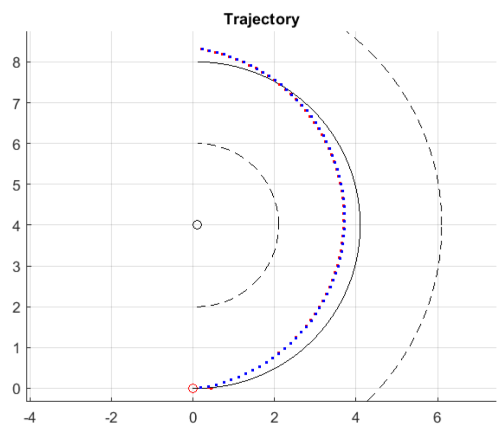
Figure 5: 2 Figures side by side



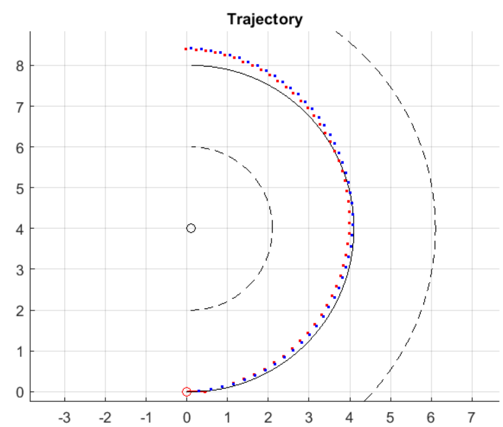
(a) v = 0.5 m/s



(b) v = 1 m/s

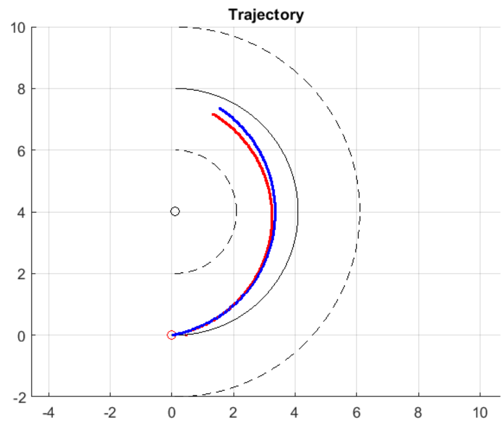


(c) v = 2 m/s

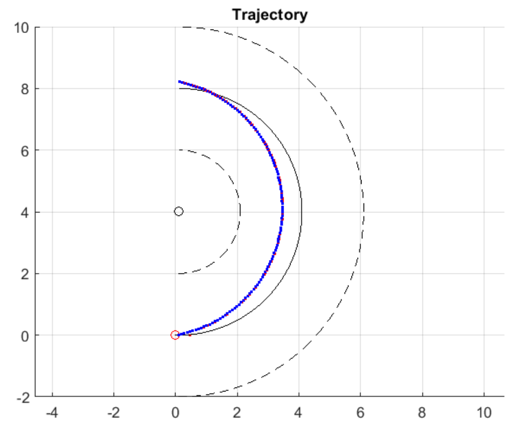


(d) v = 3 m/s

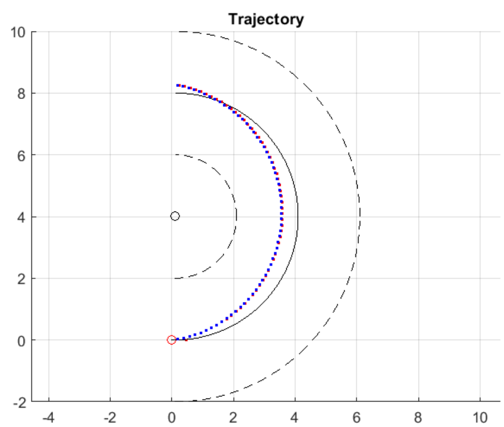
Figure 6: 2 Figures side by side



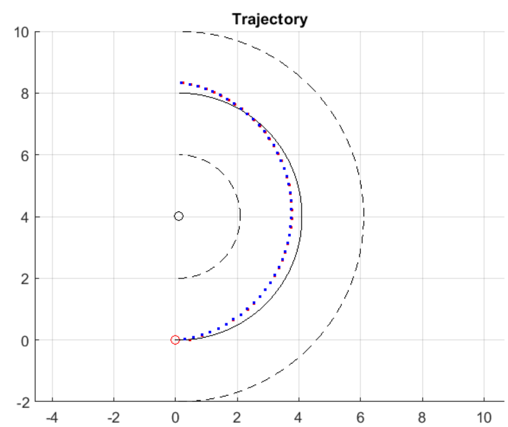
(a) v = 0.5 m/s



(b) v = 1 m/s



(c) v = 2 m/s



(d) v = 3 m/s

Figure 7: 2 Figures side by side

References

[1] R. Rajamani. *"Vehicle dynamics ancd control"*, Book, Cap2.

[2] A. Micaelli, C. Samson. *"Trajectory tracking for unicycle-type and two-steering-wheels mobile robots"*, Book, Cap2.

[3] Yiqi Gao. *"Model Predictive Control for Autonomous and Semiautonomous Vehicles"*, PhD Thesis, 2014.